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Approximately Orthogonal Additive Set-valued Mappings

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Abstract. We investigate the stability of orthogonally additive set-valued functional equation

$$F(x+y) = F(x) + F(y) \quad (x \perp y)$$

in Hausdorff topology on closed convex subsets of a Banach space.

1. Introduction

A functional equation $\mathfrak{F}$ is called stable if for any function f satisfying approximately to the equation $\mathfrak{F}$, there is a true solution of $\mathfrak{F}$ near to f. In 1940, S. M. Ulam [24] proposed the first stability problem for group homomorphisms. Hyers [9] gave the first significant partial solution to his problem for linear functions. Th. M. Rassias [20] improved Hyers' theorem by weakening the condition for the Cauchy difference controlled by $||x||^p + ||y||^p$, $p \in [0, 1)$. For some recent developments in this area, we refer the reader to the articles [5, 6, 11, 12, 15, 19] and the references therein.

In 1985, Rätz[21] gave a generalization of Birkhoff-James orthogonality [1, 10] in vector spaces. He also investigated some properties of orthogonally additive functional equation. This definition motivated some Mathematicians to discuss about the orthogonal stability of functional equations (see e. g. [8, 13, 16, 22]). On the other hand, set-valued mappings and their stability have been investigated by some authors from different point of view [2, 7, 14, 17, 23].

In the next section, we prove the stability of set-valued orthogonal additive functional equation

$$F(x+y) = F(x) + F(y) \quad (x \perp y). \tag{1}$$

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In fact, we will show if $(X, \perp)$ is an orthogonal space, Y is a Banach space and $F : X \to CC(Y)$ is an even function such that

$$\mathcal{H}\Big(F(x+y), F(x) + F(y)\Big) \le \varepsilon \quad (x, y \in X, x \perp y),$$

for some $\varepsilon > 0$. Then there exists a unique quadratic function $Q : X \to CC(Y)$ such that

$$\mathcal{H}(F(x), Q(x)) \le \frac{7\varepsilon}{4} \quad (x \in X).$$

In this case, we will show that there is a quadratic function $q : X \to Y$ such that

$$q(x) \in F(x) + \frac{7\varepsilon}{3}\,\overline{B(0,1)} \quad (x \in X).$$

2. Main Results

Throughout the paper, unless otherwise stated, we will assume that X and Y are topological vector spaces over $\mathbb{R}$. If $A, B \subset Y$ and $\lambda \in \mathbb{R}$, we use the following notions

$$A + B = \{a + b : a \in A, \quad b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}.$$

The following properties will often be used in the sequel:

For each $A, B \subset Y$ and $\lambda, \mu \ge 0$, we have

$$\lambda(A + B) = \lambda A + \lambda B, \quad (\lambda + \mu)A \subseteq \lambda A + \mu A.$$

Moreover, if A is convex, $(\lambda + \mu)A = \lambda A + \mu A$.

Definition 2.1. Let Y be a normed space and $A_1, A_2 \subseteq Y$ be non-empty closed bounded sets. Then the Hausdorff distance between A_1 and A_2 is defined by

$$\mathcal{H}(A_1, A_2) := \inf\{s > 0 : A_1 \subseteq A_2 + s\overline{B(0,1)} \text{ and } A_2 \subseteq A_1 + s\overline{B(0,1)}\}.$$

It is known that $\mathcal{H}$ defines a metric on closed convex subsets of Y, which is called Hausdorff metric topology[3, 4]. Moreover, if Y is a Banach space, $(CC(Y), \mathcal{H})$, the space of all non-empty compact convex subsets of Y with the Hausdorff metric topology is a complete metric space [3].

In 1985, Rätz [21] introduced the following notion:

Definition 2.2. Let X be a real topological vector space of dimension ≥ 2. A binary relation $\perp \subset X \times X$ is called an *orthogonal relation* if the following properties hold.

(1) $x \perp 0$, $0 \perp x$ for every $x \in X$,

(2) if $x, y \in X \setminus \{0\}$, $x \perp y$, then x and y are linearly independent;

(3) if $x, y \in X$, $x \perp y$, $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$,

(4) if P is a two dimensional subspace of X, $x \in P$, $\lambda \in \mathbb{R}^+$, then there exists some $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

The space X with an orthogonal relation $\perp$ is called an orthogonally space and is denoted by $(X, \perp)$.

Definition 2.3. Let X and Z be two sets. A function $Q : X \to Z$ is called *quadratic* if $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ for all $x, y \in X$.

We need to the following result due to Rådström [18].

Lemma 2.4. *Let A, B and C be nonempty subsets of a topological vector space Y. Suppose that B is closed and convex and C is bounded. If $A + C \subseteq B + C$, then $A \subseteq B$. If moreover, A is closed and convex and $A + C = B + C$, then $A = B$.*

Now, we are ready to state the main result of this paper.

