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Solving Volterra integral equations of the second kind by wavelet-Galerkin scheme☆

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ABSTRACT

In this paper, we apply the wavelet-Galerkin method to obtain approximate solutions to linear Volterra integral equations (VIEs) of the second kind. Daubechies wavelets are used to find such approximations. In this approach, we introduce some new connection coefficients and discuss their properties and propose algorithms to evaluate them. These coefficients can be computed just once and applied for solving every linear VIE of the second kind. Convergence and error analysis are discussed and numerical examples illustrate the efficiency of the method.

1. Introduction

In recent years, wavelets have played a crucial role in approximating the solution of a wide range of problems arising in science and engineering. Wavelets have been used in numerous areas of applied mathematics as diverse as signal analysis, statistics, computer aided geometric design, image processing and numerical analysis. Glowinski in [1] used wavelets to approximate the solution of a partial differential equation. Wavelet bases are also used for solving integral equations, in which Fredholm equations are investigated more than other types; for e.g. see [2–5]. Wavelets are mostly suited for approximating linear problems, but some numerical results were obtained for nonlinear Fredholm and Volterra equations in [6,7]. The approximate solution of two-dimensional Fredholm equations is discussed in [8].

In a wavelet-Galerkin scheme, wavelet bases are applied with the well-known Galerkin method, in place of other conventional bases like Legendre or Chebyshev bases. This method has been used for approximating PDE problems in [9–12]. The solution of integral equations by the wavelet-Galerkin method is studied by various authors such as Fang in [13], Liang in [14] and Xiao in [15]. Integro-differential equations are also considered in [16].

As we are aware, the wavelet-Galerkin scheme has not been applied for solving VIEs yet. The main difficulty in applying this procedure happens in the evaluation of the connection coefficients which arise in this method. It is difficult and unstable to compute connection coefficients by the numerical evaluation of integrals. Therefore in this paper, we propose algorithms for the exact evaluation of these coefficients.

This paper is organized as follows. In Section 2, some properties of Daubechies wavelets are reviewed. In Section 3, the wavelet-Galerkin scheme is proposed to approximate the solution of a linear VIE of the second kind. The evaluation of connection coefficients is the subject of Section 4. In Section 5, we present the error analysis of this method. The efficiency of this method is shown by providing some numerical experiments in Section 6 and a brief conclusion is presented in Section 7.

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2. Daubechies wavelets

Daubechies wavelets have gained considerable attention in the numerical analysis of partial differential and integral equations because of possessing some useful properties, such as orthogonality, compact support and ability to represent functions at different levels of resolution. In this section we review some properties of Daubechies wavelets.

Definition 2.1. A refinable function is a function $\phi : R \to C$ which satisfies a two-scale refinement equation of the form

$$\phi(x) = \sum_k a_k \phi(2x - k). \tag{2.1}$$

The $a_k \in C$ are called scaling or recursion coefficients. The exact values of some of these coefficients are evaluated in [17]. Since $\phi(\cdot)$ has compact support, the series in (2.1) reduces to a finite series as:

$$\phi(x) = \sum_{k=0}^{2g-1} a_k \phi(2x - k),$$

where g denotes the genus of the scaling function.

The refinable function ϕ is called orthogonal if

$$\langle \phi(x), \phi(x-k) \rangle = \int_R \phi(x)\phi(x-k)dx = \delta_{0,k}, \quad k \in Z, \tag{2.2}$$

where $\delta_{0,k}$ is the Kronecker delta function. In order to obtain approximations of a function $f(x) \in L^2(R)$, one can use the translated dilations of the scaling function, defined as

$$\phi_{n,l}(x) = 2^{n/2}\phi(2^n x - l), \quad n, l \in Z. \tag{2.3}$$

The set of orthogonal functions $\{\phi_{n,l}(x)\}_{l \in Z}$ for a particular n, generates a space $V_n \subset L^2(R)$. Let P_n denote the orthogonal projection $L^2(R) \to V_n$. The vector spaces $V_n (n \in Z)$ have the following properties defining a multiresolution analysis:

1. $V_n \subset L^2(R)$ and $V_n \subset V_{n+1}$
2. $\|f(x) - P_n f(x)\| = \min \|f(x) - g(x)\|$, where $g(x) \in V_n$.
3. $v(x) \in V_n \Leftrightarrow v(2x) \in V_{n+1}$.
4. The projection $P_n f(x)$ converges to $f(x)$ as n tends to infinity:

$$\lim_{n\to\infty} P_n f(x) = f(x) \quad \text{or} \quad \bigcup_{n=0}^{\infty} V_n \text{ is dense in } L^2(R).$$

Definition 2.2. The kth discrete and continuous moments of ϕ are respectively defined by

$$m_k = \frac{1}{2}\sum_l l^k a_l, \tag{2.4}$$

$$M_k = \int x^k \phi(x)dx. \tag{2.5}$$

In this paper by integral sign $\int$, we mean $\int_R$.

Theorem 2.1 ([18]). *The discrete and continuous moments are related by*

$$M_k = 2^{-k}\sum_{l=0}^{k} \binom{k}{l} m_{k-l} M_l, \tag{2.6}$$

and we let

$$\sum_k \phi(x-k) = \int \phi(x)dx = M_0 = 1.$$

3. Problem approximation

In this section we propose the wavelet-Galerkin method to approximate the solution of a linear VIE of the second kind.

