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Bi-parameter Semigroups of linear operators

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Abstract: Let $\mathcal{X}$ be a Banach space. We define the concept of a bi-parameter semigroup on $\mathcal{X}$ and its first and second generators. We also study bi-parameter semigroups on Banach algebras. A relation between uniformly continuous bi-parameter semigroups and σ-derivations is also established. It is proved that if $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ is a uniformly continuous bi-parameter semigroup on a Banach algebra $\mathcal{X}$, whose first and second generators are d and σ, respectively, and if d is also a σ-derivation then $d^n(ab) = (d+\sigma)^n(a) \star (d+\sigma)^n(b)$ and $\alpha_{t,0}(ab) = \alpha_{t,1}(a) \star \alpha_{t,1}(b)$ for all $a, b \in \mathcal{X}$.

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1. introduction

Let $\mathcal{X}$ be a Banach space and let $\mathcal{L}(\mathcal{X})$ denote the Banach space of all bounded linear operators on $\mathcal{X}$. A family $\{\alpha_t\}_{t\geqslant 0}$ in $\mathcal{B}(\mathcal{X})$ is called a uniformly (resp. strongly) continuous one-parameter semigroup on $\mathcal{X}$, if
(i) α_0 is the identity mapping I on $\mathcal{X}$;
(ii) $\alpha_{t+t'} = \alpha_t\alpha_{t'}$ for all $t, t' \in \mathbb{R}^+$;
(iii) $\lim_{t\downarrow 0} \alpha_t = I$ uniformly (resp. strongly) on $\mathcal{X}$.
Namely, α is a representation of the semigroup $(\mathbb{R}^+, +)$ into $\mathcal{B}(\mathcal{X})$ which is continuous with respect to the uniform (resp. strong) operator topology on $\mathcal{B}(\mathcal{X})$. When $\{\alpha_t\}_{t\geqslant 0}$ is

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a one-parameter semigroup on $\mathcal{X}$, the infinitesimal generator δ of α is defined by

$$\delta(x) = \lim_{t\downarrow 0} \frac{1}{t}(\alpha_t(x) - x),$$

whenever the limit exists and the domain $D(\delta)$ of δ is the set of all $x \in \mathcal{X}$ for which this limit exists. If $\{\alpha_t\}_{t\geqslant 0}$ is strongly continuous then $D(\delta)$ is a dense linear subspace of $\mathcal{X}$ and δ is a closed linear operator on this domain and if the semigroup $\{\alpha_t\}_{t\geqslant 0}$ is uniformly continuous, δ is an everywhere defined bounded linear operator on $\mathcal{X}$, see [12] for details. For example, let $\mathcal{X}$ be the Banach space (algebra) of all bounded uniformly continuous functions on $\mathbb{R}$ with the supremum norm. For each $t \in \mathbb{R}^+$, consider the linear mapping on $\mathcal{X}$ defined by $(\alpha_t(f))(h) = f(t+h), (f \in \mathcal{X})$. It is easy to see that the family $\{\alpha_t\}_{t\in\mathbb{R}^+}$ is a one-parameter semigroup satisfying $\| \alpha_t \| \leq 1$ and $(\delta(f))(h) = f'(h)$ if $f \in D(\delta)$. Obviously $D(\delta)$ is the linear subspace of $\mathcal{X}$ consisting of those f in $\mathcal{X}$ which are differentiable with $f' \in \mathcal{X}$. This example shows that the infinitesimal generator of this one-parameter semigroup, can be obtained by taking derivative when it exists.

It is easy to see that if δ is a bounded linear operator on a Banach space $\mathcal{X}$, then $\alpha_t = \exp(t\delta)$ $(t \geqslant 0)$ is a uniformly and hence strongly continuous one-parameter semigroup of operators on $\mathcal{X}$. In fact every uniformly continuous one-parameter semigroup is necessarily of this form for some bounded linear operator δ (see [12], Theorems I.2, I.3 and Corollary I.4).

If $\{\alpha_t\}_{t\geqslant 0}$ is a uniformly continuous one-parameter semigroup of homomorphisms on a Banach algebra $\mathcal{X}$, then its infinitesimal generator δ satisfies the Leibniz's rule $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{X}$. Such a linear mapping is called a derivation. Also, if δ is a bounded derivation on $\mathcal{X}$ then $\alpha_t = \exp(t\delta)$ $(t \geqslant 0)$ forms a uniformly continuous one-parameter semigroup of homomorphisms on $\mathcal{X}$, see [12, Theorems 1.2, 1.3 and Corollary 1.4] and also [1, Proposition 18.7]. The theory of one-parameter semigroups on operator algebras and their infinitesimal generators have been largely motivated by models of quantum statistical mechanics. The reader is referred to [4, 5, 13] for more details.

