A MICROSTRUCTURE-DEPENDENT TIMOSHENKO BEAM MODEL FOR VIBRATION ANALYSIS OF MICROPIPES CONVEYING FLUID BASED ON STRAIN GRADIENT THEORY

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Based on strain gradient theory, a microstructure-dependent Timoshenko beam model is presented, for the first time, to analyze the vibration and structural instability of the micro scale pipes conveying fluid. The governing equations are derived using Hamilton principle. This new model contains three material length scale parameters which interpret the size effect. The present model reduces to the modified couple stress model and the classical model when two and all of these three material length scale parameters become zero, respectively. Then, the differential quadrature method is applied to solve the equations of motion. Finally, the effects of the material length scale parameters and Poisson ratio on the vibrational characteristics and stability are explored. It is found that the natural frequencies and the critical flow velocities predicted by the present strain gradient model are higher than those predicted by the modified couple stress model and the classical model. Specially, when the outside diameter of the micropipe is comparable with the material length scale parameter, the size effect is significant. Moreover, the Poisson effect on the critical flow velocities shows an extreme point phenomenon for the two non-classical models, which is quite different from that predicted by the classical model.

1. Introduction

Micropipess/nanopipes have become prevalent in the fields of micro-electronic-mechanical systems and nanotechnology, such as those employed in sensors, actuators, fluid storage, fluid transport and drug delivery.[1-5] The inside diameter of circular micropipes considered ranged from 1 to 100 μm.[6] The study on the vibration and structural instability of micropipes is a key issue for the successful design and operation of fluidic devices. It has become an interested and challenging topic for many researchers.

In the recent decades, the size dependence of deformation behavior in micro scale beams had been experimentally observed in metals, polymers and polysilicons.[7-11] For example, in the torsion test of thin copper wires, the torsional hardening increased by a factor of three as the wire diameter reduced from 170 to 12 μm.[9] In the micro-indentation experiments, the measured indentation hardness of silver single crystal increased by more than two times when the penetration depth of the indenter decreased from 2.0 to 0.1 μm.[10] The above experimental results show that size dependence is intrinsic to certain materials with microstructures.

Due to lacking internal material length scale parameters, conventional strain-based mechanics theories fail to predict such a size dependent phenomenon. Recently, higher-order continuum theories have been developed to predict these size dependencies, in which constitutive equations introduce additional material length scale parameters in addition to classical constants.

The classical couple stress theory is one of the higher-order continuum theories, originated by Mindlin[12], Mindlin and Tiersten[13], and Toupin[14], in which contains two additional material length scale parameters besides the classical constants for an isotropic elastic material. Recently, the classical couple stress theory has been modified, in which only one additional material length scale parameter is involved and the couple stress tensor is symmetric.[15] The theory had been employed to study the mechanical and dynamical behavior of microbeams.[16-17]

The Mindlin's theory has been extended and reformulated and renamed as the strain gradient theory, in which the strain gradient tensor is decomposed into two independent parts: the stretch gradient tensor and the rotation gradient tensor, while the former tensor is not included in the couple stress theory.[18-19] This theory contains seven (five additional and two classical) constants. The modified strain gradient elasticity theory is another higher-order continuum theory, which contains a new additional equilibrium equation besides the classical equilibrium equations and also five elastic constants (two classical and three non-classical) for isotropic linear elastic materials.[20] The theory has been applied to analyze the static and dynamic problems of micro scale Bernoulli-Euler and and Timoshenko beams.[21-22]

The literature about the vibration and structural instability of micropipes conveying fluid based on higher order continuum theories are limited. More recently, microstructure-dependent Bernoulli-Euler and Timoshenko beam models have been developed to study the dynamic behaviours of micropipes containing internal fluid using the modified couple stress theory.[23-24] Also, a microstructure-dependent Bernoulli-Euler beam model has been presented based on the strain gradient theory to explore the size effect in the micropipes conveying fluid.[25] It was found that the size effect on the natural frequencies and the critical flow velocities was significant when the outside diameter of the micropipe became comparable with the material length scale parameter.

The objective of the present paper is to develop, for the first time, a microstructure-dependent Timoshenko beam model for the micropipes conveying fluid using the strain gradient theory. The rest of the paper is organized as follows. In Section 2, the equations of motion are deduced from the Hamilton principle. In Section 3, the vibration and instability structure of pinned-pinned micropipes containing fluid are investigated. Finally, the paper concludes with a summary in Section 4.