Let $\phi(x)$ be the Daubechies scaling function of genus N and the support is $[0, L-1]$, where $L = 2N$. Therefore the function $\phi_{n,i}(x) = 2^{-n/2}\phi(2^n x - i)$ has the support $[2^{-n}i, 2^{-n}(i + L - 1)]$. Consider the following VIE of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad 0 \le t \le x \le 1, \tag{3.1}$$

where u is an unknown function and $f \in L^2[0, 1]$, $K \in L^2([0, 1] \times [0, 1])$ are explicitly known. Since ϕ is orthogonal, $\phi_{n,i}(x)$ generates an orthogonal MRA, hence $u(x)$ can be approximated by scaling function series as

$$u(x) = \sum_i u_i \phi_{n,i}(x), \tag{3.2}$$

where $u_i s$ are unknown coefficients. Since $\phi_{n,i}(x)$ has compact support and we look for a solution of (3.1) in the interval $[0, 1]$, the infinite series in (3.2) reduces to a finite series

$$u(x) = \sum_{i=2-L}^{2^n-1} u_i \phi_{n,i}(x). \tag{3.3}$$

Substituting (3.3) into (3.1) leads to

$$\sum_{i=2-L}^{2^n-1} u_i \phi_{n,i}(x) = f(x) + \sum_{i=2-L}^{2^n-1} u_i \int_0^x K(x, t)\phi_{n,i}(t)dt, \quad 0 \le t \le x \le 1. \tag{3.4}$$

To solve (3.4), two different approaches are mostly applied, one of them is the collocation method and the other one is the Galerkin method. In order to solve (3.4) by the Galerkin procedure, first we must approximate functions $f(x)$ and $K(x, t)$. In the sequel we mention how to approximate these functions properly.

Let $f(x)$ is defined on R, to approximate this function we have

$$f(x) \simeq \sum_{k=-\infty}^{+\infty} a_k \phi_{n,k}(x), \quad a_k = \int f(x)\phi_{n,k}(x)dx. \tag{3.5}$$

The coefficients a_k are unknown. In order to evaluate them we apply a quadrature rule. The idea of a quadrature formula is to find weights w_i and abscissae x_i such that

$$\int g(x)\phi(x)dx = \int_0^{L-1} g(x)\phi(x)dx \simeq \sum_{i=0}^{p} w_i g(x_i). \tag{3.6}$$

We try to find unknown weights w_i such that, relation (3.6) holds exactly for polynomials of degree p. Hence we have

$$\int x^j \phi(x)dx = \int_0^{L-1} x^j \phi(x)dx = M_j = \sum_{i=0}^{p} w_i x_i^j, \quad j = 0, \dots, p,$$

which leads to an algebraic system. In case the abscissae x_i are fixed, this system is linear in the unknowns w_i. More efficient quadrature formula can be constructed by also treating the abscissae as unknowns, cf. Gauss quadrature formulae. Here we let $x_i = \frac{i(L-1)}{p}$, $i = 0 \dots p$, so the unknowns a_k in (3.5) are approximated as follows

$$a_k = \int f(x)\phi_{n,k}(x)dx = 2^{-\frac{n}{2}} \int_0^{L-1} f\left(\frac{t+k}{2^n}\right)\phi(t)dt = 2^{-\frac{n}{2}} \sum_{q=0}^{p} w_q f\left(\frac{t_q + k}{2^n}\right). \tag{3.7}$$

The function $f(x)$ in (3.1) is defined in the interval $[0, 1]$ and by considering (3.7) we need some values of this function out of this interval, hence we extend this function smoothly by the following procedure.

3.1. Smooth extension of functions

Let $f \in C^m[0, 1]$. This function can be extended to $[-\delta, 1]$, $0 < \delta \le \frac{1}{m}$, by the reflection formula (see, e.g. [19,20])

$$f(x) = \sum_{j=0}^{m} c_j f(-jx) \quad -\delta \le x < 0, \tag{3.8}$$

where c_j are chosen such that a C^m-smooth joining takes place at $x = 0$. The C^m-smooth joining at $x = 0$ happens if $\lim_{x\to 0} f^{(k)}(x) = f^{(k)}(0)$, i.e., if

$$\sum_{j=0}^{m} (-j)^k c_j = 1, \quad k = 0, \dots, m. \tag{3.9}$$

The values of c_j can be obtained by solving the system (3.9). Same c_j suit to extend f onto $[0, 1 + \delta]$:

$$f(x) = \sum_{j=0}^{m} c_j f(1 - j(x - 1)), \quad 1 < x \le 1 + \delta, \tag{3.10}$$

As the result we obtain an extended function $f \in C^m[-\delta, 1+\delta]$. It holds

$$\max_{-\delta\le x\le 1+\delta} |f(x)| \le \sum_{j=0}^{m} |c_j| \max_{0\le x\le 1} |f(x)| = (2^{m+1}-1) \max_{0\le x\le 1} |f(x)|.$$

Thus after extending the function $f(x) \in L^2[0, 1]$ by the mentioned procedure and considering (3.5) and (3.7) we have

$$f(x) \simeq 2^{-\frac{n}{2}} \sum_{k=2-L}^{2^n-1} \sum_{q=0}^{p} w_q f\left(\frac{x_q+k}{2^n}\right) \phi_{n,k}(x). \tag{3.11}$$

The kernel $K(x, t) \in L^2([0, 1] \times [0, 1])$ can be extended and approximated similarly by

$$K(x,t) \simeq 2^{-n} \sum_{j,l=2-L}^{2^n-1} \sum_{r,s=0}^{p} w_r w_s K\left(\frac{x_r+j}{2^n}, \frac{t_s+l}{2^n}\right) \phi_{n,j}(x)\phi_{n,l}(t). \tag{3.12}$$

Considering relations (3.11) and (3.12), Eq. (3.4) can be rewritten as

$$\begin{aligned}\sum_i u_i\phi_{n,i}(x) &= 2^{-\frac{n}{2}} \sum_k \sum_q w_q f\left(\frac{x_q+k}{2^n}\right) \phi_{n,k}(x) \\ &\quad + 2^{-n} \sum_{i,j,l} \sum_{r,s} w_r w_s K\left(\frac{x_r+j}{2^n}, \frac{t_s+l}{2^n}\right) u_i \phi_{n,j}(x) \int_0^x \phi_{n,i}(t)\phi_{n,l}(t)dt.\end{aligned} \tag{3.13}$$

To find the unknown coefficients u_i by the Galerkin method, we multiply (3.13) by the functions $\phi_{n,m}(x)$, $m = 2-L, \dots, 2^n-1$, then integrate over $[0, 1]$ to get

$$\begin{aligned}&\sum_i u_i \int_0^1 \phi_{n,i}(x)\phi_{n,m}(x)dx = 2^{-\frac{n}{2}} \sum_k \sum_q w_q f\left(\frac{x_q+k}{2^n}\right) \int_0^1 \phi_{n,k}(x)\phi_{n,m}(x)dx \\ &\quad + 2^{-n} \sum_{i,j,l} \sum_{r,s} w_r w_s u_i K\left(\frac{x_r+j}{2^n}, \frac{t_s+l}{2^n}\right) \int_0^1 \phi_{n,j}(x)\phi_{n,m}(x) \int_0^x \phi_{n,i}(t)\phi_{n,l}(t)dtdx.\end{aligned} \tag{3.14}$$

