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From 2-Complexes to Group Theory

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Abstract

In this talk, we introduce a topological interpretation for presentation of groups which is presented by Brent Everit in 2003. In particular, Using simplicial complexes and topological methods, we establishsome results in group theory.

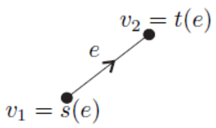
Key words:simplicial complex,fundamental group, covering space, presentation of a group.

1.Introduction

There are several topological interpretations for groups. In this talk, we use the famous onewhose main objects are simplicial complexes. This interpretation is introduced by B. Everit [1]. In section 2 we introduce complexes topologically.Section 3 deals with computing their fundamental groups.Section 4discusses the relation between complexes and presentations and focuses on the interplay between coverings of complexes and subgroups of finitely presented groups and finally section 5 is about applications of coverings to prove our main theorems in group theory. Note that all the definitions and tools which are used in this paper, come from Everit's works [1], [2] and [3].

2. The topology of complexes

A combinatorial 2-complex $K$is made up of three sets V_K, E_K and F_K (vertices, edges and faces), together with maps that describe how the pieces fit together. We have $s,t : E_K \to V_K$and $^{-1}$: $E_K \to E_K$so that $^{-1}$assigns each edge to another, called its inverse, and s,t assign start and terminal vertices to e.



We have $e^{-1} \neq e$, $(e^{-1})^{-1} = e$, $s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$. The vertex and edge sets together with these maps form a directed graph called the $1-skeleton\ K^1$ of K. (the vertices alone form $0-skeleton\ K^0$).

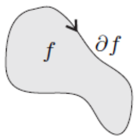
A path in K is a sequence of edges $e_1^{\varepsilon_1} \dots e_k^{\varepsilon_k}$, $\varepsilon_i = \pm 1$ with $t(e_i^{\varepsilon_i}) = s(e_{i+1}^{\varepsilon_{i+1}})$. This path is closed if $t(e_k^{\varepsilon_k}) = s(e_1^{\varepsilon_1})$.

A 2-complex is connected if there is a path between any two of its vertices.

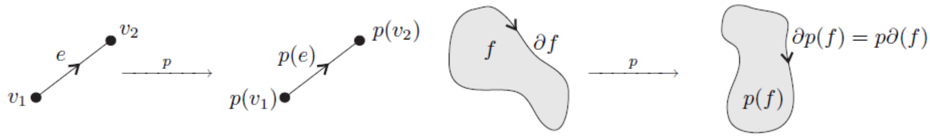
Two paths w_1 and w_2 are cyclic permutations of each other if $w_1 = e_1^{\varepsilon_1} \dots e_k^{\varepsilon_k}$then $w_2 =$

$e_j^{\varepsilon_j} \dots e_k^{\varepsilon_k} e_1^{\varepsilon_1} \dots e_{j-1}^{\varepsilon_{j-1}}$ for some k. A cycle in the $1-skeleton$ is a set consisting of a path and all of its cyclic permutations.

Finally, to define faces we consider these maps that say how the faces are glued onto the $1-skeleton$, $^{-1}: F_K \to F_K$ and $\partial: F_K \to cycles$ which must satisfy $f^{-1} \neq f$, $(f^{-1})^{-1} = f$ and $w \in \partial(f)$iff $w^{-1} \in \partial(f^{-1})$.



A map $p: K_1 \to K_2$ between 2-complexes assigns to each vertex of K_1 a vertex of K_2, to each edge of K_1 an edge or vertex of K_2 and to each face of K_1 a face, path or vertex of K_2 with these conditions: $p(e^{-1}) = p(e)^{-1}, p(f^{-1}) = p(f)^{-1}$.



This map is dimension preserving if $p: V_{K_1} \to V_{K_2}, p: E_{K_1} \to E_{K_2}$ and $p: F_{K_1} \to F_{K_2}$.

A map is an isomorphism if it preserves dimension and is bijection on the vertex, adge and face set.