2. Analytical model

Consider a circular micropipe of length L, outside diameter D, inside diameter d, external cross-sectional area A_p, mass per unit length m_p, conveying incompressible fluid of mass per unit length m_f, internal cross-sectional flow area A_f flowing axially with velocity V. The Cartesian axes for a micropipe analysis are established. The x-axis is coincident with the centroidal axis of the micropipe.

The strain gradient elasticity theory for micropipes will be reviewed first.[20,22] Compared to the modified couple stress theory, the strain gradient elasticity theory introduces additional dilatation gradient vector and the deviatoric stretch gradient tensor in addition to the symmetric rotation gradient tensor. These tensors can be specified by two classical material constants for isotropic linear elastic materials and three independent material length scale parameters. The strain energy U in a linear elastic isotropic material occupying region Ω (with a volume element υ) can be written by:

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}\chi_{ij}\right)d\upsilon \,. \tag{1}$$

in which the components of the strain tensor, the dilatation gradient vector, the deviatoric stretch gradient tensor, and the symmetric rotation gradient tensor respectively represented by ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$, χ_{ij} are defined as

$$\varepsilon_{ij}=\frac{1}{2}\left(\partial_i u_j+\partial_j u_i\right),\quad \gamma_i=\partial_i\varepsilon_{mm},\quad \chi_{ij}=\frac{1}{2}\left(e_{ipq}\partial_p\varepsilon_{qj}+e_{jpq}\partial_p\varepsilon_{qi}\right),$$
$$\eta_{ijk}^{(1)}=\frac{1}{3}\left(\partial_i\varepsilon_{jk}+\partial_j\varepsilon_{ki}+\partial_k\varepsilon_{ij}\right)-\frac{1}{15}\Big[\delta_{ij}\left(\partial_k\varepsilon_{mm}+2\partial_m\varepsilon_{mk}\right)$$
$$+\delta_{jk}\left(\partial_i\varepsilon_{mm}+2\partial_m\varepsilon_{mi}\right)+\delta_{ki}\left(\partial_j\varepsilon_{mm}+2\partial_m\varepsilon_{mj}\right)\Big]. \tag{2}$$

where in these relations, ∂_i is the differential operator, u_i represents the components of the displacement vector, δ_{ij} is the Knocker delta and e_{ipq} is the permutation symbol. In the above and in subsequent equations, the index notation will be used with repeated indices denoting summation from 1 to 3.

The stress measures: the classical stress tensor, σ_{ij}, and the higher-order stresses, p_i, $\tau_{ijk}^{(1)}$, and, m_{ij} are the work-conjugates to ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$, χ_{ij} respectively, given by

$$\sigma_{ij}=k\delta_{ij}\varepsilon_{mm}+2G\varepsilon_{ij},\quad p_i=2l_0{}^2G\gamma_i,\quad \tau_{ijk}^{(1)}=2l_1{}^2G\eta_{ijk}^{(1)},\quad m_{ij}=2l_2{}^2G\chi_{ij}\,. \tag{3}$$

where k is bulk module, G is shear module and l_0, l_1, l_2 appeared in higher order stresses, represent the additional independent material length scale parameters related to the dilatation gradients, deviatoric stretch gradients and symmetric rotation gradients, respectively.

The displacement field at any point based on the Timoshenko beam theory can be described by

$$u=-z\psi\left(x,t\right),\quad v=0,\quad w=w\left(x,t\right). \tag{4}$$

where u, v, and w are the components of the displacement vector of a point (x, y, z) on a pipe cross-section in the x-, y- and z-directions, respectively; ψ (x, t) is the rotation angle (about the y-axis) of the cross-section with respect to z-axis.