Now by introducing the following connection coefficients

$$\Gamma_k(x) = \int_0^x \phi(y)\phi(y-k)dy, \tag{3.15}$$

$$\Omega_{i,j}^{k,l} = \int_0^{2^n} \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy \tag{3.16}$$

the linear system (3.14) reduces to

$$\begin{aligned}&\sum_i u_i \left(\Gamma_{i-m}(2^n-m) - \Gamma_{i-m}(-m)\right) \\ &= 2^{-\frac{n}{2}} \sum_k \sum_q w_q f\left(\frac{x_q+k}{2^n}\right) (\Gamma_{k-m}(2^n-m) - \Gamma_{k-m}(-m)) \\ &\quad + 2^{-n} \sum_{i,j,l} \sum_{r,s} w_r w_s u_i K\left(\frac{x_r+j}{2^n}, \frac{t_s+l}{2^n}\right) \int_0^1 \phi_{n,j}(x)\phi_{n,m}(x)(\Gamma_{i-l}(2^n x - l) - \Gamma_{i-l}(-l))dx,\end{aligned} \tag{3.17}$$

consequently

$$\begin{aligned}&\sum_i u_i \left(\Gamma_{i-m}(2^n-m) - \Gamma_{i-m}(-m)\right) = 2^{-\frac{n}{2}} \sum_k \sum_q w_q f\left(\frac{x_q+k}{2^n}\right) (\Gamma_{k-m}(2^n-m) - \Gamma_{k-m}(-m)) \\ &\quad + 2^{-n} \sum_{i,j,l} \sum_{r,s} w_r w_s u_i K\left(\frac{x_r+j}{2^n}, \frac{t_s+l}{2^n}\right) \left(\Omega_{j,m}^{i,l} - \Gamma_{i-l}(-l)(\Gamma_{j-m}(2^n-m) - \Gamma_{j-m}(-m))\right) \\ &\quad (m = 2-L, \dots, 2^n-1).\end{aligned} \tag{3.18}$$

Hence the unknown parameters u_i can be obtained by solving a linear system of the form

$$A\mathbf{u} = \mathbf{b}, \tag{3.19}$$

Table 1
Some values of $\Gamma_k(x)$ for the case $L = 6$.

x	k	$\Gamma_k(x)$	x	k	$\Gamma_k(x)$	x	k	$\Gamma_k(x)$	x	k	$\Gamma_k(x)$
1	−3	−0.4413282833e−3	2	−2	0.1343887190e−3	3	−1	−0.32493517105e−4	4	0	0.9999985220887
	−2	−0.1879313571e−1		−1	0.5652294255e−2		0	0.99866006117067		1	−0.3249351711e−4
	−1	0.1233282357158		0	0.9675965284579		1	0.56522942556e−2		2	0.1343887189e−3
	0	0.4966477981054		1	0.1233282357158		2	−0.18793135706e−1		3	−0.4413282828e−3

where

$$\begin{aligned} A_{m,i} &= \Gamma_{i-m}(2^n - m) - \Gamma_{i-m}(-m) \\ &\quad - 2^{-n} \sum_{j,l=2-L}^{2^n-1} \sum_{r,s=0}^{p} w_r w_s K\left(\frac{x_r + j}{2^n}, \frac{t_s + l}{2^n}\right) \left(\Omega_{j,m}^{i,l} - \Gamma_{i-l}(-l)(\Gamma_{j-m}(2^n - m) - \Gamma_{j-m}(-m))\right), \end{aligned}$$

$$b_m = 2^{-\frac{n}{2}} \sum_{k=2-L}^{2^n-1} \sum_{q=0}^{p} w_q f\left(\frac{x_q + k}{2^n}\right) (\Gamma_{k-m}(2^n - m) - \Gamma_{k-m}(-m)). \tag{3.20}$$

It is important to note that the connection coefficients $\Gamma_k(x)$ and $\Omega_{i,j}^{k,l}$ depend only on L and not on n.

4. Evaluation of the connection coefficients

In this section we provide special algorithms to evaluate the connection coefficients which were defined in the previous section.

4.1. Evaluation of $\Gamma_k(x) = \int_0^x \phi(y)\phi(y-k)dy$

Assume that the support of the scaling function $\phi(\cdot)$ is $[0, L-1]$, therefore the support of the function $\phi(\cdot - k)$ is $[k, k+L-1]$ and generally $\phi_{n,k}(\cdot)$ has the compact support $[2^{-n}k, 2^{-n}(k+L-1)]$. Consider the following properties of $\Gamma_k(x)$:

$$\Gamma_k(x) = 0, \quad \text{if } (x \le 0 \text{ or } x \le k), \tag{4.1}$$

$$\Gamma_k(x) = 0, \quad \text{if } |k| \ge L-1, \tag{4.2}$$

$$\Gamma_k(x) = \delta_{k,0} \quad \text{if } x \ge L-1, \tag{4.3}$$

$$\Gamma_k(x) = \Gamma_{-k}(x-k), \tag{4.4}$$

$$\Gamma_k(x) = \delta_{-k,0} \quad \text{if } x-k \ge L-1, \tag{4.5}$$

$$\Gamma_k(x) = \frac{1}{2} \sum_{i,j=0}^{L-1} a_i a_j \Gamma_{2k+i-j}(2x-j), \tag{4.6}$$

