Indian J. Pure Appl. Math., **43**(4): 309-321, August 2012

EXISTENCE, UNIQUENESS AND STABILITY OF SOLUTIONS FOR A CLASS OF NONLINEAR INTEGRAL EQUATIONS UNDER GENERALIZED LIPSCHITZ CONDITION

Omid Baghani, Mortaza Gachpazan* and Hamid Baghani**

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Iran
**Center of Excellence in Nonlinear Analysis and Applications, Semnan University, Iran*
e-mails: omid.baghani@gmail.com, gachpazan@ferdowsi.um.ac.ir, h.baghani@gmail.com

(*Received* 11 *April* 2011; *after final revision* 25 *February* 2012; *accepted* 5 *May* 2012)

In this paper, we prove the existence, uniqueness and the stability of solutions for some nonlinear functional-integral equations by using generalized Lipschitz condition. We prove a fixed point theorem to obtain the mentioned aims in Banach space $X := C([a, b], R)$. As application we study some Volterra integral equations with linear, nonlinear and singular kernel.

Key words : Nonlinear functional-integral equation; Hyers-Ulam stability; iterative method; fixed point theorem.

1. Introduction

In this work, we try to prove the existence, uniqueness and Hyers-Ulam stability (for simplicity, HUs) of the solutions of the following nonhomogeneous nonlinear Volterra integral equation

$$u(x) = f(x) + \varphi\Big(\int_a^x F(x,t,u(t))dt\Big) \equiv Tu, \tag{1}$$

where $x,t \in I = [a,b],\ -\infty < a < b < \infty$ and φ is a bounded linear mapping on X. This functional-integral equation produces many integral equations which have arisen in different science fields such as theory of optimal control, economics and etc., [3], [12], [14]. Investigation on existence theorems for diverse nonlinear functional-integral equations has been presented in other references such as [1], [2], [15].

In this study, we will use the iterative method to prove that equation (1) has the mentioned cases under some appropriate conditions. On the other hand, in this paper, we prove the HUs theorem of (1) under generalized Lipschitz condition on F.

We say a functional equation is stable if for every approximate solution there exists an exact solution near it. In 1940, answering a problem of Ulam [18] affirmatively, Hyers [10] proved the following result (which is nowadays called the HUs stability theorem):

Let $S = (S,+)$ be an Abelian semigroup and assume that a function $f : S \to R$ satisfies the inequality

$$|f(x+y) - f(x) - f(y)| \le \epsilon, \qquad (x,y \in S),$$

for some nonnegative ϵ. Then there exists an additive function $h : S \to R$ such that

$$|h(x) - f(x)| \le \epsilon, \qquad (x \in S),$$

holds.

Ever since, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [13], [16], [17]). Examples of some recent developments, discussions and critiques of that idea of stability can be found, e.g., in [5], [6], [8] and [9].

In Section 2 we introduce some preliminaries and use them to obtain our aims in Section 3 and 4. Finally in section 5 we offer some examples that verify the application of this kind of nonlinear functional-integral equations.

2. BASIC CONCEPTS

In this section, we recall basic result which we will need in this paper.

Consider the nonhomogeneous nonlinear Volterra integral equation (1). We assume that $f : [a, b] \to R$ is continuous and F is a mapping on the domain $D = \{(x, t, u) : x, t \in [a, b]\ , u \in X\}$. Through this article, we consider the complete metric space (X, d), which $d(f, g) = \max_{x\in[a,b]} |f(x) - g(x)|$, for all $f, g \in X$ and as we mentioned, we assume that φ is a bounded linear mapping on X.

Note that, the linear mapping $\varphi : X \to X$ is called bounded, if there exists $M > 0$ such that $\|\varphi x\| \leq M\|x\|$; for all $x \in X$. In this case, we define $\|\varphi\| = \sup\{\frac{\|\varphi x\|}{\|x\|};\ x \neq 0,\ x \in X\}$. Thus φ is bounded if and only if $\|\varphi\| < \infty$, [7].