The strain energy of a deformed micropipe is given by[22]

$$U=\frac{1}{2}\int_0^L\left[k_1\psi''^2+k_2\psi'^2+k_3\left(w''+\psi'\right)^2+k_4\left(-w''+2\psi'\right)^2+k_5\left(w'-\psi\right)^2\right]dx\,. \tag{5}$$

where

$$k_1=I\left(2l_0{}^2G+\frac{4}{5}l_1{}^2G\right),\quad k_2=I(k+2G)+2l_0{}^2GA_p,$$
$$k_3=\frac{1}{4}l_2{}^2GA_p,\quad k_4=\frac{8}{15}l_1{}^2GA_p,\quad k_5=K_sGA_p\,. \tag{6}$$

and I is the inertia moment of the pipe. The prime represents partial derivative with respect to x. The correction factor K_s for a circular micropipe is approximately given by[24]

$$K_s=\frac{6\left(1+\nu\right)\left(1+\alpha^2\right)^2}{\left(7+6\nu\right)\left(1+\alpha^2\right)^2+\left(20+12\nu\right)\alpha^2}\,. \tag{7}$$

where $\alpha=d/D$ and ν is Poisson ratio.

The total kinetic energy of the micropipe and the internal fluid may be written as[26]

$$T = \frac{1}{2} m_p \int_0^L \left[\left(-z\dot{\psi}\right)^2 + \left(\dot{w}\right)^2 \right] dx + \frac{1}{2} m_f \int_0^L \left[\left(-z\dot{\psi} + V\right)^2 + \left(\dot{w} + Vw'\right)^2 \right] dx . \tag{8}$$

The over dot denotes partial derivative with respect to the time t.

Now, the statement of Hamilton principle for a supported fluid-conveying micropipe, in the absence of dissipative forces can be written as[26]

$$\delta \int_{t_1}^{t_2} \left(T - U\right) dt = 0 . \tag{9}$$

Finally, substituting Eqs. (5-8) into Eq. (9), after considerable manipulation, the equations of motion are obtained as

$$\begin{aligned} &\left(m_p + m_f\right)\ddot{w} + 2m_f V\dot{w}' + m_f V^2 w'' + \left(k_3 + k_4\right) w^{(4)} + \left(k_3 - 2k_4\right)\psi''' - k_5\left(w'' - \psi'\right) = 0, \\ &\left(J_p + J_f\right)\ddot{\psi} + k_1 \psi^{(4)} - \left(k_3 - 2k_4\right) w''' - \left(k_2 + k_3 + 4k_4\right)\psi'' - k_5\left(w' - \psi\right) = 0. \end{aligned} \tag{10}$$

where J_p and J_f are the mass moment of inertia per unit length for micropipe and fluid, respectively.

For a pinned-pined micropipe the boundary conditions are given by

$$w\left(0,t\right) = w\left(L,t\right) = 0, \quad \frac{\partial \psi}{\partial x}\left(0,t\right) = \frac{\partial \psi}{\partial x}\left(L,t\right) = 0 . \tag{11}$$

It should be noted that when the material length scale parameters l_0 and l_1 become zero, the present strain gradient model will reduce to the modified couple stress model. Furthermore, if all the material length scale parameters l_0, l_1 and l_2 vanish, the size effect is suppressed and the present strain gradient model will reduce to the classical model.

3. Results and discussion

To illustrate the newly derived solutions of a pinned–pinned fluid-conveying micropipe, some numerical examples have been performed. The differential quadrature method will be used to solve Eqs. (10) and (11).[27] The validity of the present analysis is confirmed by comparing the present results with those given in the literature. In addition, the size effect and the Poisson effect on the natural frequency and the structural instability of the micropipe conveying fluid are investigated.

3.1 Validation

To check the accuracy and applicability of the present analysis, the results obtained from the present analysis are compared with those obtained from the previous results. The beam considered here is taken to be made of epoxy with the rectangular cross-section. The material properties of the microbeam are chosen to be: Young's modulus E = 1.44 GPa, mass density ρ = 1220 kg/m^3, Poisson ratio ν = 0.38 and material length scale parameter l = 17.6 μm.[22] For comparison convenience, it is assumed that all three material length scale parameters are the same, i.e., $l_0 = l_1 = l_2 = l$. Also, the cross-sectional properties are taken to be b/h = 2 and L/h = 20, where b and h are the width and the thickness of the beam, respectively.[22] Figure 1 shows the comparison of the results. An excellent agreement of the results is observed in Fig. 1. It can be concluded that the present analysis is an appropriate method to predict the natural frequency.