Relation (4.1) is trivial by considering the support of functions $\phi(\cdot)$ and $\phi(\cdot - k)$, on the other hand if $|k| \ge L-1$ then the support of these functions has no interconnection so (4.2) is also valid. Relation (4.3) holds, since if $x \ge L-1$ then

$$\Gamma_k(x) = \int_R \phi(y)\phi(y-k)dy = \delta_{k,0}.$$

The property (4.4) results from choosing $y - k = t$ in (3.15) and (4.5) is the consequence of (4.3) and (4.4). The two-scale relation (4.6) follows from applying (2.1) to (3.15)

$$\begin{aligned} \Gamma_k(x) &= \sum_{i,j=0}^{L-1} a_i a_j \int_0^x \phi(2y-j)\phi(2y-2k-i)dy = \frac{1}{2} \sum_{i,j=0}^{L-1} a_i a_j \int_j^{2x-j} \phi(t)\phi(t-(2k+i-j))dt \\ &= \frac{1}{2} \sum_{i,j=0}^{L-1} a_i a_j (\Gamma_{2k+i-j}(2x-j) - \Gamma_{2k+i-j}(-j)) = \frac{1}{2} \sum_{i,j=0}^{L-1} a_i a_j \Gamma_{2k+i-j}(2x-j). \end{aligned}$$

The last equality holds because of (4.1). According to relations (4.1)–(4.5), the unknown values of $\Gamma_k(x)$ reduce to the integer values of $x \in [1, L-2]$ and $|k| \le L-2$. To find these values we apply the two-scale relation (4.5) for unknowns x and k to find a nonsingular linear system. The nonsingularity of this system is discussed in [9,12]. Some values of $\Gamma_k(x)$ for Daubechies wavelets of genus $N = 3$, i.e. $L = 6$, are gathered in Table 1.

4.2. Evaluation of integrals $\Upsilon_r^{k,l} = \int_R \phi(y)\phi(y-r)\Gamma_{k-l}(y-l)dy$

In order to evaluate values of $\Omega_{i,j}^{k,l}$, we need to define a new three parameter connection coefficient $\Upsilon_r^{k,l}$ as follows:

$$\Upsilon_r^{k,l} = \int_R \phi(y)\phi(y-r)\Gamma_{k-l}(y-l)dy. \tag{4.7}$$

Before we discuss some properties of $\Upsilon_r^{k,l}$ we present the following lemma:

Lemma 4.1. *The support of function $\Gamma_{k-l}(x-l)$ is*

$$\mathrm{supp}\Gamma_{k-l}(x-l) \subseteq \begin{cases} [\max\{k,l\}, \min\{k,l\}+L-1], & k \neq l \\ [\max\{k,l\}, +\infty), & k = l. \end{cases} \tag{4.8}$$

Proof. It is easy to see that

$$\Gamma_{k-l}(x-l) = \Gamma_{l-k}(x-k). \tag{4.9}$$

Applying (4.1) and (4.9), reveals that

$$\Gamma_{k-l}(x-l) = 0 \quad \text{if } (x-l \leq 0 \text{ or } x-k \leq 0),$$

hence, $\Gamma_{k-l}(x-l) \neq 0$ if $x \geq \max\{k,l\}$. Let $k \neq l$ then by using (4.3) and (4.9),

$$\Gamma_{k-l}(x-l) = 0 \quad \text{if } (x-l \geq L-1 \text{ or } x-k \geq L-1),$$

therefore when $k \neq l$, if $x \leq \min\{k,l\} + L - 1$ then $\Gamma_{k-l}(x-l)$ may be nonzero. On the other hand when $k = l$, $\Gamma_{k-l}(x-l) = \Gamma_0(x-l)$ thus by considering (4.3), relation (4.8) is valid. □

The integrals $\Upsilon_r^{k,l}$ satisfy the following properties

$$\Upsilon_r^{k,l} = \Upsilon_r^{l,k} \tag{4.10}$$

$$\Upsilon_r^{k,l} = 0, \quad \text{if } (|r| \geq L-1 \text{ or } |k-l| \geq L-1) \tag{4.11}$$

$$\Upsilon_r^{k,l} = 0, \quad \text{if } \max\{k,l\} \geq L-1 \tag{4.12}$$

$$\Upsilon_r^{k,l} = 0, \quad \text{if } (k \neq l \text{ and } \min\{k,l\} + L - 1 \leq 0) \tag{4.13}$$

$$\Upsilon_r^{k,k} = \delta_{r,0} \quad \text{if } k \leq 1-L \tag{4.14}$$

$$\Upsilon_r^{k,l} = \frac{1}{4} \sum_{k_1,\ldots,k_4=0}^{L-1} a_{k_1}a_{k_2}a_{k_3}a_{k_4} \Upsilon_{2r+k_2-k_1}^{2k+k_3-k_1,2l+k_4-k_1}. \tag{4.15}$$

Relation (4.10) follows immediately from (4.9). Relation (4.11) follows from the compact support of functions $\phi(\cdot)$ and $\phi(\cdot - r)$ and also (4.2). Relations (4.12) and (4.13) are valid because of (4.8) and the support of function $\phi(y)$. To verify (4.14) owing to the support of function $\phi(y)$ we have

$$\Upsilon_r^{k,k} = \int \phi(y)\phi(y-r)\Gamma_0(y-k)dy = \int_0^{+\infty} \phi(y)\phi(y-r)\Gamma_0(y-k)dy = \delta_{r,0}.$$

Last equality holds, since when $k \leq 1-L$ and $y \geq 0$ then $y - k \geq L - 1$, hence by (4.5) $\Gamma_0(y-k) = \delta_{0,0} = 1$.

To prove (4.15), we apply the two-scale relations (2.1) and (4.6) to (4.7)

$$\begin{aligned}
\Upsilon_r^{k,l} &= \frac{1}{2} \sum_{k_1,\ldots,k_4=0}^{L-1} a_{k_1}a_{k_2}a_{k_3}a_{k_4} \int \phi(2y-k_1)\phi(2y-2r-k_2)\Gamma_{2(k-l)+k_3-k_4}(2y-2l-k_4)dy \\
&= \frac{1}{4} \sum_{k_1,\ldots,k_4} a_{k_1}a_{k_2}a_{k_3}a_{k_4} \int \phi(t)\phi(t-(2r+k_2-k_1))\Gamma_{(2k+k_3)-(2l+k_4)}(t-(2l+k_4-k_1))dt \\
&= \frac{1}{4} \sum_{k_1,\ldots,k_4} a_{k_1}a_{k_2}a_{k_3}a_{k_4} \Upsilon_{2r+k_2-k_1}^{2k+k_3-k_1,2l+k_4-k_1}.
\end{aligned}$$

In the next lemma we propose a useful relation concerning integrals $\Upsilon_r^{k,l}$.