Note : As φ is a bounded linear mapping on X, then $\varphi(x) = \lambda x$ where λ does not depend on $x \in X$.

Definition 2.1 — (cf. [11], [19]). We say that equation (1) has the HUs if there exists a constant $K \geq 0$ with the following property: for every $\epsilon > 0$, $y \in X$, if

$$|y(x) - f(x) - \varphi\Big(\int_a^x F(x, t, y(t))dt\Big)| \leq \epsilon,$$

then there exists some $u \in X$ satisfying $u(x) = f(x)+\varphi(\int_a^x F(x, t, u(t))dt)$ such that

$$|u(x) - y(x)| \leq K\epsilon.$$

We call such K a HUs constant for equation (1).

Definition 2.2 — Let δ denote the class of those functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0.$$

Definition 2.3 — Let $\mathcal{B}$ denote the class of those functions $\phi : [0, \infty) \to [0, \infty)$ which satisfies the following condition:

(i) ϕ is increasing,

(ii) for each $x > 0$, $\phi(x) < x$,

(iii) $\beta(x) = \frac{\phi(x)}{x} \in \delta, \ x \neq 0.$

For example, $\phi(t) = \mu t$, where $0 \leq \mu < 1$, $\phi(t) = \frac{t}{t+1}$ and $\phi(t) = \ln(1 + t)$ are in $\mathcal{B}$.

3. Existence and Uniqueness of the Solution of Nonlinear Integral Equations

In this section, we will study the existence and uniqueness of the nonlinear functional-integral equation (1) on X.

***Theorem* 3.1** — *Consider the integral equation (1) such that,*

(i) $F : D \to R$ *and* $f : [a, b] \to R$ *are continuous.*

(ii) $\varphi : X \to X$ *is a bounded linear transformation.*

(iii) *There exists an integrable function* $p : [a, b] \times [a, b] \to R$ *and* $\phi \in \mathcal{B}$ *such that*

$$|F(x, t, u) - F(x, t, v)| \leq p(x, t)\phi(|u - v|), \tag{2}$$

for each $x, t \in [a, b]$ *and* $u, v \in X$.

(iv) $\sup_{x \in [a,b]} \int_a^b p^2(x, t)dt \leq \frac{1}{\|\varphi\|^2(b - a)}.$

Then the integral equation (1) has an unique fixed point u in X.

Note : We define (2) as a generalized Lipschitz condition.

PROOF : Consider the iterative scheme

$$u_{n+1}(x) = f(x) + \varphi\Big(\int_a^x F(x,t,u_n(t))dt\Big) \equiv Tu_n, \qquad n = 0, 1, \dots \ , \tag{3}$$

where $u_0 \in X$ is an arbitrary initial guess. So,

$$\begin{aligned} |Tu_n - Tu_{n-1}| &= |\varphi\Big(\int_a^x F(x,t,u_n(t))dt\Big) - \varphi\Big(\int_a^x F(x,t,u_{n-1}(t))dt\Big)| \\ &\le |\varphi\Big(\int_a^x F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))dt\Big)| \\ &\le \|\varphi\| \ |\int_a^x F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))dt| \\ &\le \|\varphi\| \int_a^x |F(x,t,u_n(t)) - F(x,t,u_{n-1}(t)|dt \\ &\le \|\varphi\| \int_a^b p(x,t)\phi(|u_n(t) - u_{n-1}(t)|)dt \\ &\le \|\varphi\| \Big(\int_a^b p^2(x,t)dt\Big)^{\frac{1}{2}} \Big(\int_a^b \phi^2(|u_n(t) - u_{n-1}(t)|)dt\Big)^{\frac{1}{2}}. \end{aligned} \tag{4}$$

As the function ϕ is increasing then

$$\phi(|u_n(t) - u_{n-1}(t)|) \le \phi(d(u_n, u_{n-1})),$$

so, we obtain

$$\begin{aligned} d^2(u_{n+1}, u_n) &\le \|\varphi\|^2 \Big(\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt\Big)\Big(\int_a^b \phi^2(d(u_n, u_{n-1}))dt\Big) \\ &\le \phi^2(d(u_n, u_{n-1})). \end{aligned}$$

