A brief review of the linear stationary first-order iterative methods for solving linear systems

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Abstract

This paper makes a survey on the linear stationary first-order(degree) iterative methods for solving the linear systems. Some techniques of preconditioning which improve the rate of convergence of these iterative methods are presented. Some comparison results between Jacobi iterative method with the modified preconditioned simultaneous displacement (MPSD) iteration and other iterations are given. As well as, a new iterative method based on a block splitting of the coefficient matrix is developed. Convergence analysis of the proposed method is provided. Some results comparing the Jacobi iterative method with the new method are presented. Numerical examples are also given to illustrate our results.

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1 Introduction

Assume we are given the linear system of algebraic equations

$$Ax = b, \quad A \in \mathbb{C}^{n,n}, \quad b \in \mathbb{C}^{n,n}\backslash\{0\}, \tag{1.1}$$

and $\det(A) \neq 0$ so that the uniqueness of its solution is guaranteed. The simplest iterative method for the numerical solution of (1.1) is based on a splitting of A

$$A = M - N, \tag{1.2}$$

with $\det(M) \neq 0$ and M easily inverted. Thus (1.1) is written equivalently as

$$x = Tx + c, \qquad T = M^{-1}N, \qquad c = M^{-1}b. \tag{1.3}$$

which yields the following iterative scheme for the solution of (1.1):

$$x^{(m+1)} = Tx^{(m)} + c, m = 0, 1, 2, \ldots, \tag{1.4}$$

The initial vector $x^{(0)} \in \mathbb{C}^n$ can be arbitrary; if a good guess of the solution is available, it should be used for $x^{(0)}$. We say that the iterative method in Eq. (1.4) converges if it converges for any initial vector $x^{(0)}$. A sequence of vectors $x^{(1)}, x^{(2)}, \ldots$ can be computed from Eq. (1.4), and our objective is to choose M so that

(i) the sequence $\{x^{(m)}\}$ is easily computed,

(ii) the sequence $\{x^{(m)}\}$ converges rapidly to the solution.

As is Known a sufficient and necessary condition for (1.4) to converge, to the solution $\overline{x} = A^{-1}b = (I - T)^{-1}c$ of (1.1), is $\rho(T) < 1$, where $\rho(.)$ denotes spectral radius, while a sufficient condition for convergence is $\|T\| < 1$, where $\|.\|$ denotes matrix norm induced by a vector norm (see, e.g. [33], [36], [1]). Let $\omega \in \mathbb{C} \setminus \{0\}$ be the so-called extrapolated parameter and let us, based on (1.2), consider the splitting

$$A = M_\omega - N_\omega, \qquad M_\omega = (1/\omega)M, \qquad N_\omega = (1/\omega)[(1-\omega)M + \omega N], \tag{1.5}$$

and construct, for the solution of (1.1) the iterative (or extrapolation) method

$$x^{(m+1)} = T_\omega x^{(m)} + c_\omega, \qquad m = 0, 1, 2, \ldots, \tag{1.6}$$

where

$$T_\omega = (1-\omega)I + \omega T, \quad c_\omega = \omega c$$

Our problem now is that of finding ω's for which $\rho(T_\omega) < 1$ and among them to choose the one ω_b which minimizes $\rho(T_\omega)$. Now assume:

(i) the convex hull $H(T)$ of the spectrum $\sigma(T)$ of T (namely the smallest convex polygon containing all the eigenvalues of T in the closure of its interior) is known and

(ii) $1 \notin H(T)$.

Then the optimization problem posed previously possesses a unique solution.

Assume that A in (1.1) is written as

$$A = D_A - L_A - U_A, \tag{1.7}$$

where D_A, L_A, U_A, are any matrices, with $\det(D_A) \neq 0$ and D_A^{-1} easily computed, and define

$$L == D_A^{-1} L_A, \quad U = D_A^{-1} U_A, \quad c := D_A^{-1} b.$$

Then (1.1) is written equivalently as

$$(I - L - U)x = c, \tag{1.8}$$

where, throughout this paper, L and U will be considered as strictly lower and strictly upper triangular matrices unless otherwise stated. Without loss of generality it may be assumed that

$$A = I - L - U, \tag{1.9}$$

the matrix coefficient in (1.7). In the sequel we will use either (1.7) or (1.9), whichever is the most convenient.

1.1 Some special matrices

Definition 1.1. A matrix $A \in \mathbb{C}^{n,n}$ is said to be Hermitian if and only if (iff) $A^H = A$, where the superscript H denotes complex conjugate transpose. (A real Hermitian matrix is a real symmetric matrix and there holds $A^T = A$; where T denotes transpose.)

Definition 1.2. An Hermitian matrix $A \in \mathbb{C}^{n,n}$ is said to be positive definite iff $x^H Ax > 0, \forall x \in \mathbb{C}^n \backslash \{0\}$. (For A real symmetric, the condition becomes $x^T Ax > 0, \forall x \in \mathbb{R}^n \backslash \{0\}$).

Definition 1.3. (see [33].) A matrix A is said to be irreducible if the directed graph associated with A is strongly connected.

Definition 1.4. A matrix A is irreducibly diagonally dominant if A is irreducible and

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{i=n} |a_{ij}|, \qquad i = 1, \ldots, n;$$

with strict inequality for at least one i.

Notation 1.5. Let $A, B \in \Re^{n\times n}$. If $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$), $i, j = 1, 2, \ldots, n$, we write $A \geq B$ ($A > B$). The same notation applies to vectors $x, y \in \Re^n$.

Definition 1.6. If $A \in \Re^{n\times n}$ satisfies $A \geq 0 (> 0)$ then it is said to be *nonnegative (positive).* The same terminology applies to vectors $x \in \Re^n$.

Theorem 1.7. *(see [33].). Let $A \geq 0$ be an $n \times n$ matrix. Then $\rho(A)$ is an eigenvalue of A, and there exists a nonnegative eigenvector of A associated with $\rho(A)$.*

Notation 1.8. Let $A \in \mathcal{C}^{n\times n}$. Then $|A|$ denotes the matrix whose elements are the modula of the elements of A. The same notation applies to vectors $x \in \mathcal{C}^n$.

Theorem 1.9. *(see [33], [36].) Let $A \in \mathcal{C}^{n\times n}$ and $B \in \Re^{n\times n}$ satisfy $0 \leq |A| \leq B$, then $0 \leq \rho(A) \leq \rho(|A|) \leq \rho(B)$.*

Definition 1.10. (see [36].) A matrix $A \in \Re^{n\times n}$ is called a Z-matrix if for any $i \neq j$, $a_{ij} \leq 0$; an L-matrix if it is a Z-matrix with $a_{ii} > 0$, $i = 1, 2, \ldots, n$.

Definition 1.11. A matrix $A \in \Re^{n\times n}$ is said to be an M-matrix if $a_{ij} \leq 0$, $i \neq j = 1, 2, \ldots, n$, A is nonsingular and $A^{-1} \geq 0$.

Theorem 1.12. *(see [33].) Let $A = (a_{ij})$ and $B = (b_{ij})$ satisfy $A \leq B$ and $b_{ij} \leq 0$, for $i \neq j$. If A is an M-matrix, then B is an M-matrix*

Definition 1.13. A matrix $A \in \mathcal{C}^{n\times n}$ is said to be an H-matrix if its *comparison matrix*, that is, the matrix $\langle A\rangle$ with elements $\alpha_{ii} = |a_{ii}|$, $i = 1, 2, \ldots, n$, and $\alpha_{ij} = -|a_{ij}|$, $i \neq j = 1, 2, \ldots, n$, is an M-matrix.

Lemma 1.14. *(see [14].) A is an H-matrix if and only if there exist a vector $r > 0$ such that $\langle A\rangle r > 0$.*

Theorem 1.15. *(see [33].) If $A \geq 0$ is an H-matrix, then the following are equivalent:*

(1) $\alpha > \rho(A)$

(2) $\alpha I - A$ *is nonsingular, and* $(\alpha I - A)^{-1} \geq 0$.

Definition 1.16. (see [33].) A splitting (**??**) of a nonsingular matrix $A \in \Re^{n\times n}$ is said to be *regular* if $M^{-1} \geq 0$ and $N \geq 0$.

Theorem 1.17. *(see [33].) Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $N_2 \geq N_1 \geq 0$, then,* $0 \leq \rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1$.

Definition 1.18. A matrix $A \in \mathbb{C}_{n,n}$ possesses Young's "property A" if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} D_1 & B \\ C & D_2 \end{pmatrix}, \tag{1.10}$$

where D_1, D_2 are nonsingular diagonal matrices not necessarily of the same order. A special case of Young's "property A" is what Varga calls two-cyclic consistently ordered property [33].

Definition 1.19. A matrix $A \in \mathbb{C}_{n,n}$ is said to be two-cyclic consistently ordered if $\sigma(D^{-1}(\alpha L + (1/a)U))$ is independent of $\alpha \in \mathbb{C} \setminus \{0\}$.

Among others, matrices that possess both Young's "property A" and Varga's "two-cyclic consistently ordered property" are the tridiagonal matrices, with nonzero diagonal elements, and the matrices that have already form (1.10).

Varga generalized the concept of two-cyclic consistently ordered matrices to what he called (block) p-cyclic consistently ordered.

Definition 1.20. A matrix $A \in \mathbb{C}_{n,n}$ in the block form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{pmatrix} \tag{1.11}$$

is said to be (block) p-cyclic consistently ordered if $\sigma(D^{-1}(\alpha L+(1/\alpha^{p-1})U))$ is independent of $\alpha \in \mathbb{C}^n \setminus \{0\}$.

The best representative of such a block partitioned matrix will be the following:

$$A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & A_{1p} \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ 0 & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{p,p-1} & A_{pp} \end{pmatrix} \tag{1.12}$$

The block form of generalized $(q, p-q)$-cyclic consistently ordered matrices is the following:

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & A_{1,p-q+1} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 & 0 & A_{2,p-q+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A_{qp} \\ A_{q+1,1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{p,p-q} & 0 & 0 & \cdots & A_{pp} \end{pmatrix} \tag{1.13}$$

where the diagonal blocks satisfy the same restrictions as in (1.11) and p and q are relatively prime. Obviously, for $q = 1$ the generalized $(1; p-1)$-cyclic consistently ordered matrices reduce to the block p-cyclic consistently ordered ones of the previous section.

2 Iterative methods

2.1 Richardson method

We consider the Richardson method, in which $M = I$ and $N = I - A$ is chosen to be the identity matrix of order n. Eq. (1.4) in this case is as follows:

$$x^{(m+1)} = (I - A)x^{(m)} + b,$$

and the Richardson iteration matrix is $T = (I - A)$. In this approach the sequence $\{x^{(m)}\}$ is easily computed, but the rate of convergence of the sequence $\{x^{(m)}\}$ is very slow. By this method we have

$$\rho(T) = \max_{i=1}^{n}\{|1 - \lambda_i(A)|\}.$$

So, when A is a symmetric positive definite matrix, then the Richardson method converges if and only if $\rho(A) < 2$ [13].

2.2 The extrapolated Richardson (ER) method

Another iterative method is the ER iteration, in which $M = \frac{1}{\alpha}I$ and $N = \frac{1}{\omega}(I - \omega A)$, where $\omega > 0$ is called the extrapolation parameter. In this case we have

$$x^{(m+1)} = (I - \omega A)x^{(m)} + \omega b,$$

and the ER iteration matrix as $T = (I - \omega A)$. When A is a symmetric positive definite matrix, it can be shown that the method converges for any scalar ω satisfies

$$0 < \omega < \frac{2}{M(A)}$$

where $M(A)$ is the largest eigenvalue of A. The optimal extrapolation parameter would be

$$\omega_b = \frac{2}{m(A) + M(A)}$$

where $m(A)$ and $M(A)$ are the smallest and the largest eigenvalues of A, and in this case we have

$$\rho(T) = \frac{M(A) - m(A)}{M(A) + m(A)}$$

(see [13]).

2.3 Jacobi method

The Jacobi iteration is defined by $M = I$ and $N = L + U$. So, we have

$$T_J = (L + U)$$

$$x^{(m+1)} = (L + U)x^{(m)} + b, \tag{2.1}$$

For the Jacobi method the sequence $x^{(m)}$ is easily computed, and the rate of convergence is better than the Richardson's method.

