$(\sigma, \gamma)-$ GENERALIZED DYNAMICS ON MODULES

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Abstract. In this paper we introduce the notions of generalized $d-$ derivations, $(\sigma, \gamma)-$ generalized module maps and $(\sigma, \gamma)-$ generalized dynamics on Modules. We show that if $\varphi : R \to B(M)$ is a $\sigma-$ dynamics on Banach algebra A with the infinitesimal generator d and $T : R \to B(M)$ is a $(\sigma, \gamma)-$ generalized dynamics on Banach $A-$ bimodule M with the infinitesimal generator δ, then δ is a generalized $d-$ derivation. Using the computation formula for $\delta^n(ax)$ and the concept of $\gamma-$ one parameter groups on M, we prove the converse under some more restrictions on δ.

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1. Introduction

Let A be Banach algebra, M be an $A-$ bimodule. We recall that a linear map $\delta : M \to M$ is said to be a generalized derivation if there exists a derivation $d : A \to A$ such that $\delta(ax) = a\delta(x) + d(a)x$ $(a \in A, x \in M)$. Also $\delta : M \to M$ is called generalized inner derivation if there exist $c, b \in A$ such that $\delta(x) = cx - xb$, [2].

In [1], the generalized derivations are appeared as the infinitesimal generators of dynamical systems on Hilbert C^*- modules. Such dynamical systems are in fact strongly continuous one parameter groups of a class of generalized module maps which is called unitaries, [see 4].

Let $\sigma : A \to A$ be a linear operator. following [9], we call a linear map $d : A \to A$ a $\sigma-$ derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ and a linear mapping $\varphi : A \to A$ is called a $\sigma-$ endomorphism if $(\varphi + \sigma - I)(ab) - (\varphi + \sigma - I)(a)(\varphi + \sigma - I)(b) = \sigma(ab) - \sigma(a)\sigma(b)$, for all $a, b \in A$.

It has been shown in [9], that the generator d of a $\sigma-$ dynamics (i.e. a uniformly continuous one parameter group of $\sigma-$ endomorphisms) $\{\varphi_t\}_{t\in R}$ is an every where defined $\sigma-$ derivation.

The above concepts (generalized module maps and $\sigma-$ endomorphisms) motivate us to define the $(\sigma, \gamma)-$ generalized module maps as follows:

Let $\gamma : M \to M$ be a linear operator. A linear mapping $T : M \to M$ is said to be $(\sigma, \gamma)-$ *generalized module map* if there exists a linear mapping $\varphi : A \to A$ such that $(T + \gamma - I)(ax) - (\varphi + \sigma - I)(a)(T + \gamma - I)(x) = \gamma(ax) - \sigma(a)\gamma(x)$ (for all $a \in A$ and $x \in M$).

In this paper we introduce (inner) generalized $d-$ derivations (as a generalization of $d-$ derivations), (quasi inner) $(\sigma, \gamma)-$ generalized module maps and $(\sigma, \gamma)-$ generalized dynamics on M. We prove that if $\varphi : R \to B(A)$ is a $\sigma-$ dynamics with the generator d and $T : R \to B(M)$ is a $(\sigma, \gamma)-$ generalized dynamics with the generator δ, then δ is a generalized $d-$ derivation. This is a generalization of a similar result of M. Mirzavaziri and M.S. Moslehian, [9].

Using the computation formula for $\delta^n(ax)$ and the concept of $\gamma-$ one parameter groups on M [8], we deal with the converse under some conditions.
The reader is referred to [5] and [7] for more details on Banach algebras and to [6] and [11] for more information on dynamical systems.

2. Preliminaries

Throughout the paper A and B are Banach algebras, M and N are A and $B-$ bimodule, respectively.

Definition 2.1 Let $\sigma : A \to B$ be a linear mapping. A linear mapping $\delta : M \to N$ is said to be a *generalized* $d-$ *derivation* if there exists a $\sigma-$ derivation $d : A \to B$ and a linear map $\gamma : M \to N$ such that

$$\delta(ax) = \sigma(a)\delta(x) + d(a)\gamma(x) \ (x \in M, a \in A)$$

We call δ a $(\sigma, \gamma, d)-$ *derivation*.

In the case that $B = A$, $N = M$, and σ, γ are identity operators on A and M respectively, then δ is in fact a $d-$ derivation in the usual sense.