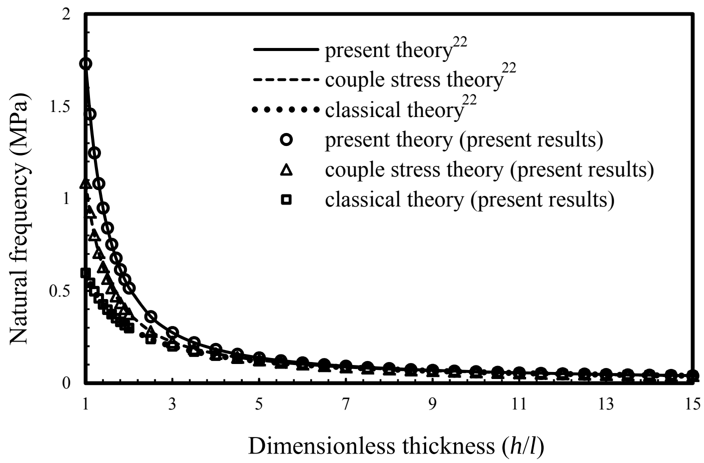

Figure 1. Comparison of the present natural frequency results with previous results.

3.2 Size effect

Figure 2 show the dimensionless natural frequency of the pinned-pinned micropipe with respect to the fluid flow velocity based on the present strain gradient model, the modified couple stress model and the classical model, for four values of outside diameter of the micropipe D = 20, 50, 100 and 200 μm, respectively. The dimensionless natural frequency is defined as $\mathrm{Im}(\omega)\sqrt{(m_p + m_f)L^4/(EI)}$, where ω is the circular frequency. The micropipe here is taken to be made of epoxy. The Young's modulus E = 1.44 GPa, mass density of micropipe ρ_p = 1220 kg/m^3, mass density of fluid ρ_f = 1000 kg/m^3, Poisson ratio ν = 0.38, material length scale parameter l = 17.6 μm, $\alpha = d/D$ = 0.8 and aspect ratio L/D = 20 are considered in the analysis.[24]

For D = 20 μm, when V = 0, the dimensionless natural frequency predicted by present strain gradient model is approximately 2.35 times greater than that predicted by the classical model and the dimensionless natural frequency predicted by the modified couple stress model is approximately 1.56 times greater than that predicted by the classical model. On the other hand, for D = 200 μm and V = 0, the ratios of dimensionless natural frequencies predicted by the two non-classical models to that predicted by the classical model are reduced to 1.02 and 1.007, respectively. Therefore, the difference between the three models results are large when the outside diameter is comparable with the material length scale parameter (i.e. $D/l \approx$ 1), but it decreases when the outside diameter increases, indicating that the size effect becomes negligible for large outside diameters. Furthermore, it is obvious that the dimensionless natural frequency predicted by the present strain gradient model is always greater than that predicted by the modified couple stress model and the dimensionless natural frequencies predicted by the two non-classical models are always greater than that predicted by the classical model. The reason is the strain gradient model contains two additional material length scale parameters in comparison with the modified couple stress model and the classical model has no material length scale parameter.

From Fig. 2, it is found that the dimensionless natural frequency decreases as the flow velocity increases. When the dimensionless natural frequency becomes zero, the micropipe loses its stability via divergence (buckling). The flow velocity at which the micropipe becomes unstable is called the critical flow velocity (V_{cr}). To show the size effect on the critical flow velocity, the variation of the critical flow velocity with the outside diameter is plotted in Fig. 3a. As shown in Fig. 3a, for both cases of ν = 0.0 and ν = 0.38, the differences in critical flow velocity for the three models are large when the outside diameter is small (e.g., D < 100 μm), where as they are decreasing or even diminishing with the outside diameter increasing. This indicates that the size effect is prominent only when the outside diameter is as small as the material length scale parameter. Moreover, the critical flow velocity predicted by the modified couple stress model is always smaller than that predicted by the present strain gradient model, while larger than that predicted by the classical model with the outside diameter varying. It is noted that the critical flow velocity with $\nu = 0.0$ is always smaller than that with $\nu = 0.38$ for the classical model. However, it is not true when $D <$ 44 μm for the present model and $D <$ 23 μm for the modified couple stress model. Therefore, Poisson effect on the critical flow velocity is complicated and will be discussed below.

