The Commuting Graph of Finite Groups

A. Jafarzadeh AND A. Iranmanesh

Abstract

The commuting graph of a group G, denoted by $\Gamma(G)$, is a simple graph whose vertices are all non-central elements of G and two distinct vertices x, y are adjacent if $xy = yx$. In [1] it is conjectured that if M is a simple group and G is a group satisfying $\Gamma(G) \cong \Gamma(M)$, then $G \cong M$. In this paper first we investigate the properties of the commuting graph of the Symmetric and Alternating groups and then prove the above conjecture for simple groups.

1 Introduction

Definition 1. The *commuting graph* of a finite group G, denoted by Γ(G), is a graph whose vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent whenever $xy = yx$. The commuting graph of a subset of a group is defined similarly. The *non-commuting graph* of a group G, denoted by $\nabla(G)$, is the complement of Γ(G), i.e., the graph with $G \setminus Z(G)$ as its vertex set and two vertices x and y are adjacent, if $xy \neq yx$ (see [1, 2]).

Definition 2. The *diameter* of a (connected or disconnected) graph Γ is defined by

$$\mathrm{diam}(\Gamma) = \max\{d(u,v) \mid u \text{ and } v \text{ are distinct vertices of } \Gamma\}.$$

Definition 3. The *prime graph* of a finite group G, denoted by $\Gamma_1(G)$, is a graph whose vertex set is $\pi(G)$, i.e., the set of all prime divisors of $|G|$ and two distinct vertices p and q are adjacent if and only if G contains an element of order pq (see [5]).

Note 6. Denote the number of components of the prime graph of G with $t(G)$, and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the vertex set of the components of $\Gamma_1(G)$ and $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, t(G)\}$. Therefore

$$\pi(G) = \bigcup_{i=1}^{t(G)} \pi_i.$$

Now $|G|$ can be expressed as a product of coprime positive integers m_i, $i = 1, 2, \ldots, t(G)$ where $\pi(m_i) = \pi_i$. These integers are called the *order components* of G and the set of order components of G is denoted by $OC(G)$:

$$OC(G) = \{m_i | i = 1, 2, \ldots, t(G)\}.$$

In [4], the authors investigated some conditions on the connectivity of the commuting graph of finite simple groups and proved that for all finite classical simple groups G, $\mathrm{diam}(\Gamma(G)) \leq 10$, where Γ(G) is connected.

In section 2, we investigate necessary and sufficient conditions for the connectivity of the commuting graph of the Symmetric and the Alternating groups, a good upper bound for their diameter and their maximum and minimum degrees.

In [1], the authors put forward the following conjecture:

Conjecture 1. *Let M be a finite simple group. If G is any finite group such that $\Gamma(M) \cong \Gamma(G)$, then we have $M \cong G$.*

In section 3, we prove Conjecture 1 for some simple groups.

[0]2000 Mathematics Subject Classifications 20D05, 20D60 (primary), 05C25 (secondary)

[0]Keywords: Commuting graph, Characterization, Simple group, Prime graph

2 Commuting graph of the Symmetric and the Alternating groups

Theorem 2. *For $n \geq 3$, $\Gamma(S_n)$ is connected if and only if n and $n-1$ are not primes and* $\mathrm{diam}(\Gamma(S_n)) \leq 6$.

Remark 3. *The above theorem shows that for each prime number $p \geq 3$, graphs $\Gamma(S_p)$ and $\Gamma(S_{p+1})$ are disconnected. A simple calculation shows that $\Gamma(S_3)$ has four components and $\Gamma(S_4)$ has five components. In the next Lemma, we obtain the number of components of $\Gamma(S_p)$ and $\Gamma(S_{p+1})$ for all prime numbers $p \geq 5$:*

Lemma 4. *Suppose $p \geq 5$ is a prime number. Then $\Gamma(S_p)$ and $\Gamma(S_{p+1})$ have exactly $(p-2)!+1$ and $(p+1)(p-2)!+1$ components.*

Let $\Delta(\Gamma)$ and $\delta(\Gamma)$ denote the maximum and minimum degrees of a simple graph Γ, respectively. We have the following Lemma:

Lemma 5. *If $n \geq 5$, then we have $\Delta(\Gamma(S_n)) = d((1\ 2)) = 2(n-2)! - 2$ and $\delta(\Gamma(S_n)) = d((1\ 2 \cdots n-1)) = n-3$.*

Theorem 6. *For $n \geq 4$, $\Gamma(A_n)$ is connected if and only if n, $n-1$ and $n-2$ are not primes and* $\mathrm{diam}(\Gamma(A_n)) \leq 6$.

Lemma 7. **(a)** *If $p \geq 11$ and $p-2$ is not prime, then $\Gamma(A_p)$ has $(p-2)!+1$ components.*
(b) *If $p \geq 13$ and $p-2$ is prime, then $\Gamma(A_p)$ has $(p-2)! + \frac{p!}{2(p-2)(p-3)} + 1$ components.*
(c) *If $p \geq 11$, then $\Gamma(A_{p+1})$ has $(p+1)(p-2)!+1$ components.*
(d) *If $p \geq 7$ and $p+2$ is not prime, then $\Gamma(A_{p+2})$ has $\frac{(p+2)!}{2p(p-1)} + 1$ components.*

Lemma 8. *We have the following equations:*
(a)

$$\Delta(\Gamma(A_n)) = \begin{cases} d((1\ 2)(3\ 4)) = 2, & n = 4; \\ d((1\ 2\ 3\ 4\ 5)) = 3, & n = 5; \\ d((1\ 2\ 3)) = 7, & n = 6; \\ d((1\ 2\ 3)) = 34, & n = 7; \\ d((1\ 2)(3\ 4)(5\ 6)(7\ 8)) = 190, & n = 8, \end{cases}$$

and if $n \geq 9$, then $\Delta(\Gamma(A_n)) = d((1\ 2\ 3)) = \frac{3}{2}(n-3)! - 2$.
(b) *For $n = 4, 5$*

$$\delta(\Gamma(A_n)) = d((1\ 2\ 3)) = 1,$$

and if $n \geq 6$ is even, then $\delta(\Gamma(A_n)) = d((1\ 2)(3\ 4 \cdots n)) = n-4$ and if $n \geq 7$ is odd, then $\delta(\Gamma(A_n)) = d((1\ 2)(3\ 4 \cdots n-1)) = n-5$.