Therefore

$$\begin{aligned} d(u_{n+1},u_n) &\leq \phi(d(u_n,u_{n-1})) = \frac{\phi(d(u_n,u_{n-1}))}{d(u_n,u_{n-1})} d(u_n,u_{n-1}) \\ &= \beta(d(u_n,u_{n-1}))d(u_n,u_{n-1}), \end{aligned} \tag{5}$$

and so the sequence $\{d(u_{n+1},u_n)\}$ is decreasing and bounded. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(u_{n+1},u_n) = r$. Assume $r > 0$. Then from (5) we have

$$\frac{d(u_{n+1},u_n)}{d(u_n,u_{n-1})} \leq \beta(d(u_n,u_{n-1})), \quad n = 1,2,\dots \ .$$

The above inequality yields $\lim_{n\to\infty} \beta(d(u_{n+1},u_n)) = 1$. Then $\beta \notin \delta$ and this is contradiction. So $r = 0$ and then $\lim_{n\to\infty} d(u_{n+1},u_n) = 0$. Now we show that $\{u_n\}$ is a Cauchy sequence. On the contrary, assume that

$$\limsup_{n,m\to\infty} d(u_n,u_m) > 0. \tag{6}$$

By the triangle inequality and relation (5) we have

$$\begin{aligned} d(u_n,u_m) &\leq d(u_n,u_{n+1}) + d(u_{n+1},u_{m+1}) + d(u_{m+1},u_m) \\ &\leq d(u_n,u_{n+1}) + (\beta(d(u_n,u_m))d(u_n,u_m)) + d(u_{m+1},u_m), \end{aligned}$$

hence

$$d(u_n,u_m) - (\beta(d(u_n,u_m))d(u_n,u_m)) \quad \leq d(u_n,u_{n+1}) + d(u_{m+1},u_m).$$

Therefore we obtain

$$d(u_n,u_m) \leq (1-\beta(d(u_n,u_m)))^{-1}[d(u_n,u_{n+1}) + d(u_{m+1},u_m)].$$

Since $\limsup_{n,m\to\infty} d(u_n,u_m) > 0$ and $\lim_{n\to\infty} d(u_{n+1},u_n) = 0$ then

$$\limsup_{n,m\to\infty}((1-\beta(d(u_n,u_m)))^{-1}) = +\infty,$$

from which we obtain $\limsup_{n,m\to\infty} \beta(d(u_n,u_m)) = 1$. But since $\beta \in \delta$, we get $\limsup_{n,m\to\infty} d(u_n,u_m) = 0$. This contradicts (6) and shows $\{u_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, then there exists $u \in X$ such that $\lim_{n\to\infty} u_n = u$. Now by taking the limit of both sides of (3), we have

$$\begin{aligned} u = \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} (f(x) + \varphi(\int_a^x F(x, t, u_n(t))dt)) \\ &= f(x) + \varphi(\int_a^x F(x, t, \lim_{n\to\infty} u_n(t))dt) \\ &= f(x) + \varphi(\int_a^x F(x, t, u(t))dt). \end{aligned}$$

So, there exists an unique solution $u \in X$ such that $Tu = u$.

Note : Theorem 3.1 was proved with the condition (ii), but there exist some nonlinear examples φ, such that by the analogue method mentioned in this theorem, the existence and uniqueness can be proved for those. For example $\varphi(x) = \sin(x)$.

4. Stability of Nonlinear Integral Equations

In this section, we prove that the nonlinear integral equation in (1) has the HUs.

***Theorem* 4.1** — *The equation $Tx = x$, where T is defined by (1), under the assumption of Theorem 3.1, has the Hyers-Ulam stability that is for every $\xi \in X$ and $\epsilon > 0$ with*

$$d(T\xi, \xi) \leq \epsilon,$$

there exists an unique solution $u \in X$ such that

$$Tu = u,$$

and

$$d(\xi, u) \leq K\epsilon,$$

for some $K \geq 0$.