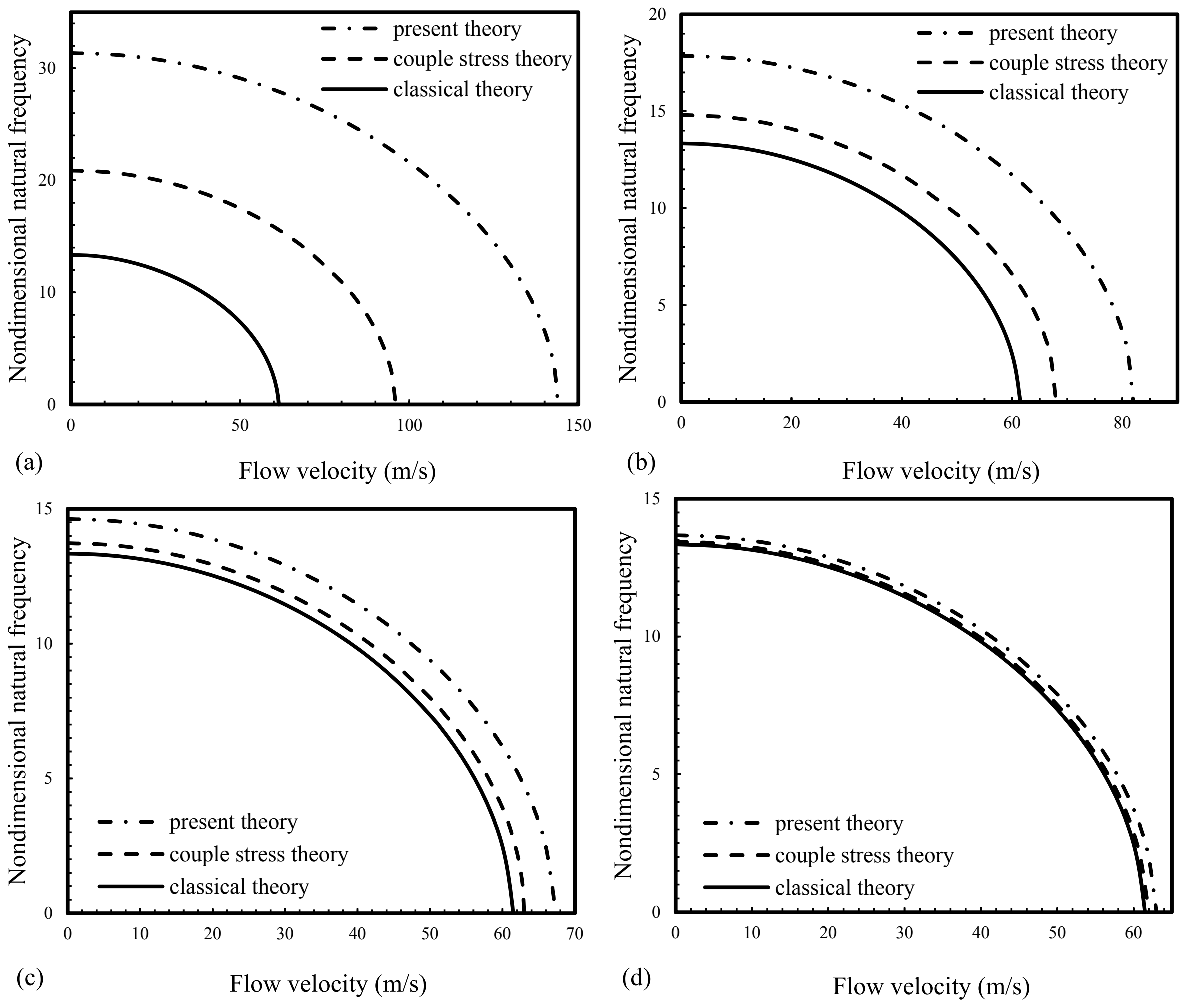

Figure 2. The nondimensional natural frequencies based on the strain gradient model, modified couple stress model and classical model, as functions of the flow velocity, for a pinned-pinned micropipe with (a) $D = 20$ μm, (b) 50 μm, (c) 100 μm and (d) 200 μm.

From Figs. 2-3, it can be seen that the results predicted by the modified couple stress model and the classical model in the present study are in good agreement with the previous results,[24] which further demonstrates the validity of the current model.