Table 2
Some values of integrals $\Upsilon_r^{k,l}$ for the case $L = 6$.

r	k	l	$\Upsilon_r^{k,l}$	r	k	l	$\Upsilon_r^{k,l}$	r	k	l	$\Upsilon_r^{k,l}$	r	k	l	$\Upsilon_r^{k,l}$
−3	−2	−3	−7.68523e−7	−1	−1	−2	3.38897e−3	1	1	−1	−1.10634e−3	3	2	0	−1.15834e−5
		−2	3.44478e−6			−1	−1.75358e−2			1	−9.70245e−2			1	1.08026e−4
		−1	−1.15834e−5			1	7.61407e−4			2	−3.38897e−3			2	−4.94291e−4
		0	−1.11526e−8			2	−3.44478e−6			3	1.37889e−5			3	−2.62214e−4
−2	0	−1	1.10634e−3	0	1	0	1.75358e−2	2	1	1	3.82301e−3	4	2	3	1.37104e−6
		0	1.48138e−2			1	3.17289e−2			2	1.10634e−3			4	2.01381e−7
		1	8.44486e−6			2	7.44154e−4			3	−2.21011e−5		3	3	−4.96920e−6
		2	−9.35083e−8			3	−5.66583e−6			4	1.11526e−8			4	−6.4296e−7

Lemma 4.2. *If $|r - l| \geq L - 1$ then*

$$\Upsilon_r^{k,l} = \delta_{k-l,0}(\delta_{r,0} - \Gamma_r(l + L - 1)), \tag{4.16}$$

and if $|r - k| \geq L - 1$ then

$$\Upsilon_r^{k,l} = \delta_{l-k,0}(\delta_{r,0} - \Gamma_r(k + L - 1)). \tag{4.17}$$

Proof. If $y - l \leq 0$ then by (4.1), $\Gamma_{k-l}(y - l) = 0$. On the other hand, by considering (4.9) and (4.5) we have $\Gamma_{k-l}(y - l) = \Gamma_{l-k}(y - k) = \delta_{k-l,0}$ if $y - l \geq L - 1$. Hence

$$\begin{aligned} \Upsilon_r^{k,l} &= \int_l^{l+L-1} \phi(y)\phi(y-r)\Gamma_{k-l}(y-l)dy + \delta_{k-l,0}\int_{l+L-1}^{+\infty} \phi(y)\phi(y-r)dy \\ &= \int_l^{l+L-1} \phi(y)\phi(y-r)\Gamma_{k-l}(y-l)dy + \delta_{k-l,0}\int_{l+L-1}^{L-1} \phi(y)\phi(y-r)dy. \end{aligned}$$

Which results from the support of function $\phi(y)$. Now, if $r \geq l + L - 1$ or $r + L - 1 \leq l$ then the first integral in the above equation vanishes. So when $|r - l| \geq L - 1$ then

$$\Upsilon_r^{k,l} = \delta_{l-k,0}(\Gamma_r(L-1) - \Gamma_r(l + L - 1)) = \delta_{k-l,0}(\delta_{r,0} - \Gamma_r(l + L - 1)).$$

Relation (4.17) can also be verified similarly. □

Based on relations (4.10)–(4.15), and supposing integer values for k, l and r, nonzero values of integrals $\Upsilon_r^{k,l}$ are limited to the cases when $|r| \leq L-2$, $k \leq L-2$, $l \leq L-2$ and if $k \neq l$, then $k \geq 2-L$ and $l \geq 2-L$. And when $k = l \leq 1-L$ then $\Upsilon_r^{k,k} = \delta_{r,0}$.

Now by applying the recursive relation (4.15) for the unknowns k, l and r, one can find a non-homogeneous system of linear equations which can be easily solved. Some values of $\Upsilon_r^{k,l}$ have been collected in Table 2.

4.3. Evaluation of integrals $\Omega_{i,j}^{k,l} = \int_0^{2^n} \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy$

The following relations can be easily verified

$$\Omega_{i,j}^{k,l} = \Omega_{j,i}^{k,l} = \Omega_{i,j}^{l,k} = \Omega_{j,i}^{l,k}, \tag{4.18}$$

$$\Omega_{i,j}^{k,l} = 0 \quad \text{if } |k - l| \geq L - 1 \text{ or } |i - j| \geq L - 1, \tag{4.19}$$

$$\Omega_{i,j}^{k,l} = 0 \quad \text{if } (i \leq 1 - L \text{ or } j \leq 1 - L \text{ or } i \geq 2^n \text{ or } j \geq 2^n \text{ or } k \geq 2^n \text{ or } l \geq 2^n), \tag{4.20}$$

$$\Omega_{i,j}^{k,l} = 0 \quad \text{if } k \neq l \text{ and } (k \leq 1 - L \text{ or } l \leq 1 - L), \tag{4.21}$$

$$\Omega_{i,j}^{k,l} = \Gamma_{j-i}(2^n - i) - \Gamma_{j-i}(-i) \quad \text{if } (k = l \leq 1 - L). \tag{4.22}$$

Relation (4.18) is valid due to (4.9) and the formulae of $\Omega_{i,j}^{k,l}$. Relation (4.19) follows from (4.2) and the compact support of functions $\phi(y-i)$ and $\phi(y-j)$. The support of these functions also reveals that when $i \leq 1-L$ or $j \leq 1-L$ or $i \geq 2^n$ or $j \geq 2^n$, there is no interconnection between their support and the interval $[0, 2^n]$ and also when $k \geq 2^n$ or $l \geq 2^n$, then by (4.8) relation (4.20) holds. Relation (4.21) is another consequence of (4.8). To verify (4.22) note that when $k = l \leq 1 - L$ and $x \geq 0$ then $x - l \geq L - 1$, so

$$\begin{aligned} \Omega_{i,j}^{k,k} &= \int_0^{2^n} \phi(y-i)\phi(y-j)\Gamma_0(y-l)dy = \int_0^{2^n} \phi(y-i)\phi(y-j)\int_0^{x-l} \phi^2(t)dtdy \\ &= \int_0^{2^n} \phi(y-i)\phi(y-j)dy = \Gamma_{j-i}(2^n - i) - \Gamma_{j-i}(-i). \end{aligned}$$