Example 2.2 Let M and N be Hilbert C^*- modules over C^*- algebras A and B of compact operators acting on Hilbert spaces H and H', respectively . By Theorem 4 of [3], M and N have orthonormal basis and so that each element x of M and y of N can be expressed as $x = \sum_\lambda < x, v_\lambda > v_\lambda$ and $y = \sum_\lambda < y, w_\lambda > w_\lambda$.
Suppose that $\sigma : A \to B$ is a linear mapping and $d : A \to B$ is a $\sigma-$ derivation. Then the mapping $\delta : M \to N$ defined by

$$\delta(x) = \sum_\lambda d(< x, v_\lambda >)w_\lambda$$

is a $(\sigma, \gamma, d)-$ derivation, where $\gamma(x) = \sum_\lambda \sigma(< x, v_\lambda >)w_\lambda$. Since trivially γ is linear and

$$\begin{aligned}
\delta(ax) &= \delta(a \sum_\lambda < x, v_\lambda > v_\lambda) \\
&= \delta(\sum_\lambda < ax, v_\lambda > v_\lambda) \\
&= \sum_\lambda d(< ax, v_\lambda >)w_\lambda \\
&= \sum_\lambda d(a < x, v_\lambda >)w_\lambda \\
&= \sum_\lambda [\sigma(a)d(< x, v_\lambda >) + d(a)\sigma(< x, v_\lambda >)]w_\lambda \\
&= \sigma(a) \sum_\lambda d(< x, v_\lambda >)w_\lambda + d(a) \sum_\lambda \sigma(< x, v_\lambda >)w_\lambda \\
&= \sigma(a)\delta(x) + d(a)\gamma(x). \square
\end{aligned}$$

Example 2.3 Let $\sigma : A \to B$ and $\gamma : M \to N$ be arbitrary linear mappings and suppose that u, v are two elements of B satisfying

(i) $u\,(\gamma(ax) - \sigma(a)\gamma(x)) = (\gamma(ax) - \sigma(a)\gamma(x))\,v$, and

(ii) $u\,(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))\,u$.

Then the mapping δ defined by $\delta(x) = u\gamma(x) - \gamma(x)v$ is a generalized $d-$ derivation.

In fact δ is a $(\sigma, \gamma, d_u^\sigma)-$ derivation where $d_u^\sigma(a) = u\sigma(a) - \sigma(a)u$. The above generalized $d_u^\sigma-$ derivation is called *inner generalized* $d_u^\sigma-$ *derivation.*

Note that if σ is an endomorphism and γ is a $\sigma-$ module map (i.e. $\gamma(ax) = \sigma(a)\gamma(x)$), then u and v can be any arbitrary elements of B.

Definition 2.4 A linear mapping $T : M \to M$ is said to be a $(\sigma, \gamma)-$ *generalized module map* if there exists a linear mapping $\varphi : A \to A$ such that

$$(T + \gamma - I)(ax) - (\varphi + \sigma - I)(a)(T + \gamma - I)(x) = \gamma(ax) - \sigma(a)\gamma(x)$$
$$\text{(for all } a \in A \text{ and } x \in M)$$

This map is called $(\sigma, \gamma, \varphi)-$ *module map.* Note that if σ and γ are identity of A and M respectively, then a $(\sigma, \gamma, \varphi)-$ module map T is a $\varphi-$ module map. Also if γ is a $\sigma-$ module map, then $T + \gamma - I$ is a $(\varphi + \sigma - I)-$ module map.

Further if $M = A$ and $\gamma = \sigma$, then T is said to be a *generalized* $\sigma-$ *endomorphism* or more precisely $(\sigma, \varphi)-$ *endomorphism.* Also if additionally $\varphi = T$, then T is called $\sigma-$ endomorphism, [9].

lemma 2.5 Let $T : M \to M$ is a linear mapping. The following are equivalent:

(i) T is a $(\sigma, \gamma, \varphi)-$ generalized module map;

(ii) $T(ax) - \varphi(a)T(x) = (\varphi(a) - a)(\gamma(x) - x) + (\sigma(a) - a)(T(x) - x)$;

$$\begin{aligned}(iii)(T + \gamma - I)(ax) - \gamma(ax) &= T(ax) - ax \\ &= \sigma(a)(T(x) - x) + (\varphi(a) - a)\gamma(x) + (\varphi(a) - a)(T(x) - x).\end{aligned}$$

Proof. Straightforward.□

Example 2.6 Let M and N be A and $B-$ bimodules, respectively. Consider $A \times B$ with the following structure:

(i) $(a, b) + (c, d) = (a + c, b + d)$

(ii) $\lambda(a, b) = (\lambda a, \lambda b)$

(iii) $(a, b).(c, d) = (ac, bd)$

(iv) $\| (a, b) \| = \| a \| + \| b \|$

Then $A \times B$ is itself a Banach algebra and $M \times N$ is a $A \times B-$ bimodule by regarding the following module structure:

$(a,b).(x,y) = (ax,by)$ and $(x,y).(a,b) = (xa,yb)$

Now define the maps T, γ, σ and φ as follows:

$T(x,y) = (0,3y)$

$\gamma(x,y) = (0,y)$

$\varphi(a,b) = (2a,b)$

$\sigma(a,b) = (0,b)$

then T is a $(\sigma,\gamma,\varphi)-$ module map.