3. Fundamental groups of complexes

Two paths w_1 and w_2 are homotopic (written $w_1 \sim_h w_2$) iff there is a finite sequence of these two moves taking one path to the other; the first inserts or deletes a spur: an edge/inverse edge pair of the form ee^{-1} or $e^{-1}e$. The second inserts or deletes the boundary of a face: a $w \in \partial(f)$ for some face f of K with $s(w) = t(w) = t\left(e_i^{\varepsilon_i}\right) = s(e_{i+1}^{\varepsilon_{i+1}})$.

If w_1 and w_2 are paths in a 2-complex K with $t(w_1) = s(w_2)$; then let $w_1 w_2$ be the path obtained by juxtaposing these two; by traversing the edges of w_1 and then the edges of w_2. In particular, if $w_1 \sim_h w_1{}'$ and $w_2 \sim_h w_2{}'$ then $w_1 w_2 \sim_h w_1{}' w_2{}'$. So we can extend a product on the homotopy classes of paths in K; if $[w_1]_h$ and$[w_2]_h$ are two such, where $t(w_1) = s(w_2)$, then $[w_1]_h [w_2]_h \coloneqq [w_1 w_2]_h$.

To get a group from homotopy classes of paths, we need to ensure we can always multiply paths. Let K be a 2-complex and fix a vertex v. Let $\pi_1(K, v)$ be the set of all homotopy classes of closed paths with start and terminal vertex v. $\pi_1(K, v)$ together with product $[w_1]_h [w_2]_h = [w_1 w_2]_h$ forms a group with identity $[v]_h$ and $[w]_h^{-1} = [w^{-1}]_h$.

A 2-complex K is called a tree iff the face set of K is empty and $\pi_1(K)$ is the trivial group.

4. Complexes, presentations and subgroups

First we obtain a group presentation for the fundamental group $\pi_1(K)$ of a complex K. Let K be a connected 2-complex and v a vertex of K. Let T be a connected tree that contains all the vertices of K. Choose an edge e_α from each edge or its inverse in $K^1 \backslash T$. Then there are unique paths w_α, $\overline{w_\alpha}$ without spurs in T, such that w_α connects v to the start vertex of e_α and $\overline{w_\alpha}$ connects v to the terminal vertex. Let $x_\alpha = w_\alpha e_\alpha \overline{w_\alpha}^{-1}$, a loop based at v, and $X = \{x_\alpha \mid e_\alpha \in K^1 \backslash T\}$. Choose f_β from each face or its inverse in K. Let $\partial(f_\beta) = e_{\alpha_1}^{\varepsilon_1} \dots e_{\alpha_k}^{\varepsilon_k}$ be the boundary label after the edges that are contained in the tree T have been removed. Take $w_\beta = x_{\alpha_1}^{\varepsilon_1} \dots x_{\alpha_k}^{\varepsilon_k}$, a word in $X \cup X^{-1}$ and $R = \{w_\beta \mid f_\beta \text{ } is \text{ } a \text{ } face\}$.

Theorem 4.1.$< X; R >$ is a presentation for the fundamental group of K.

Corollary4.2. Suppose that $K$is a graph (F_K is empty), then we can find such a presentation for $\pi_1(K)$ with $|E_K| - |V_K| + 1 = 1 - \chi(K)$ generators and no relators.

Now we want to obtain a 2-complex from a group presentation. Let $< X; R >$ be a presentation for a group G. Define a 2-complex $K = K < X; R >$ with a single vertex v. For each $x \in X$ take an $e_x^{\pm 1} \in K$ and for each $w \in R$ an $f_w^{\pm 1} \in F_K$. We have $s, t(e_x^{\pm 1}) = v$, $\partial(f_w) =$ the cyclic permutation of $e_{x_{\alpha_1}}^{\varepsilon_1} \dots e_{x_{\alpha_k}}^{\varepsilon_k}$ if $w = x_{\alpha_1}^{\varepsilon_1} \dots x_{\alpha_k}^{\varepsilon_k}$.