Theorem 2.1. *Let matrix A in Eq. (1.1) be strictly diagonally dominant or irreducibly diagonally dominant, then the Jacobi method converge to $A^{-1}b$ for any arbitrary initial value $x^{(0)}$.*

Theorem 2.2. *Let $A = I - B$ be a nonnegative H-matrix, then $T_J = B$ and*

(1) $1 < \rho(A) < 2$;

(2) $\rho(B) = \rho(A) - 1$;

(3) $\rho(B) < 1$.

(see [15]).

2.4 Jacobi Overrelaxation (JOR) method

If we use the Jacobi method with an extrapolation parameter ω, i.e., $M = \frac{1}{\omega}I$ and $N = \frac{1}{\omega}[(1-\omega)I + \omega(L+U)]$, we have the extrapolated Jacobi method which is called the JOR method by Young [36]. Thus we have

$$T_{JOR} = (1-\omega)I + \omega(L+U)$$

$$x^{(m+1)} = [(1-\omega)I + \omega(L+U)]x^{(m)} + \omega b, \tag{2.2}$$

When A is a symmetric positive definite matrix, we can show [13] that

$$\omega_b = \frac{2}{2 - m(T_J) - M(T_J)}$$

where $m(T_J)$ and $M(T_J)$ are the smallest and the largest eigenvalues of T_J, (see [13]).

Theorem 2.3. *Let matrix A in Eq. (1.1) be strictly diagonally dominant or irreducibly diagonally dominant, then the JOR methods with $0 \le \omega \le 1$ converges to $A^{-1}b$ for any arbitrary initial value $x^{(0)}$.*

2.5 Gauss-Seidel method

Let us first observe forward Gauss-Seidel iteration and then observe the backward Gauss-Seidel iteration. The forward Gauss-Seidel is defined by letting $M = I - L$ and $N = U$. So, we have

$$T_{GS} = (I - L)^{-1}U$$
$$x^{(m+1)} = (I - L)^{-1}Ux^{(m)} + (I - L)^{-1}b, \tag{2.3}$$

The backward Gauss-Seidel is defined by letting $M = I - U$, $N = L$. So, we have

$$T_{GSB} = (I - U)^{-1}L$$
$$x^{(m+1)} = (I - U)^{-1}Lx^{(m)} + (I - U)^{-1}b, \tag{2.4}$$

Theorem 2.4. *Let matrix A in Eq. (1.1) be strictly diagonally dominant or irreducibly diagonally dominant, then the Gauss-Seidel converges to $A^{-1}b$ for any arbitrary initial value $x^{(0)}$.*

2.6 The extrapolated Gauss-Seidel (EGS) method

The forward extrapolated Gauss-Seidel (EGS) method is defined by $M = \frac{1}{\omega}(I - L)$, and $N = \frac{1}{\omega}[(1 - \omega)(I - L) + \omega U]$ where ω is the extrapolation parameter. So, we have

$$T_{EGS} = (1 - \omega)I + \omega(I - L)^{-1}U,$$
$$x^{(m+1)} = ((1 - \omega)I + \omega(I - L)^{-1}U)x^{(m)} + \omega(I - L)^{-1}b. \tag{2.5}$$

Similarly the backward EGS method can be obtained by letting $M = \frac{1}{\omega}(I - U)$, and $N = \frac{1}{\omega}[(1 - \omega)(I - U) + \omega L]$

Theorem 2.5. *Let A be irreducibly diagonally dominant, then the EGS method converges for $0 < \omega \leq 1$.*

The following theorems have been proved in [24].

Theorem 2.6. *If A is a two-cyclic consistently ordered matrix with nonzero diagonal elements and T_J has real eigenvalues, then $\rho(T_{EGS}) < 1$ if and only if $\rho(T_J) < 1$ and $0 \leq \omega < 2$.*

Theorem 2.7. *Let A be a two-cyclic consistently ordered matrix with nonzero diagonal elements such that T_J has real eigenvalues with $\rho(T_J) < 1$. If we let*

$$\omega_b = \frac{2}{2 - (\rho(T_J))^2}$$

then $\rho(T_{EGS})$ is minimized and its corresponding value is $\rho(T_{EGS}) = \frac{\omega_b(\rho(T_J))^2}{2}$

Theorem 2.8. *Under hypotheses of the previous theorem and by ω_b and $R(M) = -log\rho(M)$ as the asymptotically rate of convergence of matrix M, we have*

$$\lim_{\rho(T_J)\to 1^-} \frac{R(T_{EGS})}{R(T_J)} = 2$$

2.7 The successive overrelaxation (SOR) method

The next important iterative methods are known as forward and backward Successive Over-Relaxation methods commonly abbreviated as SOR. The forward SOR iterative method is defined by $M = \frac{1}{\omega}(I - \omega L)$, $N = \frac{1}{\omega}[(1-\omega)I + \omega U]$ and we have

$$T_{SOR} = (I - \omega L)^{-1}[(1-\omega)I + \omega U],$$

$$x^{(m+1)} = (D - \omega L)^{-1}[(1-\omega)I + \omega U]x^{(m)} + \omega(I - \omega L)^{-1}b. \qquad (2.6)$$

Evidently the forward SOR method (2.6) is reduced to the forward Gauss-Seidel method for $\omega = 1$. Similarly, the backward SOR iteration is defined by letting $M = \frac{1}{\omega}(I - \omega U))$, $N = \frac{1}{\omega}[(1-\omega)I + \omega L]$ and the backward SOR method reduces to the backward Gauss-Seidel method for $\omega = 1$.

The following theorems have been proved in [33].

Theorem 2.9. *(Kahan). A necessary condition for the SOR method to converge is $|\omega - 1| < 1$. (For $\omega \in \mathbb{R}$ this condition becomes $\omega \in (0, 2)$.)*

Theorem 2.10. *(Reich-Ostrowski-Varga). Let $A = D - E - E^H \in \mathcal{C}^{n,n}$ be Hermitian, D be Hermitian and positive definite, and $\det(D - \omega E) \neq 0, \forall \omega \in (0, 2)$. Then, $\rho(T_{SOR}) < 1$ iff A is positive definite and $\omega \in (0, 2)$. (Note: Notice that except for the restrictions in the statement the matrices $D, E \in \mathbb{C}^{n,n}$ must satisfy, they can be any matrices!)*

Theorem 2.11. *Let A be an irreducibly diagonally dominant matrix. Then the SOR method converge for* $0 \leq \omega \leq 1$.

A theorem connecting spectral radii of the Jacobi and the Gauss-Seidel iteration matrices associated with an L-matrix A was given originally by Stein and Rosenberg. In Young [36] a form of it that includes the spectral radius of the SOR iteration matrix is given below. Its proof is mainly based on the Perron-Frobenius theory.

Theorem 2.12. *If $A \in \mathbb{R}^{n,n}$ is an L-matrix and $\omega \in (0, 1]$, then:*

(a) $\rho(T_J) < 1$ *iff* $\rho(T_{SOR}) < 1$.

(b) $\rho(T_J) < 1$ *iff A is an M-matrix, if* $\rho(T_J) < 1$ *then* $\rho(T_{SOR}) \leq 1 - \omega + \omega\rho(T_J)$.

(c) If $\rho(T_J) \geq 1$ *then* $\rho(T_{SOR}) \geq 1 - \omega + \omega\rho(T_J) \geq 1$*:*

Notes: (i) The original form of Stein-Rosenberg theorem restricts to $\omega = 1$ and gives four mutually exclusive relations:

(a) $0 = \rho(T_J) = \rho(T_{GS})$, (b) $0 < \rho(T_{GS}) < \rho(T_J) < 1$,

(c) $1 = \rho(T_J) = \rho(T_{GS})$; (d) $1 < \rho(T_J) < \rho(T_{GS})$:

In [36] a theorem that gives an interval of ω for which the SOR method converges for M-matrices A is based on the theory of regular splittings is stated.

Theorem 2.13. *If $A \in \mathbb{R}^{n,n}$ is an M-matrix and if $\omega \in (0, 2/(1+\rho(T_J)))$ then* $\rho(T_{SOR}) < 1$.

The following is a statement extending the previous one to H-matrices.

Theorem 2.14. *If $A \in \mathbb{C}_{n,n}$ is an H-matrix and if $\omega \in (0; 2/(1+\rho(|T_J|)))$ then* $\rho(T_{SOR}) < 1$.

There is a class of matrices for which the investigation for the optimal value of ω leads to the most beautiful theory that has been developed for the last 50 years and which is still going on. It is associated with the class of p-cyclic consistently ordered matrices.

Such matrices naturally arise, e.g., for $p = 2$ in the discretization of second-order elliptic or parabolic PDEs by finite differences, finite element or collocation methods, for $p = 3$ in the case of large-scale least-squares problems, and for any $p > 2$ in the case of Markov chain analysis. For two-cyclic consistently ordered matrices A, Young [36] discovered that the eigenvalues μ and λ of the Jacobi and the SOR iteration matrices, respectively, associated with A satisfy the functional relationship

$$(\lambda + \omega - 1)^2 = \omega^2 \mu^2 \lambda. \tag{2.7}$$

He also found that if $T_J = (L + U)$, the eigenvalues of J^2 are nonnegative and $\rho(T_J) < 1$, then there exists an optimal value of ω, ω_b, such that

$$\omega_{opt} = \frac{2}{1 + (1 - \rho^2(T_J))1/2}, \rho(T_{SOR,\omega_{opt}}) = |\omega_{opt} - 1| (< \rho(T_{SOR,\omega}) \text{ for all } \omega \neq \omega_{opt}.) \tag{2.8}$$

(Note: For more details see [36].)

Remark 2.15. The spectrum $\rho(T_J)$ of the eigenvalues of the (block) Jacobi iteration matrix associated with a p-cyclic consistently ordered matrix A (1.12), which Varga calls weakly cyclic of index p [33], presents a p-cyclic symmetry about the origin. That is, with each eigenvalue $\mu \in \sigma(T_J) \setminus \{0\}$ there are another $p - 1$ eigenvalues of J; of the same multiplicity as that of μ; given by the expressions $\mu \exp(i(2\pi k)/p), k = 1, 2, \ldots, p - 1$.

Notation 2.16. From now on the Jacobi iteration matrix associated with a block p-cyclic consistently ordered matrix will be denoted by T_{J_p}.

For such matrices Varga [31] extended Young's results (2.7)-(2.8) to any $p > 3$; namely

$$(\lambda + \omega - 1)^p = \omega^p \mu^p \lambda^{p-1}. \tag{2.9}$$

He also proved that if the pth powers of the eigenvalues $\mu \in \sigma(T_{J_p})$ are real nonnegative and $\rho(J_p) < 1$, then there exists an optimal value of ω, ω_{opt}, which is the unique positive real root in $(1, p/(p - 1))$ of the equation

$$(\rho(T_{J_p})\omega_{opt})^p = \frac{p^p}{(p-1)^{p-1}}(\omega_{opt} - 1), \tag{2.10}$$

which is such that

$$\rho(T_{SOR,\omega_{opt}})^p = (p-1)(\omega_{opt}-1)(< \rho(T_{SOR,\omega}) \text{ for all } \omega \neq \omega_b), \tag{2.11}$$

In [1] the optimal values and the regions of convergence are also given.

Theorem 2.17. *Let the matrix $A \in \mathbb{C}^{n,n}$ be p-cyclic consistently ordered and suppose that all the eigenvalues of $T_{J_p}^p$ are nonnegative (nonpositive). Let $s = 1(-1)$ if the signs of the eigenvalues of $T_{J_p}^p$ are nonnegative (nonpositive). If*

$$\rho(T_{J_p}) < \frac{p-1-s}{p-2}, \tag{2.12}$$

then the regions of convergence of the SOR method ($\rho(T_{SOR}) < 1$) are

$$\text{For } s=1, \quad \omega \in (0, \frac{p}{p-1}) \text{ and for } s=-1, \omega \in (\frac{p-2}{p-1}, \frac{2}{1+\rho(T_{J_p})}). \tag{2.13}$$

The optimal relaxation factor ω_{opt} is the unique real positive root $\omega_{opt} \in ((2p-3+s)/(2(p-1)), (2p-1+s)/(2(p-1)))$ of the equation

$$(\rho(TJ_p)\omega_{opt})^p = sp^p(p-1)^{1-p}(\omega_{opt}-1) \tag{2.14}$$

and the optimal SOR spectral radius is given by

$$\rho(T_{SOR,\omega_{opt}}) = s(p-1)(\omega_{opt}-1)(< \rho(T_{SOR,\omega}) \text{ for all } \omega \neq \omega_b). \tag{2.15}$$

Note: For $p = 2; (p-2)/(p-2)$ and $p/(p-2)$ should be interpreted as 1 and ∞, respectively.