Definition 2.7 A linear mapping $T : M \to M$ is called *quasi inner* $(\sigma,\gamma)-$ *generalized module map* if there exist elements $b,c \in A$ such that for each $a \in A, x \in M$

(i)$(T+\gamma-I)(x) = e^c\gamma(x)e^{-b}$

(ii)$c(\gamma(ax)-\sigma(a)\gamma(x)) = (\gamma(ax)-\sigma(a)\gamma(x))b$

If $c = b$, then T is said to be an *inner* $(\sigma,\gamma)-$ *generalized module map.* Also if additionally $M = A$ and $\gamma = \sigma$, then T is said to be an inner $\sigma-$ endomorphism, [9].

In the case that A is a C^*- algebra, we call $T : M \to M$ a quasi inner $(\sigma,\gamma)-$ generalized module map, if there exist unitary elements $u,v \in A$ such that for each $a \in A$ and $x \in M$

(i)$(T+\gamma-I)(x) = u\gamma(x)v^*$

(ii)$u(\gamma(ax)-\sigma(a)\gamma(x)) = (\gamma(ax)-\sigma(a)\gamma(x))v$

Example 2.8 Let $\sigma : A \to A$ be an arbitrary linear operator on A, γ an arbitrary linear $\sigma-$ module map and $a,b \in M$. Then the map $T : M \to M$ given by
$T(x) = e^b\gamma(x)e^{-a} - \gamma(x) + x$ is a quasi inner $(\sigma,\gamma)-$ generalized module map.

Lemma 2.9 Let T be a quasi inner $(\sigma,\gamma)-$ generalized module map. Then there exists a linear mapping $\varphi : A \to A$ such that T is a $(\sigma,\gamma,\varphi)-$ module map. If σ is a linear endomorphism, then φ is itself an inner $\sigma-$ endomorphism.

Proof. Since T is a quasi inner $(\sigma,\gamma)-$ generalized module map, so there exist elements $c,b \in A$ such that for each $a \in A$ and $x \in M$

$$e^c(\gamma(ax)-\sigma(a)\gamma(x)) = (\gamma(ax)-\sigma(a)\gamma(x))e^{-b}$$

Now define $\varphi : A \to A$ by $\varphi(x) = e^c\sigma(a)e^{-c} - \sigma(a) + a$. Therefore

$$\begin{aligned}(T+\gamma-I)(ax)-(\varphi+\sigma-I)(a)(T+\gamma-I)(x) &= \left(e^c\gamma(ax)e^{-b}\right)-\left(e^c\sigma(a)e^{-c}\right)\left(e^c\gamma(x)e^{-b}\right)\\ &= e^c\left(\gamma(ax)-\sigma(a)\gamma(x)\right)e^{-b}\\ &= \gamma(ax)-\sigma(a)\gamma(x)\end{aligned}$$

Hence T is a $(\sigma, \gamma, \varphi)-$ module map. The second is evident.□

3. $(\sigma, \gamma)-$ Generalized Dynamics

Definition 3.1 Let $\{T_t\}_{t\in R}$ be a one parameter group of bounded linear operators on M such that for each $t \in R$, T_t is a $(\sigma, \gamma, \varphi_t)-$ module map. If moreover $\{T_t\}_{t\in R}$ is uniformly continuous, then it is called $(\sigma, \gamma, \varphi_t)-$ *dynamics* or $(\sigma, \gamma)-$ *generalized dynamics* on M.

In the case that T_t is a $\sigma-$ endomorphism, (i.e. $M = A$, $\gamma = \sigma$, and $\varphi_t = T_t$ for all $t \in R$) T_t is called a $\sigma-$ dynamics on A, [9].