PROOF : By relation (4) we can write

$$\begin{aligned} |Tu_n - Tu_{n-1}| &\le \|\varphi\| \int_a^x |F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))|dt \\ &\le \|\varphi\| \int_a^x p(x,t)\phi(|u_n(t) - u_{n-1}(t)|)dt \\ &\le \|\varphi\| \Big(\int_a^b p^2(x,t)dt \Big)^{\frac{1}{2}} \Big(\int_a^x \phi^2(|u_n(t) - u_{n-1}(t)|)dt \Big)^{\frac{1}{2}} \\ &\le \|\varphi\| \Big(\frac{1}{\|\varphi\|^2(b-a)} \Big)^{\frac{1}{2}} \Big(\int_a^x |u_n(t) - u_{n-1}(t)|^2 dt \Big)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &\le \Big(\frac{1}{(b-a)} \int_a^x |u_n(t_1) - u_{n-1}(t_1)|^2 \, dt_1 \Big)^{\frac{1}{2}} \\ &\le \Big(\frac{1}{(b-a)^2} \int_a^x \int_a^{t_1} |u_{n-1}(t_2) - u_{n-2}(t_2)|^2 \, dt_2 \, dt_1 \Big)^{\frac{1}{2}} \\ &\le \Big(\frac{1}{(b-a)^3} \int_a^x \int_a^{t_1} \int_a^{t_2} |u_{n-2}(t_3) - u_{n-3}(t_3)|^2 \, dt_3 \, dt_2 \, dt_1 \Big)^{\frac{1}{2}} \\ &\vdots \\ &\le \Big(\frac{1}{(b-a)^n} \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} |u_1(t_n) - u_0(t_n)|^2 \, dt_n \cdots dt_2 \, dt_1 \Big)^{\frac{1}{2}} \\ &\le \Big(\frac{1}{(b-a)^n} \, d^2(Tu_0, u_0) \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} \, dt_n \cdots dt_2 \, dt_1 \Big)^{\frac{1}{2}} \\ &\le \Big(\frac{1}{(b-a)^n} \, \frac{(x-a)^n}{(n)!} \, d^2(Tu_0, u_0) \Big)^{\frac{1}{2}} \\ &\le \frac{d(Tu_0, u_0)}{(n!)^{\frac{1}{2}}}. \end{aligned}$$

Let $\xi \in X$, $\epsilon > 0$ and $d(T\xi, \xi) \le \epsilon$. In the previous section, we proved that

$$u(t) \equiv \lim_{n\to\infty} T^n \xi(t),$$

is an exact solution of the equation $Tx = x$. Clearly there is N with $d(T^N\xi, u) \leq \epsilon$, because $T^n\xi$ is uniformly convergent to u as $n \to \infty$. Without loss of generality, for sufficiently large odd number N, we have

$$\begin{aligned}
d(\xi, u) &\leq d(\xi, T^N\xi) + d(T^N\xi, u) \\
&\leq d(\xi, T\xi) + d(T\xi, T^2\xi) + d(T^2\xi, T^3\xi) + ... + d(T^{N-1}\xi, T^N\xi) \\
&\quad +d(T^N\xi, u) \\
&\leq d(\xi, T\xi) + \frac{d(\xi, T\xi)}{(1!)^{\frac{1}{2}}} + \frac{d(\xi, T\xi)}{(2!)^{\frac{1}{2}}} + \cdots + \frac{d(\xi, T\xi)}{((N-1)!)^{\frac{1}{2}}} + d(T^N\xi, u) \\
&\leq d(\xi, T\xi)(1 + \frac{1}{(1!)^{\frac{1}{2}}} + \frac{1}{(2!)^{\frac{1}{2}}} + \cdots + \frac{1}{((N-1)!)^{\frac{1}{2}}}) + \epsilon \\
&\leq \epsilon\Big(\{1 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}\} + \{(\frac{1}{2^4})^{\frac{1}{2}} + (\frac{1}{2^5})^{\frac{1}{2}} + \cdots + (\frac{1}{2^{N-1}})^{\frac{1}{2}}\}\Big) + \epsilon \\
&\leq \epsilon\Big(\{1 + 1 + 1 + \frac{1}{2}\} + \{\frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{\frac{N-1}{2}}}\} \\
&\quad +\{\frac{1}{2^{2.5}} + \frac{1}{2^{3.5}} + \cdots + \frac{1}{2^{\frac{N-2}{2}}}\}\Big) + \epsilon \\
&\leq \epsilon\Big(\frac{7}{2} + \{\frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{\frac{N-1}{2}}}\} + \{\frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{\frac{N-1}{2}}}\}\Big) + \epsilon \\
&\leq \epsilon\Big(\frac{7}{2} + \{1 - \frac{1}{2^{\frac{N-3}{2}}}\}\Big) + \epsilon = (\frac{11}{2} - \frac{1}{2^{\frac{N-3}{2}}})\epsilon = K\epsilon,
\end{aligned}$$