2.8 The symmetric SOR (SSOR) method

Each iteration step of the Symmetric SOR (SSOR) method consists of two semi-iterations the first of which is a usual (forward) SOR iteration followed by a backward SOR iteration, namely an SOR where the roles of L and U have been interchanged. More specifically

$$x^{(m+1/2)} = (I-\omega L)^{-1}[(1-\omega)I + \omega U]x^{(m)} + \omega(I-\omega L)^{-1}b$$

,

$$x^{(m+1)} = (I-\omega U)^{-1}[(1-\omega)I + \omega L]x^{(m+1/2)} + \omega(I-\omega U)^{-1}b$$

An elimination of $x^{(m+1/2)}$ from the above equations yields

$$x^{(m+1)} = T_{SSOR}x^{(m)} + c_{SSOR}, \qquad k = 0, 1, 2, \ldots, \quad x^{(0)} \in \mathbb{C}^n \ \text{ arbitrary} \tag{2.16}$$

with

$$T_{SSOR} = (I - \omega U)^{-1}[(1 - \omega)I + \omega L](I - \omega L)^{-1}[(1 - \omega)I + \omega U]$$

,

$$c_{SSOR} = \omega(2 - \omega)(I - \omega U)^{-1}(I - \omega L)^{-1}b.$$

For this method we have

$$M = \frac{1}{\omega(2 - \omega)}(I - \omega L)(I - \omega U), N = \frac{1}{\omega(2 - \omega)}[(1 - \omega)I + \omega L][(1 - \omega)I + \omega U].$$

Statements analogous to Kahan's theorem and also to Reich-Ostrowski Varga's theorem of the SOR method can be proved. Specifically we have:

Theorem 2.18. *A necessary condition for the SSOR method defined in (2.33) to converge is $|\omega - 1| < 1$. For $\omega \in \mathbb{R}$ the condition becomes $\omega \in (0, 2)$.*

Theorem 2.19. *Let $A \in \mathbb{C}^{n,n}$ be Hermitian with positive diagonal elements. Then for any $\omega \in (0, 2)$ the SSOR iteration matrix T_{SSOR} has real nonnegative eigenvalues. In addition, if A is positive definite then the SSOR method converges. Conversely, if the SSOR method converges and $\omega \in \mathbb{R}$ then $\omega \in (0, 2)$ and A is positive definite.*

For 2-cyclic consistently ordered matrices the first functional relationship between the eigenvalues μ and λ of the associated Jacobi and SSOR iteration matrices was given by D'Sylva and Miles [4] and Lynn [22] and is the following:

$$(\lambda - (\omega - 1)^2)^2 = \omega^2(2 - \omega)^2\mu^2\lambda.$$

It can be found that for A as in (1.10) the optimal ω, $\omega_b = 1$. Then $\rho(T_{SOR,1}) = \rho(T_{SSOR,1}) = \rho^2(T_J)$ The functional eigenvalue relationship in the case of block p-cyclic consistently ordered matrices was discovered by Varga, Niethammer and Cai [32], who obtained the relationship

$$(\lambda - (\omega - 1)^2)^p = \omega^p(2 - \omega)^2\mu^p\lambda(\lambda - (\omega - 1))^{p-2}.$$

The relationship above was then extended by Chong and Cai [3] to cover the class of GCO($q; p-q$) matrices in (1.13) to

$$(\lambda - (\omega-1)^2)^p = \omega^p (2-\omega)^{2q} \mu^p \lambda^q (\lambda - (\omega - 1))^{p-2q}.$$

Optimal values of the SSOR method for spectra $\sigma(J_p^p)$ nonnegative or nonpositive for any $p \geq 3$ cannot be found anywhere in the literature except in a very recent article [9], where a number of cases are covered analytically and experimentally and a number of conjectures based on strong numerical evidence are made

2.9 The unsymmetric SOR (USSOR) method

The USSOR method differs from the SSOR method in the second SOR part of each iteration where a different relaxation factor is used. It consists of the following two half steps:

$$x^{(m+1/2)} = (I - \omega_1 L)^{-1}[(1-\omega_1)I + \omega_1 U]x^{(m)} + \omega_1 (I - \omega_1 L)^{-1} b,$$
$$x^{(m+1)} = (I - \omega_2 U)^{-1}[(1-\omega_2)I + \omega_2 L]x^{(m+1/2)} + \omega_2 (I - \omega_2 U)^{-1} b$$

On elimination of $x^{(m+1/2)}$ it is produced

$$x^{(m+1)} = T_{USSOR} x^{(m)} + c_{USSOR}, \qquad k = 0, 1, 2, \ldots, \quad x^{(0)} \in \mathbb{C}^n \ \text{ arbitrary} \tag{2.17}$$

with

$$T_{USSOR} = (I - \omega_2 U)^{-1}[(1-\omega_2)I + \omega_2 L](I - \omega_1 L)^{-1}[(1-\omega_1)I + \omega_1 U],$$
$$c_{USSOR} = (\omega_1 + \omega_2 - \omega_1\omega_2)(I - \omega_2 U)^{-1}(I - \omega_1 L)^{-1} b.$$

We can obtain this method easily by

$$M = \frac{1}{\omega_1 + \omega_2) - \omega_1\omega_2}(I - \omega_1 L)^{-1}(I - \omega_2 U), N = \frac{1}{\omega_1 + \omega_2 - \omega_1\omega_2}[(1-\omega_1)I + \omega_1 L][(1-\omega_2)I + \omega_2 U].$$

Theory analogous to that of the SSOR method can be developed and the interested reader is referred to [36]. The only point we would like to make is that for p-cyclic consistently ordered and for GCO($q, p-q$) matrices A there are functional eigenvalue relationships connecting the eigenvalue spectra of the Jacobi and of the USSOR iteration matrices.

They were discovered by Saridakis [29] and the most general one below by Li and Varga [19]

$$(\lambda-(1-\omega_1)(1-\omega_2))^p = (\omega_1+\omega_2-\omega_1\omega_2)^{2q}\mu^p\lambda^q(\lambda\omega_1+\omega_2-\omega_1\omega_2)^{|\zeta_L|-q}(\lambda\omega_2+\omega_1-\omega_1\omega_2)^{|\zeta_U|-q}.$$

where $|\zeta_L|$ and $|\zeta_U|$ are the cardinalities of the sets ζ_L and ζ_U, which are the two disjoint subsets of $P = \{1, 2, \ldots, p\}$ associated with the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ as these are defined in [19].

2.10 The modified successive overrelaxation (MSOR) method

The idea of Modified SOR (or MSOR) method is to associate a different ω with each (block) row of the original linear system. The idea goes back to Russell [27] but it was mainly McDowell [23] and Taylor [30], who analyzed its convergence properties (see also [16]). It is best applied when the matrix A is 2-cyclic consistently ordered of the form (1.10). In such a case by partitioning b as to $b = [b_1^T, b_2^T]^T$, the MSOR method will be defined by the following iterative scheme:

$$\begin{cases} x_1^{m+1} = \omega_1(U_A x_2^m + b_1) + (I - \omega_1 D_1)x_1^m, \\ x_2^{m+1} = \omega_2(L_A x_1^m + b_2) + (I - \omega_2 D_2)x_2^m \end{cases}$$

Evidently we may write this iterative scheme in the form

$$x^{(m+1)} = T_{MSOR}x^{(m)} + c_{MSOR}, \tag{2.18}$$

where

$$T_{MSOR} = (D - \omega_2 L_A)^{-1}[\mathrm{diag}((1-\omega_1)D_1, (1-\omega_2)D_2) + \omega_1 U_A],$$

$$c_{MSOR} = (D - \omega_2 L_A)^{-1}\mathrm{diag}(\omega_1 I_{n_1}, \omega_2 I_{n_2})b$$

with I_{n_1}, I_{n_2} the unit matrices of the orders of D_1, D_2, respectively. In such a case the basic relationship that connects the eigenvalues μ and λ of the spectra $\sigma(J_2)$ and $\sigma(\omega_1, \omega_2)$ is

$$(\lambda + \omega_1 - 1)(\lambda + \omega_2 - 1) = \omega_1\omega_2\mu^2\lambda,$$

which reduces to the classical one by Young for the SOR method for $\omega_1 = \omega_2$. The following theorem has been proved in [10].

Theorem 2.20. *Let $A \in \mathbb{R}^{n,n}$ be a symmetric positive-definite matrix having property A and the block form*

$$A = \begin{pmatrix} I_{n_1} & -M \\ -M^T & I_{n_2} \end{pmatrix} = I - J_2, \qquad , n_1 + n_2 = n. \tag{2.19}$$

Then for any fixed $t = \rho^2(j_2) \in [0, 1)$ the pair (ω_1, ω_2) call it $(\widehat{\omega}_1, \widehat{\omega}_2)$, which yields the minimum in $\widehat{\delta} = \min_{\omega_1,\omega_2 \in (0,2)} \|T_{MSOR}\|_2$ is as follows. For $t \in [0, \frac{1}{3}]$

$$(\widehat{\omega}_1, \widehat{\omega}_2) = (\frac{1}{1+t}, \frac{1}{1-t}) \quad \text{when} \quad \widehat{\delta} = (\frac{t}{1+t})^{1/2}$$

while for $t \in [\frac{1}{3}, 1]$

$$(\widehat{\omega}_1, \widehat{\omega}_2) = (\frac{4}{5+t}, \frac{4}{3-t}) \quad \text{when} \quad \widehat{\delta} = (\frac{1+t}{3-t})$$

2.11 the extrapolated successive overrelaxation (ESOR) method

With $M = \frac{1}{\omega r}(I - rL)) = \frac{1}{\omega'}(I - rL))$ and $N = \frac{1}{\omega r}[(1 - \omega r)I + (\omega r - r)L + \omega r U] = \frac{1}{\omega'}[(1 - \omega')I + (\omega' - r)L + \omega' U]$, where $\omega' = \omega r$, we can obtain the Extrapolated SOR (ESOR) method with relaxation parameter r and extrapolation parameter ω').

Similarly, the backward ESOR iteration is defined by letting $M = \frac{1}{\omega r}(D - \alpha U)) = \frac{1}{\omega'}(D - \alpha U))$

2.12 The accelerated overrelaxation (AOR) method

The AOR method is first introduced by Hadjidimos [7]. This method uses two parameters r (called relaxation parameter) and ω (called extrapolation parameter). The forward AOR method is defined by $M = \frac{1}{\omega}(I - rL))$ and $N = [(1 - \omega)I + (\omega - r)L + \omega U]$. So, we have

$$T_{AOR} = (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U],$$

and

$$x^{(m+1)} = (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U]x^{(m)} + \omega(I - rL)^{-1}b, \tag{2.20}$$

Evidently the forward AOR method (2.10) is reduced to

(i) Jacobi method (2.1) for $r = 0$ and $\omega = 1$.

(ii) JOR method (2.2) for $r = 0$.

(iii) Forward Gauss-Seidel method (2.3) for $r = \omega = 1$.

(iv) EGS method (2.5) for $r = 1$.

(v) Forward SOR method (2.6) for $r = \omega$.

(vi) ESOR method for $\omega = \omega'$

For Hermitian matrices $A \in \mathcal{C}^{n,n}$ a statement analogous to the Reich-Ostrowski-Varga theorem holds for the AOR method as well. Here is one version of it given in [42].