Let $\mathcal{X}$ be a Banach algebra and let σ be a linear mapping on $\mathcal{X}$. A linear mapping $d : \mathcal{X} \to \mathcal{X}$ is called a σ-derivation if it satisfies the generalized Leibniz rule $d(xy) = d(x)\sigma(y) + \sigma(x)d(y)$ for all $x, y \in \mathcal{X}$. For example, if ρ is a homomorphism and $\sigma = \frac{\rho}{2}$ then ρ is a σ-derivation. Moreover, when σ is an automorphism we can consider $\delta = d\sigma^{-1}$ and find out that δ is an ordinary derivation. This shows that the theory of σ-derivations combines the two subjects of derivations and homomorphisms. σ-derivations are investigated by many physicists and mathematicians. Automatic continuity, innerness, approximately innerness and amenability are the most important subjects which are studied in the theory of derivations and σ-derivations, see [6, 7, 8, 9, 10, 11].

When δ is a derivation on a Banach algebra $\mathcal{X}$, using the parameter t we can consider $\alpha_t = \exp(t\delta)$ and construct the one parameter semigroup $\{\alpha_t\}_{t\geqslant 0}$ of homomorphisms on $\mathcal{X}$. It seems that when we are dealing with a σ-derivation d, we need to consider two parameters t and s corresponding to d and σ, respectively. In what follows we define a uniformly (resp. strongly) bi-parameter semigroup of operators and its first and second generators. We will show that each uniformly continuous bi-parameter semigroup of operators on a Banach space $\mathcal{X}$ is of the form $\alpha_{t,s} = \exp(t(d + s\sigma))$ $(t, s \geqslant 0)$, where d and σ are bounded linear operators on $\mathcal{X}$. We will also give a relation between uniformly continuous bi-parameter semigroups on Banach algebras and σ-derivations.

2. Bi-parameter Semigroups

We start with the definition of a bi-parameter semigroup.

Definition 2..1. Let $\mathcal{X}$ be a Banach space. A family $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ of bounded linear operators on $\mathcal{X}$ is called a uniformly (resp. strongly) continuous bi-parameter semigroup if

(i) for each fixed $s \geqslant 0$, the family $\{\alpha_{t,s}\}_{t\geqslant 0}$ is a uniformly (resp. strongly) continuous one parameter semigroup with infinitesimal generator δ_s;

(ii) for each $s \geqslant 0$, $D(\delta_s) = D(\delta_0)$;

(iii) for $s > 0$, the value

$$\frac{1}{s}(\lim_{t\downarrow 0} \frac{1}{t}(\alpha_{t,s}(x) - x) - \lim_{t\downarrow 0} \frac{1}{t}(\alpha_{t,0}(x) - x)) = \frac{1}{s}(\delta_s(x) - \delta_0(x))$$

is independent of s for all $x \in D(\delta_0)$.

Take $d = \delta_0$ and $D = D(\delta_0)$. Note that for $x \in D$ and $s > 0$, $\sigma(x) := \frac{1}{s}(\delta_s(x) - \delta_0(x))$ is the average growth of δ_s in the interval $[0, s]$ at x, which by definition is independent of the choice of s. Obviously σ is a linear mapping on D and $\delta_s = d + s\sigma$. The operators d and σ, defined on D, are said to be the *first and second generators* of the bi-parameter semigroup $\{\alpha_{t,s}\}_{t,s\geqslant 0}$, respectively. The ordered pair (d, σ) is simply called the *generator* of $\{\alpha_{t,s}\}_{t,s\geqslant 0}$.

If d, σ are bounded linear operators on X then as in the case of one-parameter semigroups [12], we examine $\alpha_{t,s} = \exp(t(d + s\sigma)) = \exp(t\delta_s)$ and get the following result.

Proposition 2..2. If $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ is a uniformly continuous bi-parameter semigroup, then its first and second generators are bounded. Conversely, if d and σ are two bounded linear operators on a Banach space $\mathcal{X}$ then $\alpha_{t,s} = \exp(t(d + s\sigma))$ is a uniformly continuous bi-parameter semigroup whose generator is (d, σ).

It is clear that the first and second generators of a uniformly continuous bi-parameter semigroup are unique. Also, if d and σ are bounded linear operators then $\alpha_{t,s} = \exp(t(d + s\sigma))$ is a uniformly continuous bi-parameter semigroup with generator (d, σ). Is this semigroup unique? The answer is affirmative as we see below.

