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On the Baer invariants of triples of groups

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Abstract. In this paper, we develop the theory of Baer invariants for triples of groups. First, we focus on the general properties of the Baer invariant of triples. Second, we prove that the Baer invariant of a triple preserves direct limits of directed systems of triples of groups. Moreover, we present a structure for the nilpotent multiplier of a triple of the free product in some cases. Finally, we give some conditions in which the Baer invariant of a triple is a torsion group.

Keywords: Baer invariant; Triple of groups; Simplicial groups.

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1 Introduction and motivation

Throughout the article, let $\mathcal{V}$ be an arbitrary variety of groups defined by a set of laws V and consider the functor $V(-)$ between groups which takes a group to its verbal subgroup. By a triple of groups (G, M, N) we mean a group G with two normal subgroups N, M. A homomorphism of triples $(G_1, M_1, N_1) \to (G_2, M_2, N_2)$ is a group homomorphism $G_1 \to G_2$ that sends M_1 into M_2 and N_1 into N_2.

Finding a topological interpretation for algebraic concepts can be useful to solve some algebraic problems. The Schur multiplier, $M(G)$, of an arbitrary group G can be described as the second integral homology $H_2(X)$ of a connected CW-space X, where X has the fundamental group G and contractible universal cover [9]. The Baer invariant of G, $\mathcal{V}M(G)$, has a similar topological interpretation. For this, consider the first homotopy group, $\pi_1(K_./V(K_.))$, of the factor of a free simplicial resolution of the group G [6].

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G. Ellis [5] defined the Schur multiplier of a pair of groups (G,N) denoted by $M(G,N)$, as a functorial abelian group whose principal feature is a natural exact sequence

$$\begin{aligned} \cdots \quad &\to M(G,N) \to M(G) \to M(G/N) \\ &\to N/[N,G] \to G^{ab} \to (G/N)^{ab} \to 0. \end{aligned}$$

The natural epimorphism $G \to G/N$ implies the following exact sequence of free simplicial groups

$$1 \to \ker(\alpha) \to K_. \overset{\alpha}{\to} L_. \to 1,$$

where $K_.$ and $L_.$ are free simplicial resolutions of G and G/N, respectively [6]. The authors in [13] extended the above notation to the Baer invariant of a pair of groups, $\mathcal{V}M(G,N)$, as the first homotopy group of the kernel of a map α_v, $\pi_1(\ker \alpha_v)$, where $\alpha_v : K_./V(K_.) \to L_./V(L_.)$ is induced from the simplicial map α. Also the authors obtained the following long exact sequence

$$\begin{aligned} \cdots \quad &\to \mathcal{V}M(G,N) \to \mathcal{V}M(G) \to \mathcal{V}M(G/N) \\ &\to N/[NV^*G] \to G/V(G) \to (G/N)/V(G/N) \to 0. \end{aligned}$$

The Schur multiplier of a triple was first studied by Ellis [5], who introduced it as a functorial abelian group $M(G,M,N)$ which fits into the following natural exact sequence

$$\begin{aligned} \cdots \quad &\to M(G,M,N) \to M(G,N) \to M(G/M, MN/M) \\ &\to M \cap N/[M \cap N, G][M,N] \to N/[N,G] \to MN/M[N,G] \to 0. \end{aligned}$$

In this article, we extend the concept of the Schur multiplier of triples to the Baer invariant of triples of groups with respect to an arbitrary variety $\mathcal{V}$ using homotopy theory for simplicial groups and free simplicial resolutions of groups.

Definition 1.1. Consider the following natural commutative diagram of a triple of groups (G,M,N)

$$\begin{array}{ccc} G & \to & \frac{G}{N} \\ \downarrow & & \downarrow \\ \frac{G}{M} & \to & \frac{G}{MN} \end{array}$$

which implies the following commutative diagram of simplicial groups

$$\begin{array}{ccccccc} & 1 & & 1 & & 1 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & \ker(\alpha_V, \gamma_V) & \to & \ker \beta_V & \to & \ker \delta_V & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & \ker \alpha_V & \to & \frac{K_.}{V(K_.)} & \overset{\alpha_V}{\to} & \frac{L_.}{V(L_.)} & \to 1 \\ & \downarrow & & \beta_V \downarrow & & \delta_V \downarrow & \\ 1 \to & \ker \gamma_V & \to & \frac{R_.}{V(R_.)} & \overset{\gamma_V}{\to} & \frac{S_.}{V(S_.)} & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 1 & & 1 & & 1, & \end{array} \tag{1.1}$$

where $K_.$, $L_.$, $R_.$ and $S_.$ are free simplicial resolutions of G, G/N, G/M and G/MN, respectively [1]. We define the Baer invariant of the triple (G, M, N) with respect to a variety of groups $\mathcal{V}$ as follows

$$\mathcal{V}M(G, M, N) = \pi_1(\ker(\alpha_V, \gamma_V)),$$

where $\ker(\alpha_V, \gamma_V)$ is defined in Diagram (1.1).