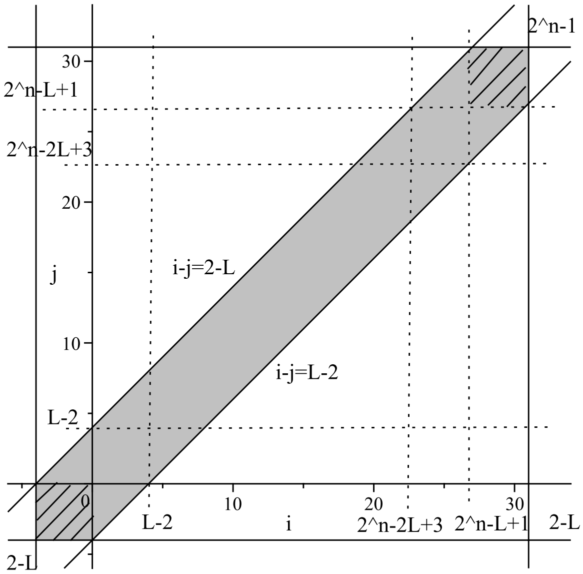

Fig. 1. Region of unknown values of integrals $\Omega_{i,j}^{k,l}$ related to variables i and j. ($n = 5$).

Another important relation for the integrals $\Omega_{i,j}^{k,l}$ is discussed in the next lemma.

Lemma 4.3. *For the integrals* $\Omega_{i,j}^{k,l}$, *if* $|i-l| \geq L-1$ *or* $|j-l| \geq L-1$ *then*

$$\Omega_{i,j}^{k,l} = \delta_{k-l,0}(\Gamma_{j-i}(2^n - i) - \Gamma_{j-i}(l - i + L - 1)), \tag{4.23}$$

and if $|i-k| \geq L-1$ *or* $|j-k| \geq L-1$ *then*

$$\Omega_{i,j}^{k,l} = \delta_{l-k,0}(\Gamma_{j-i}(2^n - i) - \Gamma_{j-i}(k - i + L - 1)). \tag{4.24}$$

Proof. As mentioned in the proof of Lemma 4.2, we have

$$\begin{aligned}\Omega_{i,j}^{k,l} &= \int_l^{l+L-1} \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy + \delta_{k-l,0}\int_{l+L-1}^{2^n} \phi(y-i)\phi(y-j)dy \\ &= \int_l^{l+L-1} \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy + \delta_{k-l,0}\int_{l+L-1-i}^{2^n-i} \phi(y)\phi(y-(j-i))dy \\ &= \int_l^{l+L-1} \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy + \delta_{k-l,0}(\Gamma_{j-i}(2^n-i) - \Gamma_{j-i}(l-i+L-1)),\end{aligned}$$

if $i \geq l+L-1$ or $j \geq l+L-1$ or $i+L-1 \leq l$ or $j+L-1 \leq l$, the integral in the last equation vanishes and (4.23) holds. Relation (4.24) can also be proved by the same discussion. □

By considering the above relations, one can find out which unknown values of integrals $\Omega_{i,j}^{k,l}$ fall in a bounded region related to variables i, j, k and l. The grey part in Fig. 1 indicates the region where the unknown values of $\Omega_{i,j}^{k,l}$ are with respect to variables i and j. The same figure can be plotted for the region where the variables k and l can vary.

To evaluate the unknowns $\Omega_{i,j}^{k,l}$ we try to find a recursive relation. By applying (2.1) and the two-scale relation (4.6) to (3.16), we get

$$\Omega_{i,j}^{k,l} = \frac{1}{4}\sum_{k_1,\ldots,k_4=0}^{L-1} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\int_0^{2^{n+1}} \phi(y-(2i+k_1))\phi(y-(2j+k_2))\Gamma_{(2k+k_3)-(2l+k_4)}(y-2l-k_4)dy. \tag{4.25}$$

The upper limit of the integral in the right-hand side of (4.25) is 2^{n+1}, hence there is not a recursive relation for $\Omega_{i,j}^{k,l}$ generally. Now we consider the following cases. Fig. 1 can also be helpful in better understanding these cases.

Case 1. If i or $j \in [0, 2^n - L + 1]$, then for each k and l

$$\Omega_{i,j}^{k,l} = \int_0^{2^n} \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy = \int_R \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy,$$

since in this case the support of functions $\phi(y-i)$ or $\phi(y-j)$ fall inside the interval $[0, 2^n]$. Now by using of (4.7), we have

$$\Omega_{i,j}^{k,l} = \int \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy = \int \phi(y)\phi(y-(j-i))\Gamma_{k-l}(y-(l-i))dy = \Upsilon_{j-i}^{k-i,l-i}. \tag{4.26}$$

Case 2. If $(i,j) \in [2-L, -1]^2$, then for each k and l and by choosing $n \geq \log_2(2L-4)$, relation (4.25) can be written in the recursive form, as

$$\Omega_{i,j}^{k,l} = \frac{1}{4} \sum_{k_1,\ldots,k_4=0}^{L-1} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\Omega_{2i+k_1,2j+k_2}^{2k+k_3,2l+k_4}. \tag{4.27}$$

This is due to the fact that the support of the function $\phi(y-(2i+k_1))$ in (4.25) is $[2i+k_1, 2i+k_1+L-1]$ and if $2i+k_1+L-1 \leq 2^n$ then the integration in relation (4.25) can be considered over the interval $[0, 2^n]$ instead of $[0, 2^{n+1}]$. On the other hand by Lemma 4.3, one can see that in this case when k or $l \geq L-2$ then the values of $\Omega_{i,j}^{k,l}$ are known. Therefore, relation (4.27) can be applied for $(i,j) \in [2-L, -1]^2$ and $(k,l) \in [2-L, L-3]^2$. After evaluating the connection coefficients $\Upsilon_r^{k,l}$ and using them in (4.27), a non-homogeneous system of linear equations is obtained which can be easily solved.