Let $\{T_t\}_{t\in R}$ be a $(\sigma, \gamma, \varphi_t)-$ dynamics on M. We define the *infinitesimal generator* δ of T as a mapping $\delta : D(\delta) \subseteq M \to M$ such that

$$\delta(x) = \lim_{t\to 0} \frac{T_t(x) - x}{t}$$

where

$$D(\delta) = \{x \in M \ \textit{such that} \ \lim_{t\to 0} \frac{T_t(x) - x}{t} \textit{exists}\}$$

Now we are ready to prove the following main theorem:

Theorem 3.2 Let $\{\varphi_t\}_{t\in R}$ be a $\sigma-$ dynamics on A with the infinitesimal generator d and $\{T_t\}_{t\in R}$ be a $(\sigma, \gamma, \varphi_t)-$ dynamics on M with the infinitesimal generator δ. Then δ is a $(\sigma, \gamma, d)-$ derivation.

proof. First note that δ and d are everywhere defined operators by [10]. Now let $a \in A$ and $x \in M$. By Lemma 2.4 we have

$$\begin{aligned}
\delta(ax) &= \lim_{t\to 0} \frac{T_t(ax) - (ax)}{t} \\
&= \lim_{t\to 0} \frac{(T_t + \gamma - I)(ax) - \gamma(ax)}{t} \\
&= \lim_{t\to 0} \sigma(a)\frac{T_t(x) - x}{t} + \lim_{t\to 0} \frac{\varphi_t(a) - a}{t}\gamma(x) + \lim_{t\to 0}(\frac{\varphi_t(a) - a}{t})(T_t(x) - x) \\
&= \sigma(a)\delta(x) + d(a)\gamma(x)
\end{aligned}$$

A similar calculation shows that d is a $\sigma-$ derivation. Thus δ is a $(\sigma, \gamma, d)-$ derivation.□

We are going to establish some conditions making the converse of the above theorem be held. More precisely we like to investigate some restrictions under which a bounded $(\sigma,\gamma,d)-$ derivation induces a $(\sigma,\gamma)-$ generalized dynamics. For, we need the following preliminaries:

Lemma 3.3 Let $\{\beta_t\}_{t\in R}$ and $\{\alpha_t\}_{t\in R}$ be uniformly continuous $\sigma-$ and $\gamma-$ one parameter groups respectively, satisfying $\beta_t(ab)-\beta_t(a)\beta_t(b)=\sigma(ab)-\sigma(a)\sigma(b)$ and $\alpha_t(ax)-\beta_t(a)\alpha_t(x)=\gamma(ax)-\sigma(a)\gamma(x)$. Then $\varphi_t=\beta_t-\sigma+I$ and $T_t=\alpha_t-\gamma+I$ are $\sigma-$ dynamics on A and $(\sigma,\gamma,\varphi_t)-$ dynamics on M with the same generators of $\{\alpha_t\}_{t\in R}$ and $\{\beta_t\}_{t\in R}$, respectively.

proof. (i) Trivially $T_0=I$ and

$$\begin{aligned}
T_t(T_s(x)) &= (\alpha_t-\gamma+I)(\alpha_s(x)-\gamma(x)+x)\\
&= (\alpha_{t+s}(x)-\alpha_t\gamma(x)+\alpha_t(x))+(-\gamma\alpha_s(x)+\gamma^2(x)-\gamma(x))+(\alpha_s(x)-\gamma(x)+x)\\
&= \alpha_{t+s}(x)-\gamma(x)+x\\
&= T_{t+s}(x)
\end{aligned}$$

and $\| T_t-I \|=\| \alpha_t-\gamma \|\to 0$ (as $t\to 0$). Therefore $\{T_t\}_{t\in R}$ is a uniformly continuous one parameter group of bounded linear operators satisfying $(T_t+\gamma-I)(ax)-(\varphi_t+\sigma-I)(a)(T_t+\gamma-I)(x)=\alpha_t(ax)-\beta_t(a)\alpha_t(x)=\gamma(ax)-\sigma(a)\gamma(x)$ $(a\in A, x\in M)$.

Hence $\{T_t\}_{t\in R}$ is a $(\sigma,\gamma,\varphi_t)-$ dynamics. Finally if δ is the generator of $\{\alpha_t\}_{t\in R}$, then

$$\lim_{t\to 0}\frac{T_t(x)-x}{t}=\lim_{t\to 0}\frac{\alpha_t(x)-\gamma(x)}{t}=\delta(x)$$

Similarly $\{\varphi_t\}_{t\in R}$ is a $\sigma-$ dynamics on A with the same generator of $\{\beta_t\}_{t\in R}$.□

Theorem 3.4 Let δ be a $(\sigma,\gamma,d)-$ derivation on M. If $\sigma d=d\sigma$ and $\delta\gamma=\gamma\delta$, then

$$\delta^n(ax)=\sum_{k=0}^{n}\binom{n}{k}d^{n-k}(\sigma^k(a))\gamma^{n-k}(\delta^k(x)) \qquad (*)$$