Theorem 2.21. *Let $A = D - E - E^H \in \mathcal{C}^{n,n}$ be Hermitian; D be Hermitian and positive definite,* $\det(D - rE) \neq 0, \omega \in (0, 2)$ *and* $r \in (\omega + (2 - \omega)/\mu_m, \omega + (2 - \omega)/\mu_M)$ *with $\mu_m < 0 < \mu_M$ being the smallest and the largest eigenvalues of $D^{-1}(E + E^H)$. Then; $\rho(\mathcal{L}_{r,\omega} < 1$ if A is positive definite. (Note: Except for the restrictions in the statement the matrices $D, E \in \mathcal{C}^{n,n}$ can be any matrices.)*

Many more theoretical results can be proved in case A is$p$-cyclic consistently ordered. For example, if A is 2-cyclic consistently ordered and $\rho(J_2^2)$ is either nonnegative or nonpositive then optimal parameters for the AOR method can be derived. They are better than the optimal ones for the corresponding SOR method if some further assumptions are satisfied. These results can be found in [1,62,27].

Theorem 2.22. *Let $\underline{\mu}$ and $\bar{\mu}$ denote the absolutely smallest and the absolutely largest of the eigenvalues of the Jacobi iteration matrix J^2 of a 2-cyclic consistently ordered matrix A. Then: For $\sigma(J_2^2)$ nonnegative and $0 < \underline{\mu} < \bar{\mu} < 1$ if $1 - \underline{\mu}^2 < (1 - \bar{\mu}^2)^{1/2}$ the optimal parameters of the AOR method are given by the expressions*

$$r_b = \frac{2}{1 + (1 - \bar{\mu}^2)^{1/2}}, \quad \omega_b = \frac{1 - \underline{\mu}^2 + (1 - \bar{\mu}^2)^{1/2}}{(1 - \bar{\mu}^2)(1 + (1 - \bar{\mu}^2)^{1/2})} \tag{2.21}$$

$$\rho(\mathcal{L}_{r_b,\omega_b}) = \frac{\underline{\mu}(\bar{\mu}^2 - \underline{\mu}^2)^{1/2}}{(1 - \underline{\mu}^2)^{1/2}(1 + (1 - \bar{\mu}^2)^{1/2})} \tag{2.22}$$

Furthermore, for $0 < \underline{\mu} = \bar{\mu} < 1$ *there are two pairs of optimal parameters*

$$(r_b, \omega_b) = (\frac{2}{1 + \varepsilon(1 - \bar{\mu}^2)^{1/2}}, \frac{\varepsilon}{(1 - \bar{\mu}^2)^{1/2})\nu} \qquad \varepsilon \pm 1, \tag{2.23}$$

both of which give $\rho(\mathcal{L}_{r_b,\omega_b}) = 0$*. For* $\sigma(J_2^2)$ *nonpositive and if* $(1 + \bar{\mu}^2)^{1/2} < 1 + \underline{\mu}^2$ *the optimal parameters of the AOR method are given by the expressions*

$$r_b = \frac{2}{1 + (1 + \bar{\mu}^2)^{1/2}}, \quad \omega_b = \frac{1 + \underline{\mu}^2 + (1 + \bar{\mu}^2)^{1/2}}{(1 + \bar{\mu}^2)(1 + (1 + \bar{\mu}^2)^{1/2})}, \tag{2.24}$$

$$\rho(\mathcal{L}_{r_b,\omega_b} = \frac{\underline{\mu}(\bar{\mu}^2 2 - \underline{\mu}^2)^{1/2}}{(1 + \underline{\mu}^2)^{1/2}(1 + (1 + \bar{\mu}^2)^{1/2})}. \tag{2.25}$$

Again for $0 < \underline{\mu} = \bar{\mu}$ *there are two pairs of optimal parameters*

$$(r_b, \omega_b) = (\frac{2}{1 + \varepsilon(1 + \bar{\mu}^2)^{1/2}}, \frac{\varepsilon}{(1 + \bar{\mu}^2)^{1/2}}), \quad \varepsilon = \pm 1, \tag{2.26}$$

both of which give $\rho(\mathcal{L}_{r_b,\omega_b}) = 0$.

2.13 The symmetric AOR (SAOR) method

The SAOR iterative method [8] is defined by

$$\left\{ x^{(m+1/2)} = (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U]x^{(m)} + \omega(I - rL)^{-1}bx^{(m+1)} = (I - rU)^{-1}[(1 - \omega)I + (\omega - \right.$$

On elimination of $x^{(m+1/2)}$ it is produced

$$x^{(m+1)} = T_{SAOR}x^{(m)} + (I - rU)^{-1}[(2 - \omega)I + (\omega - r)(L + U)(I - rL)^{-1}b \tag{2.27}$$

where

$$T_{SAOR} = U_{r,\omega}L_{r,\omega}$$

$$U_{r,\omega} = (I - rU)^{-1}[(1 - \omega)I + (\omega - r)U + \omega L]$$

$$L_{r,\omega} = (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U]$$

2.14 The unsymmetric AOR (USAOR) method

The USAOR iterative method [?],[38] is defined by

$$\left\{ \; x^{(m+1/2)} = L_{r_1,\omega_1} x^{(m)} + \omega_1 (I - r_1 L)^{-1} b x^{m+1} = U_{r_2,\omega_2} x^{(m+1/2)} + \omega_2 (I - r_2 U)^{-1} b \right.$$

On elimination of $x^{(m+1/2)}$ it is produced

$$x^{(m+1)} = T_{USAOR} x^{(m)} + (I - r_2 U)^{-1} [(\omega_1 + \omega_2 - \omega_1 \omega_2) I + \omega_2 (\omega_1 - r_1) L + \omega_1 (\omega_2 - r_2) U (I - r_1 L)^{-1} b \tag{2.28}$$

where

$$T_{USAOR} = U_{r_2,\omega_2} L_{r_1,\omega_1}$$

$$U = (I - r_2 U)^{-1} [(1 - \omega_2) I + (\omega_2 - r_2) U + \omega_2 L]$$

$$L = (I - r_1 L)^{-1} [(1 - \omega_1) I + (\omega_1 - r_1) L + \omega_1 U]$$

It is easy to see that many known iterative methods are its special cases

(1) $r_1 = 0, \omega_1 = 1, r_2 = \omega_2 = 0$, Jacobi,

(2) $r_1 = \omega_1 = 1, r_2 = \omega_2 = 0$, GaussSeidel;

(3) $r_1 = 0, \omega_1 = \omega, r_2 = \omega_2 = 0$, JOR,

(4) $r_1 = \omega_1 = \omega, r_2 = \omega_2 = 0$, SOR,

(5) $r_1 = r_2 = \omega_1 = \omega_2 = \omega$, SSOR,

(6) $r_1 = \omega_1 = \omega, r_2 = \omega_2 = \widehat{\omega}$, USSOR,

(7) $r_1 = r, \omega_1 = \omega, r_2 = \omega_2 = 0$, AOR,

(8) $r_1 = r_2 = r, \omega_1 = \omega_2 = \omega$, SAOR,

2.15 The two-parameter overrelaxation (TOR) method

Kuang [?] generalized the AOR method in the case of A Hermitian (positive definite) by splitting further the component $L_A(= U_A^H)$ of A into the sum of two other matrices ($L_A = E_A + F_A$) and used again two parameters in his "two-parameter overrelaxation (TOR) method". He found many interesting results and applied his method for the numerical solution of the biharmonic equation. The TOR method corresponds to the splitting

$$M = \frac{1}{\alpha+\beta}(2D_A - \alpha E_A - \beta F_A),$$

$$N = \frac{1}{\alpha+\beta}(2-\alpha-\beta)D_A + (\alpha+\beta)(E_A^H + F_A^H) + \beta E_A + \alpha F_A],$$

with $\alpha+\beta)\det(2D_A - \alpha E_A - \beta F_A) \neq 0$ and $\alpha, \beta \in \mathbb{R}$. For $F_A = 0, \alpha = 2r, \beta = 2(\omega - r)$ or $E_A = 0, \alpha = 2(\omega - r), \beta = 2r$, TOR reduces to the AOR method.

2.16 The extrapolated modified Aitken (EMA) method

The extrapolated modified Aitken (EMA) iterative method was first introduced by Evans [5] as a method for solving the systems of linear algebraic equations arising from discretizing of the elliptic difference equations. This method could be easily used for solving the fuzzy linear systems by taking

$$M = \frac{1}{\omega}(I - \omega L)(I - \omega U), N = \frac{1}{\omega}[(1-\omega)I - \omega^2 LU].$$

and

$$x^{(m+1)} = T_{EMA}x^{(m)} + \omega(I - \omega U)^{-1}(I - \omega L)^{-1}b \tag{2.29}$$

where

$$T_{EMA} = (I - \omega U)^{-1}(I - \omega L)^{-1}[(1-\omega)I - \omega^2 LU]$$

2.17 the Preconditioned Simultaneous Displacement (PSD) method and the modified preconditioned simultaneous displacement (MPSD) method

The PSD and MPSD method is studied in [2], [18], [25]. Let us now transform the original system into the following preconditioned form

$$R^{-l}Ax = R^{-l}b, \tag{2.30}$$

where the conditioning matrix R is non-singular and is required to satisfy the following properties:

(i) The spectral condition number of $R^{-1}A, \kappa(R^{-1}A$ becomes smaller than $\kappa(A)$.

(ii) For any vectors s and t it is "computationally convenient" to solve the system $Rs = t$, i.e. R is easily solvable.

For the numerical solution of system (2.13) we define the general iterative scheme

$$x^{(m+l)} = x^{(m)} + rR^{-l}(b - Ax^{(m)}$$

where r is a real parameter. A general form of R associated with the splitting of $A = I - L - U$ is the following

$$R = (I - \omega_1 L)(I - \omega_2 U)$$

where ω_1, ω_2 are real parameters. In this case, the iterative method (2.13, which called MPSD method, can be written as follows:

$$x^{(m+1)} = x^{(m)} + r(I - \omega_2 U)^{-1}(I - \omega_1 L)^{-1}(b - Ax^{(m)}),$$

or

$$x^{(m+1)} = T_{MPSD}x^{(m)} + r(I - \omega_2 U)^{-1}(I - \omega_1 L)^{-1}b \tag{2.31}$$

where

$$\begin{aligned} T_{MPSD} &= I - r(I - \omega_2 U)^{-1}(I - \omega_1 L)^{-1}A \\ &= I - \omega_1 L)^{-1}(I - \omega_2 U)^{-1}[(1-r)I + (r - \omega_1)L + (r - \omega_2)U + \omega_1\omega_2 LU]. \\ &= M^{-1}N, \end{aligned}$$

where $M = (I - \omega_1 L)(I - \omega_2 U), N = M - rA; \omega_1, \omega_2, r \in \mathbb{R}$ and $r \neq 0$.. For $\omega_1, \omega_2 = \omega$ we obtain, from (2.13, the PSD iterative method defined by

$$x^{(m+1)} = T_{PSD} x^{(m)} + r(I - \omega U)^{-1}(I - \omega L)^{-1} b \tag{2.32}$$

where

$$T_{PSD} = I - r(I - \omega U)^{-1}(I - \omega L)^{-1} A$$

Some special cases of MPSD method are studied in [4,5,9,14].With special values of ω_1, ω_2 and r, the corresponding iterative methods are as follows:

(1) $r = 1, \omega_1 = 0, \omega_2 = 0$, Jacobi,

(2) $r = \omega, \omega_1 = 0, \omega_2 = 0$, JOR,

(3) $r = \omega, \omega_1 = 0, \omega_2 = 0$, RF,

(4) $r = 1, \omega_1 = 1, \omega_2 = 0$, GS,

(5) $r = \omega, \omega_1 \omega, \omega_2 = 0$, SOR,

(6) $r = r, \omega_1 = \omega, \omega_2 = 0$, AOR,

(7) $r = \omega(2 - \omega), \omega_1 = \omega_2 = \omega$, SSOR,

(8) $r = \omega, \omega_1 = \omega, \omega_2 = \omega,$, EMA,

(9) $r = r, \omega_1 = \omega, \omega_2 = \omega$, PSD,

When $\tau < \omega_i \leq 1, i = 1, 2$, the following theorem is presented in [2].