Proposition 2..3. Let $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ and $\{\beta_{t,s}\}_{t,s\geqslant 0}$ be two uniformly continuous bi-parameter semigroups with the same generator (d, σ). Then $\alpha_{t,s} = \beta_{t,s}$, for every $t, s \geqslant 0$.

Proof 2..4. Fix $s \geqslant 0$, then $\{\alpha_{t,s}\}_{t\geqslant 0}$ and $\{\beta_{t,s}\}_{t\geqslant 0}$ are one parameter semigroups with infinitesimal generator δ_s. So $\alpha_{t,s} = \beta_{t,s}$ for all $t \geqslant 0$. Since s is arbitrary we have the result.

Corollary 2..5. Uniformly continuous bi-parameter semigroups are of the form $\exp(t(d + s\sigma))$ for bounded linear operators d and σ.

3. σ-Derivations and Bi-parameter Semigroups

Let d, σ be linear operators on a linear space $\mathcal{X}$. We construct a family of linear mappings $\{Q_{n,k}\}$ $(n \in \mathbb{N},\ 0 \leqslant k \leqslant 2^n - 1)$, called the *binary family* corresponding to (d, σ), as follows.

Write the positive integer k in base 2 with exactly n digits, and put the operator d in place of 1's and σ in place of 0's. For example, $7 = (111)_2$, $11 = (01011)_2$, $Q_{3,7} = ddd = d^3$ and $Q_{5,11} = \sigma d\sigma dd = \sigma d\sigma d^2$ (cf. [9]).

The following lemma is stated and proved in [9, Lemma ...]. We give the proof, for the sake of convenience.

Lemma 3..1. Let $n \in \mathbb{N}$ and let $k \in \{0, ..., 2^n - 1\}$. Then
(i) $dQ_{n,k} = Q_{n+1,2^n+k}$;
(ii) $\sigma Q_{n,k} = Q_{n+1,k}$.

Proof 3..2. Suppose that $k = (c_n \ldots c_2c_1)_2$ where $c_j \in \{0, 1\}$ for $j = 1, ..., n$, be the representation of k in the base 2 with n digits. Then
(i) $dQ_{n,k} = Q_{n+1,(1c_n\ldots c_2c_1)_2} = Q_{n+1,k+2^n}$,
(ii) $\sigma Q_{n,k} = Q_{n+1,(0c_n\ldots c_2c_1)_2} = Q_{n+1,k}$.

Lemma 3..3. If $n \in \mathbb{N}$ and $k \in \{0, ..., 2^n - 1\}$. Then

$$(d + \sigma)^n = \sum_{k=0}^{2^n-1} Q_{n,k}.$$

Proof 3..4. We prove the assertion by induction on n. For $n = 1$ the result is clear.

Now suppose that it is true for n. By Lemma 3..1, we obtain

$$\begin{aligned}(d+\sigma)^{n+1} &= (d+\sigma)(d+\sigma)^n \\ &= (d+\sigma)(\sum_{k=0}^{2^n-1} Q_{n,k}) \\ &= \sum_{k=0}^{2^n-1} dQ_{n,k} + \sum_{k=0}^{2^n-1} \sigma Q_{n,k} \\ &= \sum_{k=0}^{2^n-1} Q_{n+1,2^n+k} + \sum_{k=0}^{2^n-1} Q_{n+1,k} \\ &= \sum_{k=2^n}^{2^{n+1}-1} Q_{n+1,k} + \sum_{k=0}^{2^n-1} Q_{n+1,k} \\ &= \sum_{k=0}^{2^{n+1}-1} Q_{n+1,k}.\end{aligned}$$

Definition 3..5. Let $\mathcal{X}$ be a Banach space and let $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ be a uniformly continuous bi-parameter semigroup of bounded linear operators on $\mathcal{X}$ with generator (d,σ), that is $\alpha_{t,s} = \exp(t(d+s\sigma))$. Take $\delta_s = d + s\sigma \ \ (s \geqslant 0)$. Take

$$\mathcal{Y} = \{\sum_{n=0}^{\infty} r_n t^n {\delta_s}^n : r_n \in \mathbb{C},\ t,s \geqslant 0, \text{ and the series is convergent in norm of } \mathcal{L}(\mathcal{X})\},$$

$$\mathcal{H} = \{T(a) : T \in \mathcal{Y} \text{ and } a \in \mathcal{X}\}.$$

Let n, m be nonnegative integers and $r, w \in \mathbb{C}$. We define a mapping $\star : \mathcal{H} \times \mathcal{H} \to \mathcal{X}$ as follows