Theorem 2.5. *Let X be a topological vector space over $\mathbb{R}$ which is also an orthogonal space and let Y be a Banach space. Let $F : X \to CC(Y)$ be an even function and for some $\varepsilon > 0$,*

$$\mathcal{H}\Big(F(x + y), F(x) + F(y)\Big) \leq \varepsilon \quad (x, y \in X, x \perp y). \tag{2.1}$$

Then there exists a unique quadratic and orthogonal additive function $Q : X \to CC(Y)$ such that

$$\mathcal{H}(F(x), Q(x)) \leq \frac{7\varepsilon}{3} \quad (x \in X).$$

Proof. We divide the proof into several steps.
Step 1. *For each $x \in X$,*

$$\mathcal{H}\Big(F(2x), 4F(x)\Big) \leq 7\varepsilon. \tag{2.2}$$

Proof of step 1. By Definition 2.2, for each $x \in X$, there is some $y \in X$ such that $x \perp y$ and $x + y \perp x - y$. Take some $y \in X$ with this property. Then

$$\begin{array}{rcl} F(x) & = & F\Big(\dfrac{x+y}{2}+\dfrac{x-y}{2}\Big) \\ & \subseteq & F\Big(\dfrac{x+y}{2}\Big)+F\Big(\dfrac{x-y}{2}\Big)+\varepsilon\overline{B(0,1)} \\ & = & F\Big(\dfrac{x+y}{2}\Big)+F\Big(\dfrac{y-x}{2}\Big)+\varepsilon\overline{B(0,1)} \quad (\because F \text{ is even}) \\ & \subseteq & F\Big(\dfrac{x+y}{2}+\dfrac{y-x}{2}\Big)+2\varepsilon\overline{B(0,1)} \\ & = & F(y)+2\varepsilon\overline{B(0,1)}. \end{array}$$

Since $x+y \perp y-x$, by interchanging the role of x and y, we see that

$$F(y) \subseteq F(x)+2\varepsilon\overline{B(0,1)}.$$

On the other hand,

$$\begin{array}{rcl} F(2x) & = & F(x+y+x-y) \subseteq F(x+y)+F(x-y)+\varepsilon\overline{B(0,1)} \\ & \subseteq & 2F(x)+2F(y)+3\varepsilon\overline{B(0,1)} \\ & \subseteq & 4F(x)+7\varepsilon\overline{B(0,1)} \end{array}$$

and

$$\begin{array}{rcl} 4F(x) & = & 2F(x)+2F(x) \subseteq 2F(x)+2F(y)+4\varepsilon\overline{B(0,1)} \\ & \subseteq & F(x)+F(y)+F(x)+F(-y)+4\varepsilon\overline{B(0,1)} \ (\text{ since } x \perp y) \\ & \subseteq & F(x+y)+F(x-y)+6\varepsilon\overline{B(0,1)} \ (\text{ since } x+y \perp x-y) \\ & \subseteq & F(2x)+7\varepsilon\overline{B(0,1)}. \end{array}$$

Therefore (2.2) holds.

Step 2. *There is a unique orthogonal additive function $Q : X \to CC(Y)$ such that*

$Q(2x) = 4Q(x)$ *and*

$$\mathcal{H}\Big(F(x),\ Q(x)\Big) \le \frac{7\varepsilon}{3} \tag{2.3}$$

for each $x \in X$.

Proof of step 2. Replace x by $2^n x$ in (2.2) and multiply both sides of the obtained inequality by $4^{-(n+1)}$ to obtain the following inequality

$$\mathcal{H}\Big(4^{-(n+1)}F(2^{n+1}x),\ 4^{-n}F(2^n x)\Big) \le \frac{7\varepsilon}{4^{n+1}} \quad (n \ge 0, x \in X).$$

It follows that for each $n > m \geq 0$, we have

$$\begin{aligned}\mathcal{H}\Big(4^{-n}F(2^nx),\ 4^{-m}F(2^mx)\Big) &\leq \sum_{k=m}^{n-1}\mathcal{H}\Big(4^{-(k+1)}F(2^{k+1}x),\ 4^{-k}F(2^kx)\Big)\\ &\leq \sum_{k=m}^{n-1}\frac{7\varepsilon}{4^{k+1}} \quad (x \in X).\end{aligned} \tag{2.4}$$

Since the right hand side of the above inequality tends to zero as $n \to \infty$, $\{4^{-n}F(2^nx)\}$ is a Cauchy sequence in $(CC(Y), \mathcal{H})$. Completeness of $CC(Y)$ with respect to the Hausdorff metric topology insures that

$$Q(x) = \lim_{n\to\infty} 4^{-n}F(2^nx) \quad (x \in X)$$

defines a function from X to $CC(Y)$. Put $m = 0$ in (2.4) to obtain

$$\begin{aligned}\mathcal{H}\Big(Q(x), F(x)\Big) &= \lim_{n\to\infty}\mathcal{H}\Big(4^{-n}F(2^nx), F(x)\Big)\\ &\leq \sum_{k=0}^{\infty}\frac{7\varepsilon}{4^{k+1}} = \frac{7\varepsilon}{3} \quad (x \in X).\end{aligned} \tag{2.5}$$

Moreover, for every $x \in X$, we have

$$\begin{aligned}Q(2x) &= \lim_{n\to\infty} 4^{-n}F(2^{n+1}x)\\ &= 4\lim_{n\to\infty} 4^{-(n+1)}F(2^{n+1}x) = 4Q(x).\end{aligned} \tag{2.6}$$