Theorem 2.23. *(Chen [2]). Let A be irreducible, $B = L + U \geq 0$. Then, for $0 < \omega_i < r \leq 1, i = 1, 2$, we have*

(1) $\rho(B) > 0, \rho(S_{r,\omega_1,\omega_2}) < 1 - r$,

(2) $0 < \rho(B) < 1 \Leftrightarrow 1 - r < \rho(S_{r,\omega_1,\omega_2}) < 1$,

(3) $\rho(B) = 1 \Leftrightarrow \rho(S_{r,\omega_1,\omega_2}) = 1$,

(4) $\rho(B) > 1 \Leftrightarrow \rho(S_{r,\omega_1,\omega_2}) > 1$.

In [21], the authors considered the iterative scheme

$$x^{(m+l)} = x^{(m)} + R^{-l}T(b - Ax^{(m)} \tag{2.33}$$

where R is the nonsingular matrix

$$R = (I - \Omega L)(I - \Omega U)$$

with T and Ω the diagonal matrices $T = \text{diag}(r_1 I_1, r_2 I_2)$, with $r_1, r-2 \in \mathbb{R}\ \{0\}, \Omega = \text{diag}(\omega_1 I_1, \omega_2 I_2)$, with $r_1, r-2 \in \mathbb{R}, I_1 I_2$ identity matrices of order n_1, n_2, respectively. Alternatively, (2.33) takes the form

$$x^{(m+1)} = D_{T,\Omega}x^{(m)} + \delta_{T,\Omega} \tag{2.34}$$

where $D_{T,\Omega}$ is the iteration matrix of the MPSD method, with

$$D_{T,\Omega} = I - (I - \Omega U)^{-1}(I - \Omega L)^{-1}TA$$

and

$$\delta_{T,\Omega} = (I - \Omega U)^{-1}(I - \Omega L)^{-1}Tb$$

For various values of the parameters r_1, r_2, ω_1 and ω_2, (2.34) yields some known iterative methods.

(1) $\omega_1 = 0$, the Modified Extrapolated SOR (MESOR) method,

(2) $\omega_1 = \omega_2 = 1$, the backward Modified Extrapolated GaussSeidel (MEGS) method,

(3) $r_1 = r_2 = \omega(2 - \omega), \omega_1 = \omega_2 = \omega$,the Symmetric SOR (SSOR) method,

(4) $r_1 = r_2 = \omega_1 + \omega_2 - \omega_1\omega_2$,the Unsymmetric SOR (USSOR) method

(5) $r_1 = r_2 = r, \omega_1 = \omega_2 = \omega$, the PSD method,

3 Improving iterative methods

Transforming the original systems (1.1) into the preconditioned form

$$PAx = Pb \tag{3.1}$$

Then, we can define the basic iterative scheme:

$$M_p x^{(k+1)} = N_p x^{(k)} + Pb, \qquad k = 0, 1, \ldots,$$

where $PA = M_p^{-1} N_p$ and M_p is nonsingular. Thus (5) can also be written as

$$x^{(k+1)} = Tx^{(k)} + c, , \qquad k = 0, 1, \ldots,$$

where $T = M_p^{-1} N_p, c = M_p^{-1} b$. Assuming

$$P_A = \widehat{A} = \widehat{D} - \widehat{L} - \widehat{U},$$

where $\widehat{D}, \widehat{L}$ and $\widehat{U}$ are diagonal, strictly lower and strictly upper triangular part of $\widehat{A}$, respectively.

3.1 Improving Jacobi method

The iteration matrix of the classical Jacobi method for preconditioned system is given by $T = \widehat{D}^{-1}\widehat{B}$, where $\widehat{B}\widehat{L} + \widehat{U}$. In [improving Jaobi], the authors proposed a preconditioner of the following general form:

$$P = I + S = \begin{pmatrix} 1 & & & & -c_{1,k_1} & & \\ & 1 & & & & -c_{2,k_2} & \\ & & \ddots & & & & \\ & -c_{r,k_r} & & \ddots & & & \\ & & & & \ddots & & \\ & & -c_{n,k_n} & & & & 1 \end{pmatrix} \tag{3.2}$$

We can obtain $\widehat{A} = (\widehat{a}_{ij})_{n,n} = \widehat{D} - \widehat{B}$ and

$$\widehat{a}_{ij} = a_{ij} - c_{ik_i} a_{k_i j}, i, j = 1, \ldots, n.$$

If $c_{ik_i}a_{k_i i} \neq 1, i, j = 1, \ldots, n$, then $\widehat{D}^{-1}$ exists, and hence it is possible to define Jacobi iteration matrix for $\widehat{A}$,namely $T = \widehat{D}^{-1}\widehat{B}$.

Theorem 3.1. *If $A = I - L - U$ is a nonnegative H-matrix, let $B = L + U$ and*

$$b_i = \min_i \{\frac{2a_{ij}}{a_{k_i j} + a_{k_i i}a_{ij}}\} \qquad i = 1, \ldots, n,$$

for any i, if $a_{k_i i} \leq b_i$ and $a_{k_i j} \geq a_{k_i i}a_{ij}, j = 1, \ldots, n$, then

(1) $|\widehat{D}^{-1}A| \leq A$;

(2) $\rho(|\widehat{D}^{-1}A|) \leq \rho(A)$;

(3) $\rho(|\widehat{D}^{-1}B|) \leq \rho(B) < 1$;

(4) $\widehat{A}$ is an H-matrix.

(Note that: if $a_{k_i j} + a_{k_i i}a_{ij} = 0$ and $a_{ij} \neq 0$ we take $\frac{2a_{ij}}{a_{k_i j}+a_{k_i i}a_{ij}} = \infty$ if $a_{k_i j} + a_{k_i i}a_{ij} = 0$ and $a_{ij} = 0$, we take $\frac{2a_{ij}}{a_{k_i j}+a_{k_i i}a_{ij}} = 0$.)

In [ ], the authors proposed to take $c_{k_i i} = bi$.

3.2 Improving Gauss-Seidel method

In [Gaussimproving Jaobi], the authors proposed the preconditioners of the following general forms:

$$P_{max} = I + S_{max} + R_{max}, \quad P_R = I + S_{max} + R, \quad P_{S_{max}} = I + S_{max},$$

where

$$S_{max} = (s_{ij}) = \begin{cases} -a_{i,k_i}, & i = 1, \ldots, n-1, j > i; \\ 0 & \text{otherwise.} \end{cases}$$

with $k_i = \min\{j | \max_j |a_{i,j}, i < n\}$ and

$$R_{max} = ((R_m ax)_{ij})) = \begin{cases} -a_{n,k_n}, & i = n, j = k_n, \\ 0 & \text{otherwise.} \end{cases}$$

with $k_n = \min\{j||a_{n,j}| = max\{|a_{n,j}|, l = 1, \ldots, n-1\}\}$ and

$$R = ((R)_{ij}) = \begin{cases} -a_{i,j}, & i = n, 1 \leq j \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

For these preconditioners, the preconditioned matrices are

$$A_{max} = P_{max}A, \qquad A_R = P_R A, \qquad A_{S_{max}} = P_{S_{max}}A.$$

For matrices , $A, A_{max}, A_R, A_{S_{max}}$ if we denote the iteration matrix of the Gauss-Seidel method by $T, T_{max}, T_R, T_{S_{max}}$, respectively, we have the following comparison theorems:

Theorem 3.2. *Let A be a nonsingular M-matrix. Assume that $0 < a_{i,k_i}a_{k_i i} < 1, 1 \leq i \leq n-1$ and $0 < a_{n,k_j}a_{k_j n} < 1, k_j = 1, \ldots, n-1$. Then, we have*

$$\rho(T_{max}) \leq \rho(T) < 1 \qquad \text{and} \qquad \rho(T_{max}) \leq \rho(T_{S_{max}}) < 1$$

.

Theorem 3.3. *Let A be a nonsingular M-matrix. Assume that $0 < a_{i,k_i}a_{k_i i} < 1, 1 \leq i \leq n-1$ and $0 \leq \sum_{k=1}^{n-1} a_{n,k}a_{k,n} < 1, k_j = 1, \ldots, n-1$. Then, we have*

$$\rho(T_R) \leq \rho(T) < 1$$

.

Theorem 3.4. *Let A be a nonsingular M-matrix. Assume that $0 < a_{i,k_i}a_{k_i i} < 1, 1 \leq i \leq n-1$, $0 \leq \sum_{k=1}^{n-1} a_{n,k}a_{k,n} < 1, k_j = 1, \ldots, n-1$ and $a_{nj}\sum_{k=1}^{n-1} a_{n,k}a_{k,n} \leq \sum_{k=1}^{n-1} a_{n,k}a_{k,j}, 1 \leq j \leq n-1$. Then, we have*

$$\rho(T_R) \leq \rho(T_{S_{max}}) < 1$$

.

Theorem 3.5. *Let A be a nonsingular M-matrix. If $(a_{n,k_n}a_{k_n j} - a_{n,j})\sum_{k=1}^{n-1} a_{n,k}a_{k,n} - a_{n,k_n}a_{k_n,j} \leq (a_{n,k_n}a_{k_n,n} - 1)\sum_{k=1}^{n-1} a_{n,k}a_{k,j} - a_{n,j}a_{n,k_n}a_{k_n,n}, 1 \leq j \leq n-1$. then under the assumptions of Theorems 3.2 and 3.5 we have*

$$\rho(T_R) \leq \rho(T_{max}) \leq \rho(T_{S_{max}})$$

.

3.3 Improving SOR method

In [dehghan], the authors introduced two preconditioners $\overline{P} = I + \overline{S}$ and $\widetilde{P} = I + \widetilde{S}$ with

$$\overline{S} = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ -(a_{21}+\gamma_2) & 0 & . & . & . & 0 \\ -(a_{31}+\gamma_3) & . & . & . & . & \\ . & . & & . & & . \\ . & . & & & . & 0 \\ -(a_{n1}+\gamma_n) & 0 & . & . & . & 0 \end{pmatrix} \tag{3.3}$$

$$\overline{S} = \begin{pmatrix} 0 & 0 & . & . & . & -(a_{1n}+\delta_1) \\ 0 & 0 & . & . & . & -(a_{2n}+\delta_2) \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & -(a_{n-1,n}+\delta_{n-1}) \\ 0 & 0 & . & . & . & 0 \end{pmatrix} \tag{3.4}$$

where $\gamma_2, \gamma_3 \ldots, \gamma_n$ and $\delta_1, \delta_2, \ldots, \delta_{n-1}$ are real parameters. Now we consider two preconditioned linear systems as follows:

$$\overline{A}x = \overline{b} \qquad \text{where} \qquad \overline{A} = PA \;\; \text{and} \;\; \overline{b} = Pb$$

$$\widetilde{A}x = \widetilde{b} \qquad \text{where} \qquad \widetilde{A} = PA \;\; \text{and} \;\; \widetilde{b} = Pb$$

Suppose that

$$\overline{AD} - \overline{L} - \overline{U} \quad \text{and} \quad \widetilde{A} = \widetilde{D} - \widetilde{L} - \widetilde{U}$$

where

$$\overline{D} = I + \overline{D}_1, \quad \overline{L} = L + -\overline{S} + \bar{l}_1 \quad \overline{U} = U + \overline{U}_1$$

and

$$\widetilde{D} = I + \widetilde{D}_1, \quad \widetilde{L} = L + \widetilde{L}_1 \quad \widetilde{U} = U + \widetilde{UD}_1$$

It is not difficult to see that $\overline{D}_1, \overline{L}_1$, and $\overline{U}_1$ ($\widetilde{D}_1, \widetilde{L}_1$ and $\widetilde{U}_1$) are diagonal, strictly lower and strictly upper triangular parts of $\overline{S}U = -\overline{D}_1 + \overline{L}_1 + \overline{U}_1(\widetilde{S}A = \widetilde{S} - \widetilde{S}U - \widetilde{S}L =$