3.3 Poisson effect

The variation of critical flow velocity predicted by the present strain gradient model and the other two reduced models with respect to the Poisson ratio is shown in Fig. 3b, for two cases of outside diameter $D = 20$ μm and $D = 50$ μm. Figure 3b illustrates that the critical flow velocity predicted by the classical model increases gradually with Poisson ratio increasing. However, the critical flow velocity predicted by the present strain gradient model decreases firstly and then increases as the Poisson ratio increases which is quite different from that predicted by the classical model. Furthermore, the critical flow velocity predicted by the modified couple stress model exhibits the same behaviour as that predicted by the present model. This variable behaviour may be called as a minimum ‘‘extreme point’’ phenomenon, which is the result of the existence of the material length scale parameters in the micro scale Timoshenko models. Therefore, both the present model and the modified couple stress model have minimum extreme points for different outside diameters while the classical model exhibits a monotonically increasing behaviour with Poisson ratio increasing as shown in Fig. 3b, indicating having no extreme point.

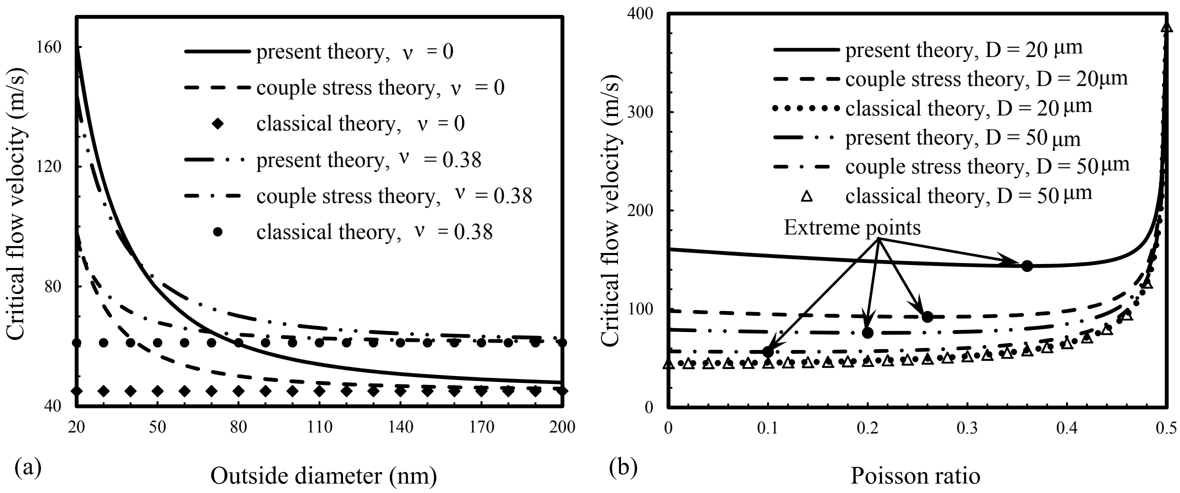

Figure 3. The critical flow velocity based on three different models varying with the, (a) outside diameter for ν = 0.0 and 0.38, (b) Poisson ratio for D = 20 and 50 μm.

4. Conclusion

In this paper, a microstructure-dependent Timoshenko beam model was developed to study the vibration and structural instability of micropipes conveying fluid using strain gradient theory. The governing equations were derived based on Hamilton principle. The present model involved three material length scale parameters to predict the size effect. After discretization via differential quadrature method, the numerical results were obtained for a simply-supported micropipe.

Results showed that the size effect was significant when the outside diameter of the micropipe became as small as the material length scale parameters, while diminished or even vanished for large outside diameters. Furthermore, it was observed that both natural frequencies and critical flow velocities predicted by the present model were not only larger than those predicted by the classical model but also larger than those predicted by the modified couple stress model. The reason is the inclusion of the additional dilatation gradient tensor and the deviatoric stretch gradient tensor in the former model. The Poisson effect on the critical flow velocity was also studied. It was found that the results obtained by the two non-classical models had the minimum extreme points with Poisson ratio varying but those obtained by the classical model had no extreme point. In the other word, the critical flow velocities predicted by the non-classical models decreased firstly and then increased but those predicted by the classical model increased monotonically with Poisson ratio increasing.

Finally, we hope that the present article will lead to the improvement in the design of micro scale sensors, actuators and fluidic devices.

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