If $x \perp y$, we have

$$\begin{aligned}&\mathcal{H}\Big(Q(x) + Q(y), Q(x+y)\Big)\\ &= \lim_{n\to\infty}\mathcal{H}\Big(4^{-n}F(2^nx) + 4^{-n}F(2^ny),\ 4^{-n}F(2^n(x+y))\Big) \leq \lim_{n\to\infty} 4^{-n}\varepsilon = 0.\end{aligned}$$

Hence Q is orthogonal additive. Suppose that $Q' : X \to CC(Y)$ satisfies the following properties:

(i) $\mathcal{H}\Big(Q'(x), F(x)\Big) \leq \frac{7\varepsilon}{3}$ and

(ii) $Q'(2x) = 4Q'(x)$ for each $x \in X$.

Then for each $x \in X$, we have

$$\begin{aligned}\mathcal{H}\Big(Q'(x), Q(x)\Big) &= \lim_{n\to\infty}\mathcal{H}\Big(4^{-n}Q'(2^nx), 4^{-n}F(2^nx)\Big)\\ &= \lim_{n\to\infty} 4^{-n}\mathcal{H}\Big(Q'(2^nx), F(2^nx)\Big) \leq \lim_{n\to\infty} 4^{-n}\frac{7\varepsilon}{3} = 0.\end{aligned}$$

Thus the uniqueness assertion of step 2 follows.

Step 3. *The function $Q : X \to CC(Y)$ is quadratic.*

Proof of step 3. Let $x, y \in X$. Then the following cases may happen.
(i) $y = \alpha x$, where $\alpha \geq 0$. In this case, by property (4) of Definition 2.2, for each $x \in X$, there is some $z \in X$ such that $x \perp z$ and $x + z \perp \alpha x - z$. Therefore

$$Q(x+y) + Q(x-y) = Q(x+\alpha x) + Q(x-\alpha x) = Q(x+z+\alpha x - z) + Q(\alpha x - x).$$

It follows that

$$\begin{aligned} Q(x+\alpha x) + Q(x-\alpha x) + Q(2z) &= Q(x+z) + Q(\alpha x - z) + Q(\alpha x - x + 2z) \\ &= Q(x) + 2Q(z) + Q(\alpha x) + Q(x+z+z-\alpha x) \\ &= Q(x) + 2Q(z) + Q(\alpha x) + Q(x+z) + Q(z-\alpha x) \\ &= 2Q(x) + 2Q(\alpha x) + 4Q(z) \\ &= 2Q(x) + 2Q(\alpha x) + Q(2z). \end{aligned}$$

Thanks to Lemma 2.4, the result follows in this case.
(ii) $y = \alpha x$, where $\alpha < 0$. Let $\beta = -\alpha$. Then $\beta > 0$. Hence,

$$\begin{aligned} Q(x+\alpha x) + Q(x-\alpha x) &= Q(x-\beta x) + Q(x+\beta x) \\ &= 2Q(x) + 2Q(\beta x) = 2Q(x) + 2Q(\alpha x) \end{aligned}$$

since Q is even.
(iii) x and y are linearly independent.
By Definition 2.2, there is some z in linear span of $\{x, y\}$ such that $x \perp z$. Let $y = \alpha x + \beta z$. Then

$$\begin{aligned} Q(x+y) + Q(x-y) &= Q[(x+\alpha x) + \beta z] + Q[x - (\alpha x + \beta z)] \\ &= Q(x+\alpha x) + Q(\beta z) + Q(x-\alpha x) + Q(-\beta z) \\ &= 2Q(x) + 2Q(\alpha x) + 2Q(\beta z) \\ &= 2Q(x) + 2Q(\alpha x + \beta z) = 2Q(x) + 2Q(y). \end{aligned}$$

This completes the proof of the theorem. □

Example 2.6. Let X be an inner product space and $\varepsilon > 0$. Define $F : X \to CC(\mathbb{R})$ by $F(x) = [0, ||x||^2 + \varepsilon]$. It is easy to see that F is $[0, \varepsilon]$-orthogonal additive even function. According to Theorem 2.5, there is a quadratic function $Q : X \to CC(\mathbb{R})$ such that

$$\mathcal{H}(F(x), Q(x)) \leq \frac{7\varepsilon}{3} \quad (x \in X).$$

Definition 2.7. Let X and Y be two sets. By a selection of a set-valued function $F : X \to 2^Y$, we mean a single-valued mapping $f : X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$.

Corollary 2.8. Under conditions of Theorem 2.5, there is a quadratic function $q : X \to Y$ such that

$$q(x) \in F(x) + \frac{7\varepsilon}{3}\,\overline{B(0,1)} \quad (x \in X).$$

Proof. It is known that if X is an abelian group with division by two and Y is a topological vector space, then every subquadratic set-valued function $Q : X \to CC(Y)$ admits a quadratic selection $q : X \to Y$ [4, Theorem 35.2]. So the result follows from Theorem 2.5. □

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