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ACCELERATED NORMAL AND SKEW-HERMITIAN SPLITTING METHODS FOR POSITIVE DEFINITE LINEAR SYSTEMS

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Abstract. For solving large sparse non-Hermitian positive definite linear equations, Bai, Golub and NG proposed the Hermitian and skew-Hermitian splitting methods (HSS). They recently generalized this technique to the normal and skew-Hermitian splitting methods (NSS). In this paper, we present an accelerated normal and skew-Hermitian splitting methods (ANSS) which involve two parameters for the NSS iteration. We theoretically study the convergence properties of the ANSS method. Moreover, the construction factor of ANSS iteration is derived. Numerical examples illustrating the effectiveness of ANSS iteration are presented.

1. Introduction and Preliminaries

Many problems in scientific computation given rise to solving the linear system

$$Ax = b, \tag{1.1}$$

with $A \in \mathbb{C}^{n\times n}$ a large non-Hermitian positive definite matrix and $x, b \in \mathbb{C}^n$. We observe that the coefficient matrix A naturally possesses the Hermitian/skew-Hermitian (HS) splitting

$$A = H + S$$

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where

$$H = \frac{1}{2}(A + A^*) \quad and \quad S = \frac{1}{2}(A - A^*)$$

with A^* being the conjugate transpose of A. Bai, Golub and NG [2] presented the HSS iteration method: given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$, until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{1.2}$$

where α is a given positive constant. They have also proved for any positive α the HSS method converges unconditionally to the unique solution of the system of linear equations.

Bai, Golub and Ng [2] recently generalized this technique to the normal and skew-Hermitian splitting methods (NSS). They split the coefficient matrix A into

$$A = N + S$$

where $N \in \mathbb{C}^{n\times n}$ is a normal matrix and $S \in \mathbb{C}^{n\times n}$ is a skew-Hermitian matrix, and obtained the following normal/skew Hermitian splitting (NSS) method to iteratively compute a reliable and accurate approximate solution for the system of linear equation (1.1):

The NSS iteration method: Given an initial guess $x^{(0)} \in \mathbb{C}^n$. For $k = 0, 1, 2 \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + N)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - N)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given positive constant. They have also proved that for any positive α the NSS method converges unconditionally to the unique solution of the system of linear equations.

In this paper, we introduce two constants for the NSS iteration and present different approach to solve Eq. (1.1), called the *accelerated normal and skew-Hermitian splitting iteration*, shortened to the ANSS iteration. Moreover, we analyze the convergence properties of the ANSS iteration and present the numerical examples for illustrating the effectiveness of ANSS iteration.

2. THE ANSS METHOD

Throughout the paper, the non-Hermitian matrix $A \in \mathbb{C}^{n\times n}$ is positive definite if its Hermitian part is Hermitian positive definite.

The ANSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2 \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + N)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\beta I + S)x^{(k+1)} = (\beta I - N)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given nonnegative constant and β is a given positive constant, and $N \in \mathbb{C}^{n\times n}$ is a normal matrix and $S \in \mathbb{C}^{n\times n}$ a skew-Hermitian such that $A = N + S$.

In matrix-vector form, the ANSS iteration method can be equivalently rewritten as

$$x^{(k+1)} = M(\alpha,\beta)x^{(k)} + G(\alpha,\beta), \quad k = 0, 1, 2 \ldots$$

where

$$M(\alpha,\beta) = (\beta I + S)^{-1}(\beta I - N)(\alpha I + N)^{-1}(\alpha I - S)$$

and

$$G(\alpha,\beta) = (\alpha+\beta)(\beta I + S)^{-1}(\alpha I + N)^{-1}$$

Here, $M(\alpha,\beta)$ is the iteration matrix of the ANSS iteration. The following theorem describes the convergence properties of the ANSS iteration.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $N \in \mathbb{C}^{n\times n}$ be a normal matrix and $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix such that $A = N + S$, and α be a nonnegative constant and β be a positive constant. Then the spectral radius $\rho(M(\alpha,\beta))$ of the iteration matrix $M(\alpha,\beta)$ of the ANSS iteration is bounded by*

$$\delta(\alpha,\beta) \equiv \max_{\sigma_j \in \sigma(s)} \frac{\sqrt{\alpha^2+\sigma_j^2}}{\sqrt{\beta^2+\sigma_j^2}} \max_{\gamma_j + i\eta_j \in \lambda(N)} \sqrt{\frac{(\beta-\gamma_j)^2+\eta_j^2}{(\alpha+\gamma_j)^2+\eta_j^2}}$$

where $\lambda(N)$ is the spectral set of N and $\sigma(s)$ is the singular-value set of S. And, for any given parameter α, if β satisfies

$$\max\left\{\frac{\alpha(\gamma_{min}^2+\eta_{max}^2)}{2\alpha\gamma_{min}+\gamma_{min}^2+\eta_{max}^2}, \frac{\alpha(\gamma_{max}^2+\eta_{max}^2)}{2\alpha\gamma_{max}+\gamma_{max}^2+\eta_{max}^2}\right\} < \beta \le \alpha + 2\gamma_{min}$$

then $\delta(\alpha,\beta) < 1$*, and or if* β *satisfies*

$$\alpha + 2\gamma_{min} \leq \beta \qquad and \qquad \sigma_{max} \leq \sqrt{\gamma_{min} + \eta_{min} + 2\gamma_{min}\alpha}$$

then $\delta(\alpha,\beta) < 1$*, i.e., the ANSS iteration converges, where* γ_{min} *and* γ_{max}*,* η_{min} *and* η_{max} *are the lower and the upper bound of the real, the absolute values of the imaginary parts of the eigenvalues of the matrix* N*, respectively, and* σ_{min}*,* σ_{max} *are the lower and the upper bound of the singular-value set of the matrix* S*, respectively.*

Theorem 2.1 mainly discusses the available β for a convergent ANSS iteration for any given nonnegative α. It also shows that the choice of β is dependent on the choice of α, the spectrum of the matrix N, the singular-values of S, but is not dependent on the spectrum of A.

References

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