$$rt^n(d+s\sigma)^n(a) \ \star \ wt^m(d+s\sigma)^m(b) = \begin{cases} 0 & n \neq m \text{ or } r \neq w \\ rt^n s^n \sum_{k=0}^{2^n-1} Q_{n,k}(a) Q_{n,2^n-1-k}(b) & n = m, r = w \end{cases}$$

and for $r_i, w_i \in \mathbb{C}$

$$\sum_{i=0}^{\infty} r_i t^i (d+s\sigma)^i(a) \ \star \ \sum_{i=0}^{\infty} w_i t^i (d+s\sigma)^i(b) = \sum_{i=0}^{\infty} \big(r_i t^i (d+s\sigma)^i(a) \star w_i t^i (d+s\sigma)^i(b)\big)$$

whenever the limit exists; otherwise we define

$$\sum_{i=1}^{\infty} r_i t^i (d+s\sigma)^i(a) \star \sum_{i=1}^{\infty} w_i t^i (d+s\sigma)^i(b) = 0.$$

In particular,

$$\alpha_{t,s}(a) \star \alpha_{t,s}(b) = \sum_{n=0}^{\infty}(\frac{t^n(d+s\sigma)^n}{n!}(a) \star \frac{t^n(d+s\sigma)^n}{n!}(b)). \tag{1}$$

Since d and σ are bounded operators, the series in (1) converges.

Lemma 3..6. Let $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ be a uniformly continuous bi-parameter semigroup with generator (d,σ). Then

$$\alpha_{t,1}(a) \star \alpha_{t,1}(b) - ab = (\alpha_{t,1}(a) - a) \star (\alpha_{t,1}(b) - b). \tag{2}$$

Proof 3..7. By definition of $\star$, we have

$$\begin{aligned}
&\alpha_{t,1}(a) \star \alpha_{t,1}(b) - ab \\
&= \sum_{n=0}^{\infty} \frac{t^n(d+\sigma)^n}{n!}(a) \star \frac{t^n(d+\sigma)^n}{n!}(b) - ab \\
&= ab + t(d+\sigma)(a) \star t(d+\sigma)(b) + \frac{t(d+\sigma)^2}{2!}(a) \star \frac{t(d+\sigma)^2}{2!}(b) + \cdots - ab \\
&= t(d+\sigma)(a) \star t(d+\sigma)(b) + \frac{t(d+\sigma)^2}{2!}(a) \star \frac{t(d+\sigma)^2}{2!}(b) + \cdots .
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(\alpha_{t,1}(a) - a) \star (\alpha_{t,1}(b) - b) \\
&= \sum_{n=1}^{\infty} \frac{t^n(d+\sigma)^n}{n!}(a) \star \sum_{n=1}^{\infty} \frac{t^n(d+\sigma)^n}{n!}(b) \\
&= t(d+\sigma)(a) \star t(d+\sigma)(b) + \frac{t(d+\sigma)^2}{2!}(a) \star \frac{t(d+\sigma)^2}{2!}(b) + \cdots .
\end{aligned}$$

Thus we have the equality in (2).

Lemma 3..8. Let $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ be a uniformly continuous bi-parameter semigroup with generator (d,σ). If $\sigma = I$, the identity mapping, then

$$\alpha_{t,1}(a) \star \alpha_{t,1}(b) = \alpha_{t,0}(a) \cdot \alpha_{t,0}(b).$$

Proof 3..9. We have

$$
\begin{aligned}
\alpha_{t,1}(a)\star\alpha_{t,1}(b) &= \exp^{t(d+I)}(a)\star\exp^{t(d+I)}(b)\\
&= \big(\sum_{n=1}^{\infty}\frac{t^n(d+I)^n}{n!}(a)\big)\star\big(\sum_{n=1}^{\infty}\frac{t^n(d+I)^n}{n!}(b)\big)\\
&= \sum_{n=1}^{\infty}\big(\frac{t^n(d+I)^n}{n!}(a)\star\big(\frac{t^n(d+I)^n}{n!}(b)\big)\\
&= \sum_{n=1}^{\infty}\big(\frac{t^n\big(\sum_{k=0}^{n}\binom{n}{k}d^k(a)\big)}{n!}\big)\star\big(\frac{t^n\big(\sum_{k=0}^{n}\binom{n}{k}d^k(b)\big)}{n!}\big)\\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{t^n\binom{n}{k}d^k(a)d^{n-k}(b)}{n!}\\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{t^kd^k(a)}{k!}\frac{t^{n-k}d^{n-k}(b)}{(n-k)!}\\
&= \big(\sum_{n=1}^{\infty}\frac{t^nd^n(a)}{n!}\big)\cdot\big(\sum_{n=1}^{\infty}\frac{t^nd^n(b)}{n!}\big)\\
&= \alpha_{t,0}(a)\cdot\alpha_{t,0}(b).
\end{aligned}
$$

Taking idea from the relation between uniformly continuous one parameter semigroups and derivations, we now are ready to state a relation between uniformly continuous bi-parameter semigroups and σ-derivations.