Definition 4. Let T_n and I_n denote the sets of all transpositions and involutions in S_n, respectively.

Theorem 9. *For $n \geq 5$, $\Gamma(T_n)$ is a connected graph and* $\mathrm{diam}(\Gamma(T_n)) = 2$. *In addition, $\Gamma(T_n)$ is a regular graph of degree $\binom{n-2}{2}$ and so $\delta(\Gamma(T_n)) = \Delta(\Gamma(T_n)) = \binom{n-2}{2}$.*

Theorem 10. *For $n \geq 4$, $\Gamma(I_n)$ is a connected graph with* $\mathrm{diam}(\Gamma(I_n)) = 3$.

3 On Conjecture 1

First we state the following Theorem:

Theorem 11 **([2]).** *Let G and M be two finite simple non-abelian groups. If $\Gamma(G) \cong \Gamma(M)$, then $G \cong M$.*

In the rest of this section, we show the relation between the commuting graph and the prime graph of finite groups and then give a positive answer to Conjecture 1 for some groups using their characterization by their prime graph. Note that this conjecture is not true if we suppose that M is an arbitrary finite group. In particular, Dihedral group and Quaternion group of order 8 are not isomorphic while $\Gamma(D_8) \cong \Gamma(Q_8)$.

Note 7. For a group H, $\nabla(H)$ and $\Gamma(H)$ are complement graphs, therefore for all groups H and K we have $\nabla(H) \cong \nabla(K)$ if and only if $\Gamma(H) \cong \Gamma(K)$.

Therefore the following Theorem is a corollary of [3]:

Theorem 12. *Let G be one of the groups S_n, A_n, D_{2n}, a sporadic simple group or a simple group of Lie type with disconnected prime graph such that $\Gamma(M) \cong \Gamma(G)$, then $|M| = |G|$.*

Definition 5. For a group G, let $N(G) = \{ n \mid G$ has a conjugacy class of size $n\}$.

Theorem 13 ([2]). *Let G_1 and G_2 be finite groups such that $|G_1| = |G_2|$ and $\Gamma(G_1) \cong \Gamma(G_2)$. Then $N(G_1) = N(G_2)$ and $OC(G_1) = OC(G_2)$.*

Corollary 14 ([2]). *Let p be a prime number and q a prime power. Suppose that M is one of the groups $PSL(p,q)$, $PSU(p,q)$, or a simple group of one of the following types:*

(a) $B_n(q)$;

(b) $C_n(q)$;

(c) $E_6(q)$ *or* $E_8(q)$;

(d) $F_4(q)$ *where* $q > 2$;

(e) ${}^2D_n(q)$ *where* $n = 2^m \geq 4$;

(f) ${}^2D_p(3)$ *where* $p = 2^n + 1 \geq 5$;

(g) ${}^2E_6(q)$ *where* $q > 2$;

(h) ${}^3D_4(q)$;

(i) *A Suzuki-Ree group, i.e., a group of type* ${}^2B_2(q)$, ${}^2F_4(q)$ *or* ${}^2G_2(q)$;

(j) *A sporadic simple group.*

If G is any group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.

Thompson's Conjecture 1. If G is a finite group with $Z(G) = 1$ and M a finite non-abelian simple group such that $N(G) = N(M)$, then $M \cong G$.

Lemma 15 ([2]). *Suppose M is a finite non-abelian simple group with disconnected prime graph which satisfies Thompson's Conjecture. If G is a group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.*

Corollary 16 ([2]). *Let M be a simple group of type $G_2(q)$ where $q > 2$. If G is any group such that $\Gamma(M) \cong \Gamma(G)$, then $M \cong G$.*

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, 'Non-commuting graph of a group', *J. Algebra*, 298 (2006), 468–492.

[2] A. Iranmanesh AND A. Jafarzadeh, 'Characterization of finite groups by their commuting graph' To appear in *Acta Math. Acad. Paedagog. Nyházi.*

[3] A. R. Moghaddamfar, W. J. Shi, W. Zhou and A. R. Zokayi, 'On the noncommuting graph associated with a finite group *Sib. Math. J.*, 46 (2005), No. 2, 325–332.

[4] Y. Segev and G. M. Seitz, 'Anisitropic groups of type A_n and the commuting graph of finite simple groups', *Pacific J. Math.*, 202 (2002), 125–225.

[5] J. S. Williams, 'Prime graph components of finite groups', *J. Algebra*, 69 (1981), No. 2, 487–513.

A. Jafarzadeh
Department of Mathematics, Tarbiat Modarres University, Tehran, Iran,
and
Department of Mathematics, Faculty of Science, Zanjan University, Zanjan, Iran
email: ajafarza@modares.ac.ir

A. Iranmanesh
Department of Mathematics, Tarbiat Modarres University
P.O.Box: 14115-137, Tehran, Iran
email: iranmana@modares.ac.ir