$\widetilde{D}_1 - \widetilde{L}_1 - \widetilde{U}_1 + U1)$, respectively. Two different forms of SOR iteration matrix associated with $\overline{A}$ and $\widetilde{A}$ can be denoted by

$$\overline{T}_1(\omega) = (\overline{D} - \omega\overline{L})^{-1}[(1-\omega)\overline{D} + \omega\overline{U}], \quad \overline{T}_2(\omega) = (I - \omega\overline{L})^{-1}[(1-\omega)I + \omega(\overline{U} - \overline{D}_1)]$$

and

$$\widetilde{T}_1(\omega) = (\widetilde{D}\omega\widetilde{L}^{-1}[(1-\omega)\widetilde{D} + \omega\widetilde{U}], \quad \widetilde{T}_2(\omega) = (I - \omega\widetilde{L})^{-1}[(1-\omega)I + \omega(\widetilde{U} - \widetilde{D}_1)]$$

respectively. we have the following comparison theorems:

Theorem 3.6. *Let $T(\omega), \overline{T}_1(\omega)$ be defined by (2) and (11). If $\gamma_q \in (1 - a_{1q}a_{q1})/a_{1q}, -a_{q1}) \bigcap (0, -a_{q1}), 0 < \omega < 1$ and A is an irreducible L-matrix with $a_{1q}a_{q1} > 0$ for $q = 2, 3, \ldots, n$, then*

(1) $\rho(\overline{T}_1(\omega)) < \rho(T(\omega))$, *if* $\rho(T(\omega)) < 1$;

(2) $\rho(\overline{T}_1(\omega)) = \rho(T(\omega))$, *if* $\rho(T(\omega)) = 1$;

(1) $\rho(\overline{T}_1(\omega)) > \rho(T(\omega))$, *if* $\rho(T(\omega)) > 1$;

Theorem 3.7. *Let $T(\omega), \overline{T}_2(\omega)$ be defined by (11) and (11). If $\gamma_q \in (1 - a_{1q}a_{q1})/a_{1q}, -a_{q1}) \bigcap (0, -a_{q1}), 0 < \omega < 1$ and A is an irreducible L-matrix with $a_{1q}a_{q1} > 0$ for $q = 2, 3, \ldots, n$, then*

(1) $\rho(\overline{T}_2(\omega)) < \rho(T(\omega))$, *if* $\rho(T(\omega)) < 1$;

(2) $\rho(\overline{T}_2(\omega)) = \rho(T(\omega))$, *if* $\rho(T(\omega)) = 1$;

(1) $\rho(\overline{T}_2(\omega)) > \rho(T(\omega))$, *if* $\rho(T(\omega)) > 1$;

Theorem 3.8. *Let $T(\omega), \widetilde{T}_1(\omega)$ be defined by (2) and (12). If $\delta_s \in (1 - a_{sn}a_{ns})/a_{ns}, -a_{sn}) \bigcap (0, -a_{sn}), 0 < \omega < 1$ and A is an irreducible L-matrix with $a_{ns}a_{sn} > 0$ for $s = 1, 2, \ldots, n-1$, then*

(1) $\rho(\widetilde{T}_1(\omega)) < \rho(T(\omega))$, *if* $\rho(T(\omega)) < 1$;

(2) $\rho(\widetilde{T}_1(\omega)) = \rho(T(\omega))$, *if* $\rho(T(\omega)) = 1$;

(1) $\rho(\widetilde{T}_1(\omega)) > \rho(T(\omega))$, *if* $\rho(T(\omega)) > 1$;

Theorem 3.9. *Let $T(\omega), \widetilde{T}_2(\omega)$ be defined by (2) and (12). If $\delta_s \in (1 - a_{sn}a_{ns})/a_{ns}, -a_{sn}) \bigcap (0, -a_{sn}), 0 < \omega < 1$ and A is an irreducible L-matrix with $a_{ns}a_{sn} > 0$ for $s = 1, 2, \ldots, n-1$, then*

(1) $\rho(\widetilde{T}_2(\omega)) < \rho(T(\omega))$, *if* $\rho(T(\omega)) < 1$;

(2) $\rho(\widetilde{T}_2(\omega)) = \rho(T(\omega))$, *if* $\rho(T(\omega)) = 1$;

(1) $\rho(\widetilde{T}_2(\omega)) > \rho(T(\omega))$, *if* $\rho(T(\omega)) > 1$;

3.4 Improving SSOR method

In [ ] the authors proposed a multi-parameters preconditioned SSOR iterative method with a preconditioner as following:

$$\widetilde{P} = I + \widetilde{S}_\alpha$$

where

$$\widetilde{S}_\alpha = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -\alpha_2 a_{21} & 0 & 0 & \ddots & \vdots \\ 0 & \alpha_3 a_{32} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha_n a_{nn-1} & 0 \end{pmatrix} \tag{3.5}$$

Now, we consider the preconditioned linear system

$$\widetilde{A}x = \widetilde{b}$$

where$\widetilde{A} = (I + \widetilde{S}_\alpha)A$ and $\widetilde{b} = (I + \widetilde{S}_\alpha)$ We express the coefficient matrix $\widetilde{A}$ of (5) as

$$\widetilde{A} = \widetilde{D} - \widetilde{L}\widetilde{U}$$

where $\widetilde{D}$ is the diagonal matrix, $\widetilde{L}$ and $\widetilde{A}$ are strictly lower and strictly upper triangular matrices, respectively. Then the corresponding iterative matrix of the above preconditioned SSOR method is

$$\widetilde{T}_{SSOR} = (\widetilde{D} - \omega\widetilde{U})^{-1}[(1-\omega)\widetilde{D} + \omega\widetilde{L}](\widetilde{D} - \omega\widetilde{L})^{-1}[(1-\omega)\widetilde{D} + \omega\widetilde{U}]$$

Theorem 3.10. *Let A be a nonsingular M-matrix, T_{SSOR} and $\widetilde{T}_{SSOR}$ are be defined by (4) and (6), respectively. Assume that $0 < \omega \leq 1$ and $0 \leq \alpha_i \leq 1, i = 2, 3, \ldots, n$, then*

$$\rho(\widetilde{T}_{SSOR}) \leq T_{SSOR} < 1$$

3.5 Improving AOR method

In [ ], the authors considered the preconditioned linear system as follows:

$$\overline{A}x = \overline{b} \qquad \text{where} \qquad \overline{A} = PA \;\text{ and }\; \overline{b} = Pb$$

and split $\overline{A}$

$$\overline{A} = \overline{D} - \overline{L} - \overline{U}$$

with $\overline{D}, \overline{L}$ and $\overline{U}$ being diagonal, strictly lower and strictly upper triangular matrices, respectively. The preconditioned AOR iterative method of (1.1), i.e., the AOR iterative method of (2.1), is defined as

$$x^{k+1} = \overline{T}_{AOR}x^{(k)} + (\overline{D} - r\overline{L})^{-1}\omega b, \quad k = 0, 1, 2, \ldots,$$

where

$$\overline{T}_{AOR} = (\overline{D} - r\overline{L})^{-1}[(1-\omega)\overline{D} + (\omega - r)\overline{L} + \omega\overline{U}]$$

is the preconditioned AOR iteration matrix.

Theorem 3.11. *Let $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ be a nonsingular Z-matrix. Assume that $0 < r < \omega < 1$, and $P = (p_{i,j}) \geq 0$ is a nonsingular preconditioner with $p_{i,i} = 1$ for $1 \leq i \leq n$, and*

$$p_{i,j} + \sum_{k=1,k\neq j}^{n} P_{i,k}a_{k,j} \leq 0, 1 \leq i \neq j \leq n.$$

(1) if $\rho(T_{SOR}) < 1$, then $\rho(\overline{T}_{SOR}) < \rho(T_{SOR}) < 1$

(2) if $\rho(T_{SOR}) > 1$, and P satisfies

$$1 + \sum_{k=1,k\neq i}^{n} P_{i,k}a_{k,i} > 0, 1 \leq i \neq j \leq n.$$

then $\rho(\overline{T}_{SOR}) > \rho(T_{SOR}) > 1$

Theorem 3.12. *Let $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ be a nonsingular M-matrix. Assume that $0 < r < \omega < 1$, and $P = (p_{i,j}) \geq 0$ is a nonsingular preconditioner with $p_{i,i} = 1$ for $1 \leq i \leq n$, and $p_{i,j} = \alpha_{i,j}a_{i,j}, 0 \leq \alpha_{i,j} \leq 1$, for $1 \leq i \neq j \leq n$. Then we have*

$$\rho(\overline{T}_{SOR}) < \rho(T_{SOR}) < 1$$

3.6 Block Gauss elimination followed by a classical iterative method for the solution of linear systems

Theorem 3.13. *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular M-matrix partitioned as in (1.3). Then n_1 successive applications of the Gauss elimination process on A are equivalent to premultiplying A by the (preconditioning) matrix*

$$P = \begin{pmatrix} L_{11}^{-1} & O_{12} & \cdots & O_{1p} \\ -A_{21}A_{11}^{-1} & I_{22} & \cdots & O_{2p} \\ \vdots & \vdots & \ddots & \\ -A_{p1}A_{11}^{-1} & O_{p2} & \cdots & I_{pp} \end{pmatrix} = Q + S, \tag{3.6}$$

$$Q = \text{diag}(L_{11}^{-1}, I_{22}, \ldots, I_{pp}) \geq 0, I_{ii} \in \mathbb{R}^{n_i,n_i}, i = 2, 3, \ldots, p$$

$$S = \begin{pmatrix} O_{11}^{-1} & O_{12} & \cdots & O_{1p} \\ -A_{21}A_{11}^{-1} & I_{22} & \cdots & O_{2p} \\ \vdots & \vdots & \ddots & \\ -A_{p1}A_{11}^{-1} & O_{p2} & \cdots & O_{pp} \end{pmatrix} \geq 0, \tag{3.7}$$

with L_{11} being the lower triangular matrix in the LU triangular decomposition of A_{11}. Moreover, $\bar{A} = PA$ and the matrix $\bar{A}_1$, obtained from $\bar{A}$ by deleting its first n_1 rows and columns, are also nonsingular M-matrices.(Note: If A is irreducible then so is $\bar{A}_1$ while if A is, in addition, singular then so are $\bar{A}$ and $\bar{A}_1$.)

Let

A = D - L - U

(3.8)

$$D = \text{diag}(A_{11}, A_{22}, \ldots, A_{pp}),$$

$$L = \begin{pmatrix} O_{11} & O_{12} & \cdots & O_{1p} \\ -A_{21} & I_{22} & \cdots & O_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{p1} & -A_{p2} & \cdots & O_{pp} \end{pmatrix}, U = \begin{pmatrix} O_{11} & -A_{12} & \cdots & -A_{1p} \\ O21 & O_{22} & \cdots & -A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ O_{p1} & O_{p2} & \cdots & O_{pp} \end{pmatrix}.$$

To solve (2.6) using a classical iterative method we consider various splittings of A. For this we define the following matrices:

$$SU = \widehat{L} + \widehat{D} + \widehat{U}$$

where

$$D = \mathrm{diag}(O_{11}, A_{21}A_{11}^{-1}, \ldots, A_{p1}A_{11}^{-1}A_{1p}) \geq 0,$$

$$\widehat{L} = \begin{pmatrix} O_{11} & O_{12} & \cdots & O_{1p} \\ O_{21} & O_{22} & \cdots & O_{2p} \\ O_{31} & A_{31}A_{11}^{-1}A_{12} & \cdots & O_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ O_{p1} & A_{p1}A_{11}^{-1}A_{12} & \cdots & O_{pp} \end{pmatrix} (\geq 0),$$

$$\widehat{U} = \begin{pmatrix} O_{11} & O_{12} & O_{13} & \cdots & O_{1p} \\ O_{21} & O_{22} & A_{21}A_{11}^{-1}A_{13} & \cdots & A_{21}A_{11}^{-1}A_{1p} \\ O_{31} & O_{32} & O_{33} & \cdots & A_{31}A_{11}^{-1}A_{1p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{p1} & O_{p2} & O_{p3} & \cdots & O_{pp} \end{pmatrix} (\geq 0),$$

Having in mind (2.10) and (2.9), we consider the following splittings of $\overline{A}$:

$$\overline{A} = (Q+S)(D-L-U) = \begin{cases} QD - (PL - SD + \widehat{L} + \widehat{D} + QU + \overline{U}), \\ (QD - \widehat{D}) - (PL - SD + \widehat{L} + QU + \widehat{U}) : \end{cases}$$