Proof. We proof the assertion by induction on n.
For $n=1$ there is nothing to do. Assume that $(*)$ holds for n. We have

$$\begin{aligned}
\delta^{n+1}(ax) &= \delta(\delta^n(ax)) \\
&= \delta\left[\sum_{k=0}^{n}\tbinom{n}{k}d^{n-k}\left(\sigma^k(a)\right)\gamma^{n-k}(\delta^k(x))\right] \\
&= \sum_{k=0}^{n}\tbinom{n}{k}\left[\sigma\left(d^{n-k}(\sigma^k(a))\right)\delta\left(\gamma^{n-k}(\delta^k(x)\right)+d\left(d^{n-k}(\sigma^k(a))\right)\gamma\left(\gamma^{n-k}(\delta^k(x))\right)\right] \\
&= \sum_{k=0}^{n}\tbinom{n}{k}\left[d^{n-k}\left(\sigma^{k+1}(a)\right)\gamma^{n-k}\left(\delta^{k+1}(x)\right)+d^{n+1-k}\left(\sigma^k(a)\right)\gamma^{n+1-k}\left(\delta^k(x)\right)\right] \\
&= \sum_{k=0}^{n}\tbinom{n}{k}d^{n-k}\left(\sigma^{k+1}(a)\right)\gamma^{n-k}\left(\delta^{k+1}(x)\right)+\sum_{k=0}^{n}\tbinom{n}{k}d^{n+1-k}\left(\sigma^k(a)\right)\gamma^{n+1-k}\left(\delta^k(x)\right) \\
&= \sum_{k=1}^{n+1}\tbinom{n}{k-1}d^{n+1-k}\left(\sigma^k(a)\right)\gamma^{n+1-k}\left(\delta^k(x)\right)+\sum_{k=0}^{n}\tbinom{n}{k}d^{n+1-k}\left(\sigma^k(a)\right)\gamma^{n+1-k}\left(\delta^k(x)\right) \\
&= \sum_{k=1}^{n}[\tbinom{n}{k}+\tbinom{n}{k-1}]d^{n+1-k}\left(\sigma^k(a)\right)\gamma^{n+1-k}\left(\delta^k(x)\right) \\
&+ \tbinom{n}{n}d^0\left(\sigma^{n+1}(a)\right)\gamma^0\left(\delta^{n+1}(x)\right)+\tbinom{n}{0}d^{n+1}\left(\sigma^0(a)\right)\gamma^{n+1}\left(\delta^0(x)\right) \\
&= \sum_{k=1}^{n}\tbinom{n+1}{k}d^{n+1-k}\left(\sigma^k(a)\right)\gamma^{n+1-k}\left(\delta^k(x)\right) \\
&+ \tbinom{n+1}{0}d^{n+1-0}\left(\sigma^0(a)\right)\gamma^{n+1-0}\left(\delta^0(x)\right)+\tbinom{n+1}{n+1}d^{n+1-(n+1)}\left(\sigma^{n+1}(a)\right)\gamma^{n+1-(n+1)}\left(\delta^{n+1}(x)\right) \\
&= \sum_{k=0}^{n+1}\tbinom{n+1}{k}d^{n+1-k}\left(\sigma^k(a)\right)\gamma^{n+1-k}\left(\delta^k(x)\right).\square
\end{aligned}$$

Corollary 3.5 Suppose that $\sigma : A \to A$ is an idempotent linear operator and d is a $\sigma-$ derivation such that $d^0=\sigma$ and $d\sigma=\sigma d=d$. If $\gamma : M \to M$ is an idempotent linear operator and δ is a $(\sigma,\gamma,d)-$ derivation such that $\delta^0=\gamma$ and $\delta\gamma=\gamma\delta=\delta$, then

$$\delta^n(ax)=\sum_{k=0}^{n}\tbinom{n}{k}d^{n-k}(a)\delta^k(x) \qquad (a\in A, x\in M)$$

.