Theorem 4.3. $< X, R >$ is a presentation for $\pi_1(K < X; R >, v)$.

Corollary4.4. A group is free if and only if it is the fundamental group of a graph.

A map $p: \widetilde{K} \to K$ of 2-complexes is a covering iff

4. p preserves dimension;
5. If $p: \tilde{v} \to v$ then p is a bijection from the set of edges in $\widetilde{K}$ with initial vertex $\tilde{v}$ to the set of edges in K with initial vertex v.
6. If f is a face and v a vertex of K, let $m(f, v)$ be the number of times that v appears in the boundary of f. Then for any $\tilde{v}$ with $p(\tilde{v}) = v$, we have $\sum_{p(\tilde{f})=f} m(\tilde{f}, \tilde{v}) = m(f, v)$.

Now we express a main theorem;

Theorem 4.5. (Subgroup theorem) Let $p: \widetilde{K} \to K$ be a covering, then $p_*: \pi_1(\widetilde{K}, \tilde{v}) \to \pi_1(K, v)$ is injective. Moreover if K is a 2-complex and H a subgroup of $\pi_1(K, v)$, then there is a connected 2-complex $\widetilde{K}$ and a covering $p: \widetilde{K} \to K$ with $H \cong \pi_1(\widetilde{K}, \tilde{v})$, where $p(\tilde{v}) = v$.

5.Main Results

In this section, we use the above tools to prove some famous group theoretical theorems.

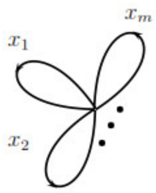
Theorem 5.1. (Nielsen-Schreier theorem) A subgroup of a free group is free.

Proof. If F is free then $F \cong < X; - >$, hence $K < X; - >$ is a bouquet of circles. For a subgroup H of F there is a covering $p: \widetilde{K} \to K < X; - >$, where the complex $\widetilde{K}$ must be a graph and $H \cong \pi_1(\widetilde{K})$. But the fundamental group of a graph is also free, hence H is free .

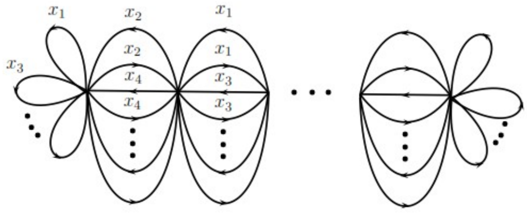
Now we deals with our theorems;

Theorem 5.2. For any $n \geq 2$, the free group of rank $m$has a free subgroup of rank $nm - n + 1$and index n.

Proof. First we consider the presentation 2-complex K for free group $G = < x_1, \dots, x_m; - >$. So we have



Now we give a covering $\widetilde{K}$ for K;



which has n vertices and the fiber of each edge has n edges. The fundamental group of $\widetilde{K}$ is free of rank $nm - n + 1$. Since the fiber of vertex v has n elements, so the index of $H \cong \pi_1(\widetilde{K}, \tilde{v})$ in G is n.

Theorem 5.3. If $G \cong < X; R >$ with $|X| = m$, $|R| = k$ and H a subgroup of G of index n, then the presentation for H has $nm - n + 1$ generators and nk relators.

Proof. With respect to the proof of last theorem, we must only prove that the presentation of the fundamental group of $\widetilde{K}$has nk relators. We know for each $r \in R$ and each vertex v_i in $\widetilde{K}$ ($i = 1, \dots, n$), we have a path starting at $v_i$with label r. Now we attach a face to $\widetilde{K}$ whose boundary label is this path. As there are n vertices in $\widetilde{K}$, then for any $r \in R$ we have n faces in $\widetilde{K}$. By existing k relators in R, we have nk faces in $\widetilde{K}$ and finally, the presentation of the fundamental group of $\widetilde{K}$ has nk relators.

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