The block Jacobi and GaussSeidel as well as the block Jacobi and GaussSeidel-type iteration matrices associated with the two splittings in (2.14) are:

$$\begin{aligned} B \quad &= D^{-1}(L+U)(\mathrm{for} A), \\ B' \quad &= (QD)^{-1}(PL - SD + \widehat{L} + \widehat{D} + QU + \widehat{U}), \\ B" \quad &= (QD - \widehat{D})^{-1}(PL - SD + \widehat{L}L + QU + \widehat{U}), \\ H \quad &= (D-L)^{-1}U(\mathrm{for} A), \\ H' \quad &= (P(D-L) - \widehat{L})^{-1}(\widehat{D} + QU + \widehat{U}), \\ H" \quad &= (P(D-L) - \widehat{L} - \widehat{D})^{-1}(QU + \widehat{U}), \end{aligned}$$

Theorem 3.14. *Under the notation and the definitions so far, suppose that A is a nonsingular M-matrix and let $\rho(B) > 0$. Let $B_1', B_1, H_1, H_1', H_1$ denote the $(n - n_1) \times (n - n_1)$ bottom right corner submatrices of $B_1', B_1, H_1, H_1', H_1$, respectively. Then the following relationships hold:*

$$\rho(B_1) = \rho(B) \leq \rho(B') = \rho(B_1') < 1,$$

$$\rho(H_1) = \rho(H) \leq \rho(H') = \rho(H_1') \leq rho(H) = rho(H_1) < 1.$$

Theorem 3.15. *Under the notation and the de9nitions used in above Theorem, suppose that A is a nonsingular M-matrix. Let $B'^{(k)}, B(k), H'^{(k)}; H^{(k)}, k = 1(1)n_1$, denote the "point" iteration matrices (Jacobi and GaussSeidel type) associated with the matrix $\overline{A}^{(k)}(\overline{A}^{(0)} = A)$ of above Theorems after the kth elimination step $k = 1(1)n1$. Let also $B^{(0)}, H^{(0)}$ be the point Jacobi and GaussSeidel iteration matrices associated with A. Then, there will hold*

$$\rho(H) \leq \rho(H^{n_1}) \leq \rho(H'^1) \leq \rho(H^0)(< 1). \tag{3.9}$$

If, in addition, A is irreducible, then there will also hold

$$\rho(B) < \rho(B^{n_1}) < \rho(B'^1) < \rho(B^0)(< 1),$$

and all the inequalities in (3.9) will be strict

4 Comparison results between Jacobi and other iterative methods

In [], the authors showed that spectral radius of Jacobi iteration matrix B is less than that of several iteration matrices introduced.

Theorem 4.1. *Let A be a nonsingular matrix. $B \geq 0$ the Jacobi iteration matrix in (1.2). If $0 \leq \omega \leq 1$ and $\rho(B) \leq (1 - \omega)^2$, we have that*

$$\rho(B) \leq \rho(T_{SSOR},$$

Theorem 4.2. *Let A be a nonsingular matrix. $B \geq 0$ the Jacobi iteration matrix in (1.2). If $0 \leq \omega \leq 1$ and $\rho(B) \leq 1 - \omega$, we have that*

$$\rho(B) \leq \rho(T_{JOR},$$

$$\rho(B) \leq \rho(T_{SOR},$$

5 The BSCM method

Some effective splitting iterative methods and preconditioning methods were presented for solving the linear system of equations (**??**) (see [1-14]). Here, we consider A as a block matrix in the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \text{ if } n = 2l, \qquad A = \begin{pmatrix} A_1 & c_1 & A_2 \\ d_1^T & 1 & d_2^T \\ A_3 & c_2 & A_4 \end{pmatrix} \text{ if } n = 2l+1, \tag{5.1}$$

where $c_1, c_2, d_1, d_2 \in \Re^l$ and $A_i \in \Re^{l \times l}$, $i = 1, 2, 3, 4$. By assuming that A has unit diagonal elements, we split A into

$$A = V - B, \tag{5.2}$$

where V is a block matrix in the form of

$$V = \begin{pmatrix} I & D_2 \\ D_3 & I \end{pmatrix} \text{ if } n = 2l, \qquad V = \begin{pmatrix} I & 0 & D_2 \\ 0 & 1 & 0 \\ D_3 & 0 & I \end{pmatrix} \text{ if } n = 2l+1, \tag{5.3}$$

in which $D_2 = \text{diag}(A_2)$, $D_3 = \text{diag}(A_3)$, and $I \in \Re^{l \times l}$ is identity matrix. By assuming that $I - D_2 D_3$ is a nonsingular matrix, it is easy to see that

$$V^{-1} = \begin{pmatrix} (I - D_2 D_3)^{-1} & 0 \\ 0 & (I - D_2 D_3)^{-1} \end{pmatrix} \begin{pmatrix} I & -D_2 \\ -D_3 & I \end{pmatrix} \tag{5.4}$$

if $n = 2l$, and

$$V^{-1} = \begin{pmatrix} (I - D_2 D_3)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (I - D_2 D_3)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 & -D_2 \\ 0 & 1 & 0 \\ -D_3 & 0 & I \end{pmatrix} \tag{5.5}$$

if $n = 2l + 1$. This split leads to the new iterative method

$$x^{(k+1)} = V^{-1}Bx^{(k)} + V^{-1}b, \quad k = 0, 1, 2, \ldots. \tag{5.6}$$

In section 2, We present the convergence analysis for this new iterative method, called the BSCM (Block Splitting of the Coefficient Matrix) method, and provide a comparison of the spectral radii for the Jacobi iterative method and this method. In section 3, numerical examples are given to illustrate our results. Section 4 is devoted to concluding remarks.

Now we give the main results.

Theorem 5.1. *Let $A = V - B$ be a strictly diagonally dominant or irreducibly diagonally dominant matrix with unit diagonal entries and partitioned as in (5.1). If $I - D_2D_3$ is a nonsingular matrix, then the associated BSCM iteration converges for any initial value $x^{(0)}$.*

Proof. Only a sketch of the proof will be given, since the main line of reasoning is analogous to that found in [28] for the Gauss-Seidel method. Let $\overline{\lambda}$ be the dominant eigenvalue of the iteration matrix $V^{-1}B$. Let $x = (x_i)$ be an eigenvector associated with $\overline{\lambda}$, with $|x_m| = 1$ and $|x_i| \le 1$ for $i \ne m$. If $m \le l$, from equation $Bx = \overline{\lambda}Vx$, we have

$$-\sum_{j \ne m, l+j} a_{mj}x_j = \overline{\lambda}(x_m + a_{m,l+j}x_{l+j})$$

which yields the inequality

$$|\overline{\lambda}| \le \frac{\sum_{j \ne m,l+j} |a_{mj}||x_j|}{1 - |a_{m,l+j}||x_{l+j}|} \le \frac{\sum_{j \ne m,l+j} |a_{mj}|}{1 - |a_{m,l+j}|} = \frac{\sum_{j \ne m,l+j} |a_{mj}|}{(1 - \sum_{j \ne m} |a_{mj}|) + \sum_{j \ne m,l+j} |a_{mj}|}$$

In the case when the matrix A is strictly diagonally dominant, from the last term of the above inequality, we have $|\overline{\lambda}| < 1$.

In the case when the matrix A is only irreducibly diagonally dominant, the last term of the above inequality only shows $|\overline{\lambda}| \le 1$. As in [28], by contradiction, we can show that in fact $|\overline{\lambda}| < 1$. The case $m > l$ can be proved in a similar way. □

Theorem 5.2. *Let $A = V - B$ be a nonsingular M-matrix and partitioned as in (5.1). If $I - D_2D_3$ is nonsingular, then $\rho(V^{-1}B) < 1$.*

Proof. By the assumptions, we have $B \geq 0$ and $A \leq V$. So, by Theorem **??**, V is an M-matrix and we have $V^{-1} \geq 0$. Thus, by Definition **??**, $A = V - B$ is a regular splitting. Since A is an M-matrix, we have $A^{-1} \geq 0$. Thus, by Theorem **??**, we have $\rho(V^{-1}B) < 1$. □

Theorem 5.3. *Let $A = V - B$ be a nonsingular H-matrix with unit diagonal entries and partitioned as in (5.1). If $D_2 \geq 0, D_3 \geq 0$ and $I - D_2D_3$ has positive diagonal entries, then $V^{-1}A$ is an H-matrix and $\rho(V^{-1}B) < 1$.*

Proof. We will prove for the case $n = 2l$, the case $n = 2l+1$ can be proved in a similar way. Since A is an H-matrix, we have from Definition 2.11 that $\langle A\rangle^{-1} \geq 0$. Denote $r = \langle A\rangle^{-1}e$, where $e = (1, 1, \ldots, 1)^T \in \Re^{2l}$. Then $r > 0$. Let $r = (r_1^T, r_2^T)$, where $r_1, r_2 \in \Re^l$. By using the definition of comparison matrix (Definition **??**), we have

$$\begin{aligned} \langle A\rangle r &= \begin{pmatrix} I - |A_1 - I| & -|A_2| \\ -|A_3| & I - |A_4 - I| \end{pmatrix} r \\ &= \begin{pmatrix} (I - |A_1 - I|)r_1 - |A_2|r_2 \\ (I - |A_4 - I|)r_2 - |A_3|r_1 \end{pmatrix} \\ &= \begin{pmatrix} e_1 \\ e_1 \end{pmatrix} \end{aligned} \tag{5.7}$$

where $e_1 = (1, 1, \ldots, 1)^T \in \Re^l$. We now show that $\langle V^{-1}A\rangle r > 0$ which will be useful to show that $V^{-1}A$ is an H-matrix. From (5.1) and (5.4), we have

$$V^{-1}A = D \begin{pmatrix} A_1 - D_2A_3 & A_2 - D_2A_4 \\ A_3 - D_3A_1 & A_4 - D_3A_2 \end{pmatrix},$$

where

$$D = \begin{pmatrix} (I - D_2D_3)^{-1} & 0 \\ 0 & (I - D_2D_3)^{-1} \end{pmatrix}.$$

From the assumption $I - D_2D_3$ has positive diagonal entries, we have $D \geq 0$. So, from the definition of comparison matrix (Definition **??**), we have

$\langle V^{-1}A\rangle$

$$= D\begin{pmatrix} |I - D_2D_3| - |(A_1 - D_2A_3) - (I - D_2D_3)| & -|A_2 - D_2A_4| \\ -|A_3 - D_3A_1| & |I - D_2D_3| - |(A_4 - D_3A_2) - (I - D_2D_3)| \end{pmatrix}$$

$$= D\begin{pmatrix} |I - D_2D_3| - |(A_1 - I) - D_2(A_3 - D_3)| & -|(A_2 - D_2) - D_2(A_4 - I)| \\ -|(A_3 - D_3) - D_3(A_1 - I)| & |I - D_2D_3| - |(A_4 - I) - D_3(A_2 - D_2)| \end{pmatrix}$$

Using the assumption $I - D_2D_3 \geq 0$ and $D_2 \geq 0, D_3 \geq 0$, we obtain

$\langle V^{-1}A\rangle$

$$\geq D\begin{pmatrix} (I - D_2D_3) - |A_1 - I| - |D_2(A_3 - D_3)| & -|A_2 - D_2| - |D_2(A_4 - I)| \\ -|A_3 - D_3| - |D_3(A_1 - I)| & (I - D_2D_3) - |A_4 - I| - |D_3(A_2 - D_2)| \end{pmatrix}$$

$$= D\begin{pmatrix} I - |A_1 - I| - D_2(D_3 + |A_3 - D_3|) & D_2(I - |A_4 - I|) - (D_2 + |A_2 - D_2|) \\ D_3(I - |A_1 - I|) - (D_3 + |A_3 - D_3|) & I - |A_4 - I| - D_3(D_2 + |A_2 - D_2|) \end{pmatrix}$$

$$= D\begin{pmatrix} I - |A_1 - I| - D_2|A_3| & D_2(I - |A_4 - I|) - |A_2| \\ D_3(I - |A_1 - I|) - |A_3| & I - |A_4 - I| - D_3|A_2| \end{pmatrix}$$