Theorem 3.6 Let σ be an idempotent bounded linear endomorphism and d be a bounded $\sigma-$ derivation on A such that $d^0=\sigma$, $d\sigma=\sigma d=d$. Let γ be an idempotent bounded linear $\sigma-$ module map and δ be a bounded $(\sigma,\gamma,d)-$ derivation on M such that $\delta^0=\gamma$, $\delta\gamma=\gamma\delta=\delta$. Then δ induces a uniformly continuous $\gamma-$ one parameter group $\{\alpha_t\}_{t\in R}$ of generalized module maps on M. More precisely there exists a uniformly continuous $\sigma-$ one parameter group $\{\beta_t\}_{t\in R}$ of linear endomorphisms on A such that d is its generator and $\alpha_t(ax) = \beta_t(a)\alpha_t(x)$ $(a \in A, x \in M)$.

proof. For each $t \in R$, $a \in A$ and $x \in M$, define $\beta_t(a) = e^{td}(a)$ and $\alpha_t(x) = e^{t\delta}(x)$. Clearly $\{\beta_t\}_{t\in R}$ and $\{\alpha_t\}_{t\in R}$ are $\sigma-$ and $\gamma-$ one parameter groups, respectively. Also the method has been used in the proof of Theorem 1.2.1 of [10] shows that $\{\beta_t\}_{t\in R}$ and $\{\alpha_t\}_{t\in R}$ are uniformly continuous with the generators d and δ, respectively. Therefore by Corollary 3.5 we have $\delta^n(ax) = \sum_{k=0}^{n}\binom{n}{k}d^{n-k}(a)\delta^k(x)$. Thus

$$
\begin{aligned}
\alpha_t(ax) &= e^{t\delta}(ax)\\
&= \sum_{k=0}^{\infty}\frac{t^k}{k!}\delta^k(ax)\\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{(k-r)+r}}{k!}\frac{k!}{r!(k-r)!}d^r(a)\delta^{k-r}(x)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m.t^n}{n!m!}d^n(a)\delta^m(x)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}d^n(a).\sum_{m=0}^{\infty}\frac{t^m}{m!}\delta^m(x)\\
&= \beta_t(a).\alpha_t(x)
\end{aligned}
$$

It is enough to show that $\beta_t(ab) = \beta_t(a)\beta_t(b)$. For this aim note that if $M := A$, then the $\sigma-$ derivation d is a $(\sigma,\sigma,d)-$ derivation in the sense of generalized $d-$ derivations. Hence letting $M = A$ and $\delta = d$ in the above, we have $d^n(ab) = \sum_{k=0}^{n}\binom{n}{k}d^{n-k}(a)d^k(b)$ and therefore $\beta_t(ab) = \beta_t(a)\beta_t(b)$.□

The following result as an immediate consequence of Theorem 3.6 gives us the conditions under which a bounded $(\sigma,\gamma,d)-$ derivation induces a $(\sigma,\gamma)-$ generalized dynamics as we required.

Corollary 3.7 Let σ be an idempotent bounded linear endomorphism and d be a bounded $\sigma-$ derivation on A such that $d^0 = \sigma$, $d\sigma = \sigma d = d$. Let γ be an idempotent bounded linear $\sigma-$ module map and δ be a bounded $(\sigma,\gamma,d)-$ derivation on M such that $\delta^0 = \gamma$, $\delta\gamma = \gamma\delta = \delta$. Then δ induces a $(\sigma,\gamma)-$ generalized dynamics $\{T_t\}_{t\in R}$ on M. More precisely there exists a $\sigma-$ dynamics $\{\varphi_t\}_{t\in R}$ on A such that d is its generator and $\{T_t\}_{t\in R}$ is a $(\sigma,\gamma,\varphi_t)-$ dynamics with the generator δ.

proof. Take for each $t \in R$, $\varphi_t = e^{td} - \sigma + I$ and $T_t = e^{t\delta} - \gamma + I$. Theorem 3.6 implies that φ_t is a $\sigma-$ endomorphism and T_t is a $(\sigma, \gamma, \varphi_t)-$ module map. Using Lemma 3.3 it follows that $\{\varphi_t\}_{t\in R}$ is a $\sigma-$ dynamics with the generator d and $\{T_t\}_{t\in R}$ is a $(\sigma, \gamma, \varphi_t)-$ dynamics with the generator δ.□

In the next Theorem we show that if $\sigma : A \to A$ is a linear endomorphism then each one parameter group $\{T_t\}_{t\in R}$ of quasi inner $(\sigma, \gamma)-$ generalized module maps of the following form has an inner generalized $d-$ derivation as its generator.