which completes the proof.

5. Illustrative Examples

In this section we present some examples of classical integral and functional equations which are particular cases of equation (1) and consequently, the existence, uniqueness and stability of their solutions can be established using Theorems 3.1 and 4.1.

Example 5.1 : Consider the following linear Volterra integral equation

$$u(x) = x^5 - \frac{x^8}{7} + \int_0^x (t - x)u(t)dt, \qquad x, t \in [0, 1]. \tag{7}$$

We have

$$\begin{aligned}|F(x,t,u)-F(x,t,v)| &= |(t-x)u-(t-x)v| = |(t-x)(u-v)| \\ &\leq |t-x||u-v| = \Big(\frac{|t-x|}{\mu}\Big)\Big(\mu|u-v|\Big),\end{aligned}$$

where $\frac{1}{2\sqrt{3}} \leq \mu < 1$. Now we put $p(x,t) = \frac{t-x}{\mu}$ and $\phi(t) = \mu t$. Because $\sup_{x\in[0,1]} \int_0^1 p^2(x,t)dt = \frac{1}{12\mu^2} \leq 1$, then by applying the result obtained in Theorems 3.1 and 4.1, we deduce that the equation (7) has a stable unique solution in Banach space C[0,1].

Example 5.2 : Consider the following nonlinear Volterra integral equation

$$u(x) = \frac{1}{10}\sin(\frac{1}{1+x}) + \frac{x}{9}\int_0^x \frac{\arctan(x^2t)}{(1+xt)^2}\sin(u(t))dt, \quad x,t\in[0,1]. \tag{8}$$

We write

$$\begin{aligned}|F(x,t,u)-F(x,t,v)| &= |\frac{x\arctan(x^2t)}{9(1+xt)^2}(\sin(u)-\sin(v))| \\ &\leq |\frac{x\arctan(x^2t)}{(1+xt)^2}||\frac{u-v}{9}|.\end{aligned}$$

Take $p(x,t) = \frac{x\arctan(x^2t)}{(1+xt)^2}$ and $\phi(t) = \frac{t}{9}$. Since $\sup_{x\in[0,1]} \int_0^1 p^2(x,t)dt \leq 1$, then the equation (8) has a stable unique solution in C[0,1].

Example 5.3 : Consider the following singular Volterra integral equation

$$u(x) = f(x) + \lambda\int_0^x (x-t)^{-\alpha}u(t)dt, \qquad x,t\in[0,T], \tag{9}$$

where $|\lambda| < 1$ and $0 < \alpha < \frac{1}{2}$. Then

$$|F(x,t,u)-F(x,t,v)| = |\lambda(u-v)(x-t)^{-\alpha}| \leq |\lambda||u-v||(x-t)|^{-\alpha}.$$

Put $p(x,t) = (x-t)^{-\alpha}$ and $\phi(t) = \lambda t$. On the other hand

$$\sup_{x\in[0,T]}\int_0^T p^2(x,t)dt = \sup_{x\in[0,T]}\int_0^T |(x-t)|^{-2\alpha}dt = \frac{T^{1-2\alpha}}{1-2\alpha}.$$

Therefore if $T^{1-\alpha} \leq (1-2\alpha)^{\frac{1}{2}}$, then the equation (9) has a stable unique solution in complete metric space $C[0,T]$. Note that we can't obtain these results by using the usual Lipschitz condition.