Case 3. If $(i,j) \in [2^n-L+2, 2^n-1]^2$, then for each k and l note that

$$\begin{aligned}\Omega_{i-2^n,j-2^n}^{k-2^n,l-2^n} &= \int_0^{2^n} \phi(y-i+2^n)\phi(y-j+2^n)\Gamma_{k-l}(y-l+2^n)dy \\ &= \int_{2^n}^{2^{n+1}} \phi(t-i)\phi(t-j)\Gamma_{k-l}(t-l)dt.\end{aligned} \tag{4.28}$$

Adding (3.16) to (4.28), leads to

$$\Omega_{i,j}^{k,l} + \Omega_{i-2^n,j-2^n}^{k-2^n,l-2^n} = \int_0^{2^{n+1}} \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy. \tag{4.29}$$

Since $i \in [2^n-L+2, 2^n-1]$, by choosing $n \geq \log_2(L-2)$, the support of function $\phi(y-i)$ falls in the interval $[0, 2^{n+1}]$, so (4.29) can be written as

$$\Omega_{i,j}^{k,l} + \Omega_{i-2^n,j-2^n}^{k-2^n,l-2^n} = \int \phi(y-i)\phi(y-j)\Gamma_{k-l}(y-l)dy,$$

which, by applying (4.7), yields

$$\Omega_{i,j}^{k,l} = -\Omega_{i-2^n,j-2^n}^{k-2^n,l-2^n} + \Upsilon_{j-i}^{k-i,l-i}. \tag{4.30}$$

This means that in this case, all values of $\Omega_{i,j}^{k,l}$ can be obtained directly from the other values that have been evaluated in previous cases. In fact Lemma 4.3 shows that when $(i,j) \in [2^n-L+2, 2^n-1]^2$, if k or $l \leq 2^n-2L+2$ the values of $\Omega_{i,j}^{k,l}$ are known. Hence relation (4.30) can be used for $(i,j) \in [2^n-L+2, 2^n-1]^2$ and $(k,l) \in [2^n-2L+3, 2^n-1]^2$.

Considering the above cases, all of unknowns $\Omega_{i,j}^{k,l}$ can be computed. In Table 3 we have proposed some values of these integrals.

5. Error analysis

Consider the following VIE of the second kind

$$u(x) = f(x) + \int_0^x K(x,t,u(t))dt, \quad x \in I := [0,1], \tag{5.1}$$

where $f : I \to R$ and $K : D \times R \to R$ (with $D := \{(x,t) : 0 \leq t \leq x \leq 1\}$) are known functions. The problem (5.1) has a unique solution if $g \in C(I)$ and K is continuous for all $(x,t) \in D$ and all u and also satisfies the (uniform) Lipschitz conditions:

$$|K(x,t,u_1) - K(x,t,u_2)| \leq l_1|u_1-u_2|, \tag{5.2}$$

$$|K_x(x,t,u_1) - K_x(x,t,u_2)| \leq l_2|u_1-u_2|, \tag{5.3}$$

for all $x \in I$, $(x,t) \in D$ and $u_1, u_2 \in R$, with Lipschitz constants l_1, l_2 being independent of u_1 and u_2.

Define a nonlinear integral operator $G : C(I) \to C(I)$ by

$$(Gv)(x) := f(x) + \int_0^x K(x,t,v(t))dt.$$

Table 3
Some values of integrals $\Omega_{i,j}^{k,l}$ for the case $L = 6$.

i	j	k	l	$\Omega_{i,j}^{k,l}$	i	j	k	l	$\Omega_{i,j}^{k,l}$	i	j	k	l	$\Omega_{i,j}^{k,l}$
−4	−4	−4	−4	1.4779e−6	0	0	0	−1	9.7025e−2	9	9	10	9	1.7536e−2
		−2	−2	1.4433e−6				0	4.9999e−1				10	3.1729e−2
−3	−3	−2	−2	1.3144e−3	4	3	6	3	−3.4448e−6	12	11	11	10	3.3889e−3
		−1	−2	1.0299e−4		4	−1	−1	1			13	10	−1.0802e−4
−2	−2	0	−2	−1.6977e−4			3	2	5.3638e−3		12	14	14	1.2727e−3
		0	−1	7.4415e−4				3	9.6827e−1	15	14	16	14	7.6140e−4
0	−1	0	0	−9.7025e−2	9	7	9	8	1.1063e−3				16	−5.3638e−3
		2	1	−4.3049e−5				9	1.4814e−2		15	14	12	2.0386e−4
16	14	17	15	−2.2101e−5	23	20	22	22	−4.9429e−4	27	27	22	22	1
	15	18	17	−4.3049e−5			23	22	−2.6221e−4			23	23	9.9999e−1
	16	7	7	1		21	23	23	1.4814e−2	29	29	16	16	9.9866e−1
		19	19	1.0987e−5	25	22	25	24	−2.6221e−4			31	31	3.8723e−4
19	17	20	19	8.4448e−6			25	25	7.3695e−5	30	28	9	9	1.3438e−4
	18	19	16	2.6221e−4		24	27	27	2.5982e−5			29	29	3.9573e−3
	19	20	17	4.9429e−4	27	25	26	26	3.8230e−3	31	30	−2	−2	1.2332e−1
		21	19	−1.6976e−4		26	29	27	1.3788e−5		31	31	30	6.9671e−2

Then the problem (5.1) reads: Find $u = u(x)$ such that

$$u(x) = (Gu)(x), \quad x \in I, \tag{5.4}$$

and its weak form is to find $u \in L^2(I)$ such that

$$(u, v) = (Gu, v), \quad v \in L^2(I), \tag{5.5}$$

where $(\cdot, \cdot)$ denotes the usual inner product in L^2-space. The Galerkin approximation of (5.5) is now defined as: Find $u^N \in V_N$ such that

$$(u^N, v) = (Gu^N, v), \quad v \in V_N. \tag{5.6}$$

Let $P_N : L^2(I) \to V_N$ denote the orthogonal projection operator defined by

$$(u, v) = (P_N, v), \quad \forall v \in V_N.$$

Then the problem (5.6) can be equivalently written as: Find $u^N \in V_N$ such that

$$u^N = P_N G u^N. \tag{5.7}$$

The following theorem shows the L^2 error approximation of a function with its orthogonal projection and the proof can be found in [21,22].