Theorem 3.8 Let $\sigma : A \to A$ be a linear endomorphism and $T_t : M \to M$ be the one parameter group $T_t(x) = e^{tc}\gamma(x)e^{-tb} - \gamma(x) + x$ of quasi inner $(\sigma, \gamma)-$ generalized module maps with the generator δ. Then δ is an inner generalized $d_c^\sigma-$ derivation.

proof. First note that $\{T_t\}_{t\in R}$ is uniformly continuous, since taking $u_t = e^{tc}$ and $v_t = e^{tb}$ we have

$$\begin{aligned} \| T_t(x) - x \| &= \| u_t\gamma(x)v_{-t} - \gamma(x) \| \\ &= \| (u_t\gamma(x) - \gamma(x)v_t)v_{-t} \| \\ &\leq \| u_t\gamma(x) - \gamma(x)v_t \| \\ &\leq \| u_t\gamma(x) - \gamma(x) \| + \| \gamma(x) - \gamma(x)v_t \| \\ &\leq (\| u_t - I \| + \| I - v_t \|) \| \gamma \| \| x \| \end{aligned}$$

Thus

$$\| T_t - \gamma \| \leq (\| u_t - I \| + \| I - v_t \|) \| \gamma \| \to 0 \ (as\ t \to 0)$$

and applying the L'Hopital rule we have

$$\begin{aligned} \delta(x) &= \lim_{t\to 0} \frac{T_t(x) - x}{t} \\ &= \lim_{t\to 0} \frac{e^{tc}\gamma(x)e^{-tb} - \gamma(x)}{t} \\ &= \lim_{t\to 0}(ce^{tc}\gamma(x)e^{-tb} - e^{tc}\gamma(x)be^{-tb}) \\ &= c\gamma(x) - \gamma(x)b \end{aligned}$$

therefore by Example 2.3. δ is an inner generalized $d_c^\sigma-$ derivation.□

Now we show that the inner generalized $d_c^\sigma-$ derivation $\delta : M \to M$ of the form $\delta(x) = c\gamma(x) - \gamma(x)b$ under the following limitations is the generator of a uniformly continuous one parameter group of quasi inner $(\sigma, \gamma)-$ generalized module maps.

Theorem 3.9 Let $\delta : M \to M$ be the generalized $d_c^\sigma-$ derivation $\delta(x) = c\gamma(x) - \gamma(x)b$. If $\gamma : M \to M$ is an idempotent bounded linear operator such that $\gamma(cx) = c\gamma(x)$ and $\gamma(xb) = \gamma(x)b$, then there exists a uniformly continuous $\gamma-$ one parameter group $\{\alpha_t\}_{t\in R}$ on M such that $\alpha_t - \gamma + I$ is quasi inner $(\sigma,\gamma)-$ generalized module map and δ is its generator.

proof. First note that we have by induction $\gamma(c^n x) = \gamma(c^n)x$ and $\gamma(xb^n) = \gamma(x)b^n$. Also the continuity of γ implies that $\gamma(e^{sc}x) = e^{sc}\gamma(x)$ and $\gamma(xe^{sb}) = \gamma(x)e^{sb}$.

Take $\alpha_t(x) = e^{tc}\gamma(x)e^{-tb}$. Then $\alpha_0 = \gamma$ and

$$\begin{aligned}\alpha_t(\alpha_s(x)) &= e^{tc}\gamma(e^{sc}\gamma(x)e^{-sb})e^{-tb}\\ &= e^{tc}.e^{sc}\gamma(\gamma(x)e^{-sb})e^{-tb}\\ &= e^{(t+s)c}\gamma(\gamma(x))e^{-sb}.e^{-tb}\\ &= e^{(t+s)c}\gamma(x)e^{-(t+s)b}\\ &= \alpha_{t+s}(x)\end{aligned}$$

Taking $u_t = e^{itc}$ and $v_t = e^{itb}$, we have u_t and v_t are uniformly continuous one parameter groups of unitaries and $\alpha_t(x) = u_t\gamma(x)v_{-t}$. Now the method has been used in the proof of Theorem 3.8, implies that $\{\alpha_t\}_{t\in R}$ is a uniformly continuous $\gamma-$ one parameter group and δ is its generator. Finally it is easy to see that $\alpha_t - \gamma + I$ is quasi inner $(\sigma,\gamma)-$ generalized module map. In fact it is a $\Big(\sigma,\gamma,(\beta_t - \sigma + I)\Big)-$ module map, where $\beta_t(a) = e^{tc}\sigma(a)e^{tc}$.□

Lemma 3.10 Let $\delta : M \to M$ be the hyper generalized $d_c^\sigma-$ derivation $\delta(x) := c\gamma(x) - \gamma(x)b$. If $\gamma : M \to M$ is an idempotent bounded linear operator such that $\gamma(cx) = c\gamma(x)$ and $\gamma(xb) = \gamma(x)b$. Then

$$(-1)^k\delta^k(x) = \sum_{r=0}^{k}(-1)^r\tbinom{k}{r}c^r\gamma(x)b^{k-r}(*)$$

for all $x \in M$, $0 \le k \le r$ and $r \ge 1$.