By using the vector $r = (r_1^T, r_2^T) > 0$ and the equation (5.7), we have

$$\begin{aligned} \langle V^{-1}A\rangle r &\geq D\begin{pmatrix} (I - |A_1 - I|)r_1 - |A_2|r_2 + D_2((I - |A_4 - I|)r_2 - |A_3|r_1) \\ (I - |A_4 - I|)r_2 - |A_3|r_1 + D_3((I - |A_1 - I|)r_1 - |A_2|r_2) \end{pmatrix} \\ &= D\begin{pmatrix} e_1 + D_2e_1 \\ e_1 + D_3e_1 \end{pmatrix} > 0. \end{aligned}$$

By Lemma **??**, it follows that $V^{-1}A$ is an H-matrix. By using (5.4) and the definition of B, we obtain

$$V^{-1}B = D\begin{pmatrix} (I - A_1) - D_2(D_3 - A_3) & (D_2 - A_2) - D_2(I - A_4) \\ (D_3 - A_3) - D_3(I - A_1) & (I - A_4) - D_3(D_2 - A_2) \end{pmatrix}.$$

This relation shows that the diagonal entries of $V^{-1}B$ are zeros. So, from $V^{-1}A = I - V^{-1}B$, we have

$$\langle V^{-1}A\rangle = I - |V^{-1}B|.$$

Since $\langle V^{-1}A\rangle$ is an H-matrix, $(I - |V^{-1}B|)$ is nonsingular and $(I - |V^{-1}B|)^{-1} \geq 0$. Finally, by using Theorem **??** and Theorem **??**, one can obtain

$$\rho(V^{-1}B) \leq \rho(|V^{-1}B|) < 1.$$

□

For comparing the asymptotic rate of convergence or equivalently the spectral radii of the iteration matrices of the Jacobi and the BSCM methods, we suppose that A has unit diagonal elements and define

$$S = I - V, \tag{5.8}$$

So, from the definition of the Jacobi matrix J, we have

$$J = B + S \tag{5.9}$$

We now state the following theorem.

Theorem 5.4. *Let $A = V - B$ be a nonsingular L-matrix and partitioned as in (5.1). If $I - D_2D_3$ has positive diagonal entries, then*

(a) $\rho(V^{-1}B) < 1$ if and only if $\rho(J) < 1$ and $\rho(V^{-1}B) \le \rho(J) < 1$.

(b) $\rho(V^{-1}B) \ge 1$ if and only if $\rho(J) \ge 1$ and $\rho(V^{-1}B) \ge \rho(J) \ge 1$.

Proof. By using the assumptions, we have $V^{-1}B \ge 0$ and $J = B + S \ge 0$. Let $\bar{\lambda} = \rho(V^{-1}B)$ and $\bar{\mu} = \rho(J)$. By Theorem **??**, $\bar{\lambda}$ is an eigenvalue of $V^{-1}B$ and for some $x \neq 0$, we have $V^{-1}Bx = \bar{\lambda}x$, which implies that

$$(\bar{\lambda}S + B)x = \bar{\lambda}x.$$

Since $\bar{\lambda}$ is an eigenvalue of $\bar{\lambda}S + B$, we have

$$\bar{\lambda} \le \rho(\bar{\lambda}S + B).$$

If $\bar{\lambda} \le 1$, then by Theorem **??**, $\rho(\bar{\lambda}S + B) \le \rho(S + B) = \bar{\mu}$, which implies that $\bar{\lambda} \le \bar{\mu}$. So, we have

(i) If $\bar{\lambda} \le 1$, then $\bar{\lambda} \le \bar{\mu}$.

On the other hand, if $\bar{\lambda} \ge 1$, then by Theorem **??**, we have

$$\bar{\lambda} \le \rho(\bar{\lambda}S + B) \le \rho(\bar{\lambda}S + \bar{\lambda}B) = \bar{\lambda}\bar{\mu},$$

which implies that $\bar{\mu} \ge 1$. So, we have

(ii) if $\bar{\lambda} \geq 1$, then $\bar{\mu} \geq 1$.

Assume that $\bar{\mu} \geq 1$. By the definition of S, $(I - \frac{1}{\bar{\mu}}S)$ is nonsingular for $\bar{\mu} \geq 1$. Since $J = B + S \geq 0$, it follows, by Theorem **??**, $\bar{\mu}$ is an eigenvalue of J. Therefore for some $y \neq 0$, we have $(B + S)y = \bar{\mu}y$ and

$$(I - \frac{1}{\bar{\mu}}S)^{-1}By = \bar{\mu}y. \tag{5.10}$$

In addition for $\bar{\mu} \geq 1$, we have

$$0 \leq (I - \frac{1}{\bar{\mu}}S)^{-1} \leq (I - S)^{-1} = V^{-1},$$

and

$$0 \leq (I - \frac{1}{\bar{\mu}}S)^{-1}B \leq V^{-1}B.$$

This together with Theorem **??** and equation (5.10) implies that

$$\bar{\mu} \leq \rho((I - \frac{1}{\bar{\mu}}S)^{-1}B) \leq \rho(V^{-1}B) = \bar{\lambda}.$$

Therefore,

(iii) if $\bar{\mu} \geq 1$ then $\bar{\lambda} \geq \bar{\mu} \geq 1$.

Now, by (i) and (iii), we have (a) and by (ii) and (iii), we have (b). □

6 Numerical results

In this section we give the numerical examples to illustrate the results obtained in Section 2. In our implementation, the initial approximation $x^{(0)}$ is taken as the zero vector, and the right hand side vector b is chosen so that $x = (1, 1, \ldots, 1)^T$ is the solution of the consider system. The stopping criterion $\|x^{(k)} - x^*\|_\infty \leq 10^{-7}$ is used, where $x^{(k)}$ is the kth iterative vector for the corresponding iterative method, while x^* is the solution of the given linear system. The maximum number of iterations is set to $10,000$. We compare the numerical behaviors of the BSCM method with the Jacobi and Gauss-Seidel methods. We report the spectral radius of corresponding iteration matrix and the number of iterations in the following tables. In these tables n represents the dimension of matrices and a † is used to indicate that there was no convergence in 10000 iterations. The numerical results in the following tables were computed using MATLAB 7.9.

Example 6.1. The coefficient matrices A of (**??**) is given by

$$A_1 = \begin{pmatrix} 1 & -0.01 & 0.19 & -0.25 & -0.04 \\ -0.28 & 1 & 0.17 & -0.22 & 0.22 \\ -0.25 & 0.27 & 1 & 0.19 & -0.09 \\ -0.14 & 0.01 & -0.25 & 1 & 0.13 \\ -0.26 & 0.08 & -0.14 & 0.05 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0.1 & 0.2 & 0.0 & 0.2 & -0.5 \\ 0.2 & 1 & 0.3 & 0.0 & -0.4 & 0.1 \\ 0.0 & 0.2 & 1 & -0.6 & 0.2 & 0.0 \\ 0.2 & -0.3 & 0.1 & 1 & 0.1 & 0.3 \\ 0.0 & 0.3 & 0.2 & 0.1 & 1 & 0.2 \\ 0.2 & -0.3 & 0.0 & -0.3 & 0.1 & 1 \end{pmatrix}$$

where A_1 and A_2 are strictly diagonally dominant and irreducibly diagonally dominant matrices, respectively. Numerical results for these matrices are given in Table 1.

Example 6.2. (see [26].) The coefficient matrices A of is given by

$$A_3 = \begin{pmatrix} 1 & q & r & s & q & r & \cdots & \\ s & 1 & q & r & & & & \\ r & \ddots & \ddots & \ddots & \ddots & & & \\ q & \ddots & & & & & & \\ s & \ddots & & \ddots & & & \ddots & \\ r & \ddots & & & & & & \\ \vdots & \ddots & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix},$$

where $q = -\frac{1}{n+1}$, $r = -\frac{1}{n}$, $s = -\frac{1}{n+1}$ and A_3 is an M-matrix. The numerical results for different values of n are given in Table 2. By choosing $q = \frac{1}{n-1}$, $r = \frac{1}{n}$, and $s = \frac{1}{n+1}$, A_3 is an H-matrix. The numerical results for this matrix and different values of n are presented in Table 3

Table 1. Numerical results for Example 6.1

	Jacobi		Gauss-Seidel		BSCM	
Coefficient matrix	IT	ρ	IT	ρ	IT	ρ
A_1	9	0.3389	5	0.1107	7	0.2415
A_2	21	0.6513	10	0.3509	16	0.5539

Example 6.3. The coefficient matrices A of (**??**) is given by

$$A_4 = \begin{pmatrix} 1 & -0.2 & -0.1 & -0.6 \\ -0.2 & 1 & -0.3 & -0.6 \\ -0.3 & -0.2 & 1 & -0.1 \\ -0.1 & -0.1 & -0.1 & 1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 1 & -0.2 & -0.7 & -0.3 \\ -0.2 & 1 & -0.9 & -0.4 \\ -0.7 & -0.5 & 1 & -0.5 \\ -0.3 & -0.4 & -0.3 & 1 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 1 & -0.2 & -0.5 & -0.3 \\ -0.2 & 1 & -0.4 & -0.4 \\ -0.2 & -0.3 & 1 & -0.5 \\ -0.3 & -0.4 & -0.3 & 1 \end{pmatrix} \quad A_7 = \begin{pmatrix} 1 & 1 & 4 & 2 \\ 1 & 1 & -1 & 4 \\ 4 & 1 & 1 & -1 \\ 1 & 4 & 1 & 1 \end{pmatrix}$$

where A_4, A_5, and A_6 are L-matrices, but A_7 is not. Numerical results for these matrices are given in Table 4

Remark 6.4. From Tables 1-4, it is easy to verify that the numerical results are consistent with the theorems in Section 2. We observe that when the methods converge, the spectral radius of the BSCM method is smaller than that of the Jacobi Method and is larger than that of the Gauss-Seidel method. In the case of L-Matrix, the BSCM iteration matrix $V^{-1}B$ and the Jacobi iteration matrix J are either both convergent, or both divergent(Table 4, the matrices A_4, A_5, and A_6). Finally, Table 4 also shows that there is a coefficient matrix (A_7) for which the BSCM method converges, but the Jacobi and the Gauss-Seidel methods diverge.

Table 2. Numerical results for Example 6.2, when A_3 is an M-matrix

	Jacobi		Gauss-Seidel		BSCM	
n	IT	ρ	IT	ρ	IT	ρ
10	55	0.8455	29	0.7199	49	0.8283
20	111	0.9198	57	0.8477	105	0.9158
50	277	0.9672	140	0.9358	271	0.9665
100	553	0.9835	278	0.9673	547	0.9833

Table 3. Numerical results for Example 6.2, when A_3 is an H-matrix

	Jacobi		Gauss-Seidel		BSCM	
n	IT	ρ	IT	ρ	IT	ρ
10	94	0.9061	7	0.2088	30	0.7328
20	186	0.9516	7	0.2099	61	0.8584
50	463	0.9803	7	0.2131	153	0.9414
100	923	0.9901	7	0.2145	307	0.9704

Table 4. Numerical results for Example 6.3

	Jacobi		Gauss-Seidel		BSCM	
Coefficient matrix	IT	ρ	IT	ρ	IT	ρ
A_4	22	0.6361	12	0.4069	16	0.5308
A_5	†	1.3758	†	1.8668	†	1.8969
A_6	†	1.0	†	1.0	†	1.0
A_7	†	5.29	†	16.40	15	0.5211

7 Conclusion

In this paper, we have presented a new iterative method based on a block splitting of coefficient matrix A for solving linear system $Ax = b$. The new method, called the BSCM method, uses a nonsingular block matrix for splitting the coefficient matrix A. We proved its convergence when A is a strictly diagonally dominant, an irreducibly diagonally dominant matrix, an M-matrix, or an H-matrix. We provided a comparison of spectral radii for the BSCM method and the Jacobi method when A is an L-matrix. Numerical examples showed that the rate of convergence of the BSCM method is faster than that of the Jacobi method, but is larger than that of the Gauss-Seidel method.

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