Remark 5.4 : The unique solution $u \in C[0,T]$ of the Volterra integral equation (9) is given by

$$u(x) = E_{1-\alpha}(\lambda\Gamma(1-\alpha)x^{1-\alpha})y_0, \qquad x \in [0,T],$$

where $f(x) = y_0$ and

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\beta)}, \qquad \beta > 0,$$

denotes the Mittag-Leffler function [4].

CONCLUSIONS

Let $I = [a,b]$ be a finite interval, $X := C([a,b],R)$ and $y = Ty$ be a functional equation in which $T : X \to X$ is a nonlinear integral map in the form (1). In this paper we showed that T has an unique stable solution in complete metric space X under the generalized Lipschitz condition. The stability by means that, if y° be an approximate solution of the integral equation and $d(y^\circ, Ty^\circ) \leq \varepsilon$ for all $t \in I$, $\varepsilon \geq 0$, then $d(y^*, y^\circ) \leq K\varepsilon$, which y^* is the exact solution of (1) and K is positive constant.

ACKNOWLEDGMENT

The authors thank the referees for their appreciation, valuable comments, and suggestions.

REFERENCES

1. R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential. Difference and Integral Equations*, Dordrecht, Kluwer Academic, (1999).

2. R. P. Agarwal and D. O'Regan, Existence of Solutions to Singular Integral Equations. *Comput. Math. Appl*., **37** (1999), 25-29.

3. J. Banas and B. Rzepka, On existence and asymptotic stability of solutions of a nonlinear integral equation. *J. Math. Anal. Appl*., **284** (2003), 165-173.

4. H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, (2004).

5. J. Brzdek, On a method of proving the Hyers–Ulam stability of functional equations on restricted domains. *Aust. J. Math. Anal. Appl*., **6** (2009), 1-10.

6. L. P. Castro and A. Ramos, Stationary Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations, *Banach J. Math. Anal*., **3** (2009), 36-43.

7. G. B. Folland, *Real Analysis Modern Techniques and Their Application*, University of Washington, 1984.

8. M. Gachpazan and O. Baghani, Hyers-Ulam stability of Volterra integral equation. *Int. J. Nonlinear Anal. Appl*., **1** (2010), 19-25.

9. M. Gachpazan and O. Baghani, *Hyers-Ulam stability of nonlinear integral equation*. Fixed Point Theory and Applications, ID 927640 (2010).

10. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. USA*, **27** (1941), 222-224.

11. S. M. Jung, *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory and Applications, ID 57064 (2007).

12. K. Maleknejad, K. Nouri and R. Mollapourasl, Existence of solutions for some nonlinear integral equations. *Commun. Nonlinear Sci. Numer. Simulat*., **14** (2009), 2559-2564.

13. Z. Moszner, On the stability of functional equations, *Aequationes Math*., **77** (2009), 33-88.

14. M. Meehan and D. O'Regan, Existence theory for nonlinear Volterra integro-differential and integral equations. *Nonlinear Anal. Theor*., **31** (1998), 317-341.

15. D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integro differential Equations*, Dordrecht, Kluwer Academic, (1998).

16. B. Paneah, A new approach to the stability of linear functional operators, *Aequationes Math*., **78** (2009), 45-61.

17. W. Prager and J. Schwaiger, Stability of the multi-Jensen equation. *Bull. Korean Math. Soc*., **45** (2008), 133-142.

18. S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI. Wiley, New York, (1960).

19. G. Wang, M. Zhou and L. Sun, Hyers–Ulam stability of linear differential equation of first order. *Appl. Math. Lett*., **21** (2008), 1024-1028.