Theorem 5.1. *For an orthogonal Daubechies wavelet system of degree n with the scaling function $\phi(x)$ and the scaling vector α, assume α has finite length. If $f(x) \in C_0^n(R)$, then*

$$\|f(x) - P_j(f)(x)\|_{L^2} \le C2^{-jn}, \tag{5.8}$$

where C depends only on $f(x)$ and the scaling vector α.

Let $e^N := u - u^N$ be the error corresponding to the wavelet-Galerkin solution u^N of (5.1). Applying the mean value theorem for the kernel function K implies that there exist a function ξ, whose value $\xi(x)$ at x is between $u(x)$ and $u^N(x)$, such that

$$K(x, t, u) - K(x, t, u^N) = K_u(x, t, \xi)e^N(x). \tag{5.9}$$

set

$$\begin{aligned} (G'v)(x) &:= \int_0^x K_u(x, t, u(t))v(t)dt, \\ (G'_N v)(x) &:= \int_0^x K_u(x, t, \xi(t))v(t)dt. \end{aligned} \tag{5.10}$$

Then we have the following lemma similar to Lemma 2.2 in [23].

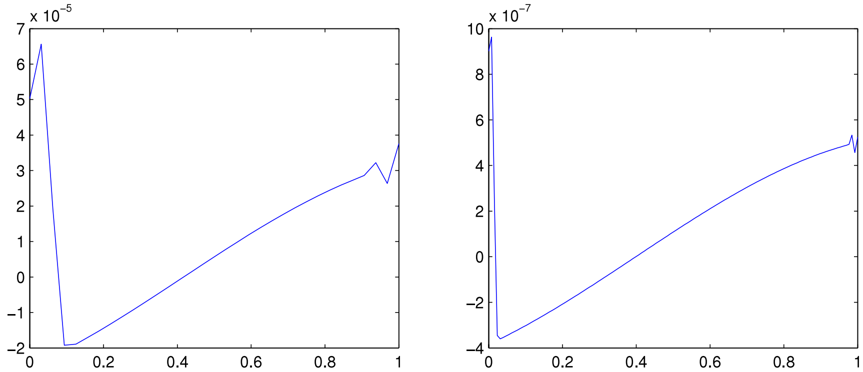

Fig. 2. Graphs of e^n, for Example 6.1, with $n = 5$ (left), $n = 7$ (right).

Lemma 5.1. *We have*

$$\lim_{N\to\infty} \|G'_N - G'\|_{L^2(I)\to L^2(I)} = 0,$$

where

$$\|A\|_{L^2(I)\to L^2(I)} := \sup_{v\in L^2(I)} \frac{\|Av\|_{L^2(I)}}{\|v\|_{L^2(I)}}.$$

Assuming $(I - G')^{-1}$ always exist and is bounded on $C(I)$ and $L^2(I)$, we get the following global convergence result for the problem (5.1).

Theorem 5.2. *In (5.1), assume that $f \in C^r(I)$ and $K \in C^r(D \times R)$ such that the VIE in (5.1) possess a unique solution $u \in C^r(I)$. Then the error e^N satisfies*

$$\|e^N\|_{L^2(I)} \le C2^{-Nr}.$$

Proof. The proof of this theorem is completely similar to the proof of Theorem 2.1 in [23] and is omitted here. □

6. Numerical results

In this section we provide some numerical examples to show the efficiency of the wavelet-Galerkin method in solving VIEs. We apply Daubechies wavelets of genus 3, i.e. $L = 6$. We emphasize that the connection coefficients $\Gamma_k(x)$ and $\Omega_{i,j}^{k,l}$ are evaluated just once and we use them in each example to construct the corresponding linear system (3.19).

Example 6.1. Consider the following VIE of the second kind

$$u(x) = -2x + 4\sin(x^2)(\sin 2x - \cos 2x) + (\sin 2x + \cos 2x)(1 + 2x\cos(x^2)) + \int_0^x (2x^2 - 8)\sin(xt)u(t)dt. \tag{6.1}$$

The exact solution of this equation is $u(x) = \sin(2x) + \cos(2x)$. In Fig. 2 we have plotted the error graph of $e^n = u(x) - u^n(x)$, where $u^n(x) = \sum_{i=2-L}^{2^n-1} u_i\phi_{n,i}(x)$ is the approximate solution obtained by applying wavelet-Galerkin method.

Example 6.2. Consider the following VIE

$$u(x) = \frac{1}{\cos(x^2)} - xe^x\tan(x^2) + \int_0^x \frac{(1 + x^2)\cos(xt)}{\cos(x^2)}u(t)dt, \tag{6.2}$$

with exact solution $u(x) = e^x$. The error graph is shown in Fig. 3.

7. Conclusion

The wavelet-Galerkin method based on Daubechies wavelets has been applied for solving linear VIEs of the second kind. In this approach some new connection coefficients have been introduced which need to be evaluated just once and this leads to a reduction in the cost of the calculations. The numerical results show the efficiency of this method.

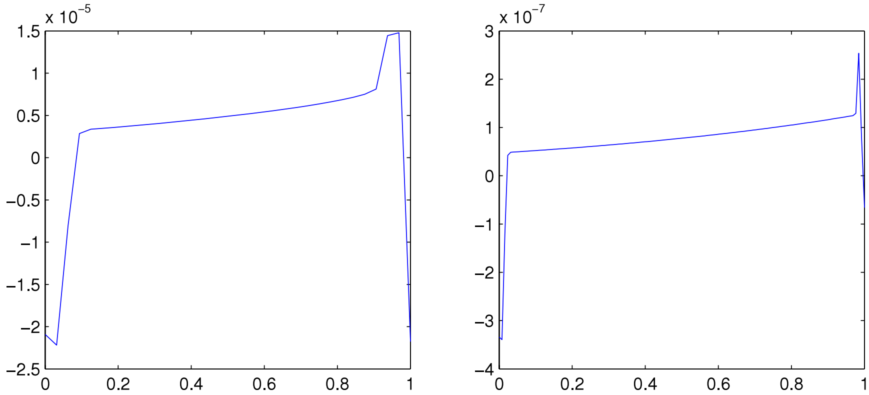

Fig. 3. Graphs of e^n, for Example 6.2, with $n = 5$ (left), $n = 7$ (right).

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