Theorem 3..10. Let $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ be a uniformly continuous bi-parameter semigroup with generator are (d,σ). If d is also a σ-derivation then
 (i) $d^n(ab)=(d+\sigma)^n(a)\star(d+\sigma)^n(b)$;
(ii) $\alpha_{t,0}(ab)=\alpha_{t,1}(a)\star\alpha_{t,1}(b)$.
In particular, if $\sigma=I$ and d is a derivation then

$$\alpha_{t,0}(ab)=\alpha_{t,0}(a)\cdot\alpha_{t,0}(b), \tag{3}$$

i.e., $\alpha_{t,0}$ is a homomorphism.

Proof 3..11. We prove (i) by induction. For $n=1$ the result is obvious. Now suppose it is true for n. From Definition 3..5 and Lemmas 3..1, 3..3 we have

$$\begin{aligned}
d^{n+1}(ab) &= d\big(d^n(ab)\big)\\
&= d\big((d+\sigma)^n(a)\star(d+\sigma)^n(b)\big)\\
&= d\Big(\sum_{k=0}^{2^n-1} Q_{n,k}(a)Q_{n,2^n-1-k}(b)\Big)\\
&= \sum_{k=0}^{2^n-1}\big(dQ_{n,k}(a)\sigma Q_{n,2^n-1-k}(b)+\sigma Q_{n,k}(a)dQ_{n,2^n-1-k}(b)\big)\\
&= \sum_{k=0}^{2^n-1}\big(Q_{n+1,k+2^n}(a)Q_{n+1,2^n-1-k}(b)+Q_{n+1,k}(a)Q_{n+1,2^n-1-k+2^n}(b)\big)\\
&= \sum_{k=0}^{2^n-1}\big(Q_{n+1,k+2^n}(a)Q_{n+1,2^{n+1}-1-(k+2^n)}(b)\big)+\sum_{k=0}^{2^n-1}\big(Q_{n+1,k}(a)Q_{n+1,2^n-1-k+2^n}(b)\big)\\
&= \sum_{k=2^n}^{2^{n+1}-1}\big(Q_{n+1,k}(a)Q_{n+1,2^{n+1}-1-k}(b)\big)+\sum_{k=0}^{2^n-1}\big(Q_{n+1,k}(a)Q_{n+1,2^n-1-k+2^n}(b)\big)\\
&= \sum_{k=0}^{2^{n+1}-1} Q_{n+1,k}(a)Q_{n+1,2^{n+1}-1-k}(b)\\
&= \Big(\sum_{k=0}^{2^{n+1}-1} Q_{n+1,k}(a)\Big)\star\Big(\sum_{k=0}^{2^{n+1}-1} Q_{n+1,k}(b)\Big)\\
&= (d+\sigma)^{n+1}(a)\star(d+\sigma)^{n+1}(b).
\end{aligned}$$

The assertion (ii) follows by (i) and the definition of $\star$.

Theorem 3..12. Let $\{\alpha_{t,s}\}_{t,s\geqslant 0}$ be a uniformly continuous bi-parameter semigroup with generator (d,σ). If

$$\alpha_{t,0}(ab)=\alpha_{t,1}(a)\star\alpha_{t,1}(b),$$

then d is a σ-derivation. In particular, if $\sigma=I$ then d is a derivation.

Proof 3..13. By assumption and the definition of $\star$ we have

$$\begin{aligned}
d(ab) &= \lim_{t\to 0}\frac{\alpha_{t,0}(ab)-ab}{t}\\
&= \lim_{t\to 0}\frac{\alpha_{t,1}(a)\star\alpha_{t,1}(b)-ab}{t}\\
&= \lim_{t\to 0}\frac{(\alpha_{t,1}(a)-a)\star(\alpha_{t,1}(b)-b)}{t}\\
&= \lim_{t\to 0}\frac{\alpha_{(t,1)}(a)-a}{t}\star\lim_{t\to 0}\frac{\alpha_{(t,1)}(b)-b}{t}\\
&= \big(d(a)+\sigma(a)\big)\star\big(d(b)+\sigma(b)\big)\\
&= d(a)\sigma(b)+\sigma(a)d(b).
\end{aligned}$$

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