proof. We use induction on k. For $k = 1$ there is nothing to do. Assume that $(*)$

holds for k. We have

$$\begin{aligned}
(-1)^{k+1}\delta^{k+1}(x) &= (-1)^{k+1}\delta(\delta^k(x)) \\
&= -c\gamma\left((-1)^k\delta^k(x)\right)+\gamma\left((-1)^k\delta^k(x)\right)b \\
&= -c\gamma\left[\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}c^r\gamma(x)b^{k-r}\right]+\gamma\left[\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}c^r\gamma(x)b^{k-r}\right]b \\
&= -\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}c^{r+1}\gamma(x)b^{k-r}+\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}c^r\gamma(x)b^{k-r+1} \\
&= -\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}c^{r+1}\gamma(x)b^{k-r}+\sum_{r=-1}^{k-1}(-1)^{r+1}\tbinom{k}{r+1}c^{r+1}\gamma(x)b^{k-r} \\
&= (-1)^{k+1}c^{k+1}\gamma(x)+\gamma(x)b^{k+1}-\sum_{r=0}^{k-1}(-1)^r\left[\tbinom{k}{r}+\tbinom{k}{r+1}\right]c^{r+1}\gamma(x)b^{k-r} \\
&= (-1)^{k+1}c^{k+1}\gamma(x)+\gamma(x)b^{k+1}-\sum_{r=0}^{k-1}(-1)^r\tbinom{k+1}{r+1}c^{r+1}\gamma(x)b^{k-r} \\
&= (-1)^{k+1}c^{k+1}\gamma(x)+\gamma(x)b^{k+1}+\sum_{r=0}^{k-1}(-1)^{r+1}\tbinom{k+1}{r+1}c^{r+1}\gamma(x)b^{k-r} \\
&= (-1)^{k+1}c^{k+1}\gamma(x)+\gamma(x)b^{k+1}+\sum_{r=1}^{k}(-1)^r\tbinom{k+1}{r}c^r\gamma(x)b^{k-r+1} \\
&= \sum_{r=0}^{k+1}(-1)^r\tbinom{k+1}{r}c^r\gamma(x)b^{k-r+1}.\square
\end{aligned}$$

Remark 3.11 It is recalled from one parameter group theory, that a bounded derivation δ on a Banach algebra A induces the uniformly continuous one parameter group $\alpha_t(a) = e^{t\delta}(a)$ of operators on A. In particular, if $\delta(a) = ha - ah$, then it is the generator of the uniformly continuous one parameter group $\beta_t(a) = e^{th}ae^{-th}$ of inner automorphism and by uniqueness theorem [10], we have $e^{t\delta}(a) = e^{th}ae^{-th}$. Now under assumption of Theorem 3.9, as we show below a direct calculation applying the formula in Lemma 3.10, gives us the formula $e^{t\delta}\gamma(x) = e^{itc}\gamma(x)e^{-itb}$ for $\gamma-$ one parameter groups similar to the form as stated above for one parameter case.

$$\begin{aligned}
e^{t\delta}\gamma(x) &= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}(-1)^k\delta^k(\gamma(x))\\
&= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\left[i^k\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}c^r\gamma(x)b^{k-r}\right]\\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(-it)^{(k-r)+r}}{k!}(-1)^r\frac{k!}{r!(k-r)!}c^r\gamma(x)b^{k-r}\\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(-it)^{k-r}.(it)^r}{(k-r)!r!}c^r\gamma(x)b^{k-r}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-it)^m.(it)^n}{m!n!}c^n\gamma(x)b^m\\
&= (\sum_{n=0}^{\infty}\frac{(it)^n}{n!}c^n)\gamma(x)(\sum_{m=0}^{\infty}\frac{(-it)^m}{m!}b^m)\\
&= e^{itc}\gamma(x)e^{-itb}.\square
\end{aligned}$$

we end this paper with the following Theorem as an immediate consequence of Theorem 3.9 whose proof is exactly similar to the method has been used in the proof of Corollary 3.7.

Theorem 3.11 Let $\delta : M \to M$ be the hyper generalized $d_c^\sigma-$ derivation $\delta(x) = c\gamma(x) - \gamma(x)b$. If $\gamma : M \to M$ is an idempotent bounded linear operator such that $\gamma(cx) = c\gamma(x)$ and $\gamma(xb) = \gamma(x)b$,then there exists a $(\sigma,\gamma)-$ generalized dynamics $\{T_t\}_{t\in R}$ on M such that δ is its generator.

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