The article is organized as follows. In Section 2, we give a brief review for simplicial groups and some of their properties needed for the rest. In Section 3, we first give some useful general results for computation of Baer invariant of triple of groups. In particular, we obtain long exact sequences that contain Baer invariants of triples and pairs of groups. Second, we prove that the Baer invariant of a triple preserves direct limits of directed systems of triples of groups. Moreover, we study the behavior of c-nilpotent multiplier with respect to the free product in some cases. Finally, we obtain some conditions in which the c-nilpotent multiplier of a triple of groups is a torsion group.

2 Preliminaries and notation

In this section we recall several concepts and results about the simplicial groups which will be needed in the sequel. A detailed can be found in [3] or [7].

A *simplicial group* $K_.$ is a sequence of groups $K_0, K_1, K_2, \ldots$ together with homomorphisms $d_i : K_n \to K_{n-1}$ (faces) and $s_i : K_n \to K_{n+1}$ (degeneracies), for each $0 \leq i \leq n$, such that the following conditions hold:

$$\begin{array}{rcll} d_j d_i & = & d_{i-1} d_j & \text{for } j < i \\ s_j s_i & = & s_{i+1} s_j & \text{for } j \leq i \\ d_j s_i & = & \begin{cases} s_{i-1} d_j & \text{for } j < i; \\ identity & \text{for } j = i, i+1; \\ s_i d_{j-1} & \text{for } j > i+1. \end{cases} & \end{array}$$

A *simplicial homomorphism* $f : K_. \to L_.$ is a sequence of homomorphisms $f_n : K_n \to L_n$, for each $n \geq 0$, such that $f \circ d_i = d_i \circ f$, and $f \circ s_i = s_i \circ f$, that is the following diagram commutes:

$$\begin{array}{ccccc} K_{n+1} & \overset{s_i}{\leftarrow} & K_n & \overset{d_i}{\to} & K_{n-1} \\ f_{n+1} \downarrow & & f_n \downarrow & & f_{n-1} \downarrow \\ L_{n+1} & \overset{s_i}{\leftarrow} & L_n & \overset{d_i}{\to} & L_{n-1}. \end{array}$$

From a simplicial group $K_.$ we can derive a chain complex $(NK_., \partial)$ which is called the *Moore complex* as follows: $(NK)_n = \cap_{i=0}^{n-1} Kerd_i$ and $\partial_n : NK_n \to NK_{n-1}$ to be the restriction of d_n.

We use the following properties of simplicial groups frequently in the rest of the paper.

Theorem 2.1. (i) For every simplicial group $K_.$ the homotopy group $\pi_n(K_.)$ is abelian even for $n = 1$ [3, Proposition 3.2].
(ii) Every epimorphism between simplicial groups is a fibration [3, Lemma 3.2].
(iii) If $K_.$ is a simplicial group, then $\pi_*(K_.) \cong H_*(NK_.)$ [3, Theorem 3.7].
(iv) $H_n\big(N(K_. \otimes L_.)\big) \cong H_n\big(N(K_.) \otimes N(L_.)\big)$ [3, Proposition 5.6].

A simplicial group $K_.$ is said to be *free* if each K_n is a free group and degeneracy homomorphisms s_i's send the free basis of K_n into the free basis of K_{n+1}. A *free simplicial resolution* of G consists of a free simplicial group $K_.$ with $\pi_0(K_.) = G$ and $\pi_n(K_.) = o$ for $n \geq 1$ (see [10]).

3 Baer invariants of triples

In this section, we study the behavior of Baer invariants of triples of groups. First, we give some basic and essential properties of the Baer invariant of a triple of groups.

Theorem 3.1. The Baer invariant of a triple of groups is a functor from the category of triples of groups to the category of abelian groups.

Proof. Theorem 2.1 (i) guarantees that the Baer invariant of triple of groups is an abelian group. Let $f : (G_1, M_1, N_1) \to (G_2, M_2, N_2)$ be a homomorphism of triples of groups, functorial property of free simplicial resolutions yields the proof completed. □

Theorem 3.2. Let (G, M, N) be a triple of groups and $\mathcal{V}$ be an arbitrary variety of groups defined by a set of laws V. Then $\mathcal{V}M(G, M, N)$ satisfies in the following long exact sequences

$$\begin{aligned} \cdots \quad &\to \mathcal{V}M(G,M,N) \to \mathcal{V}M(G,N) \to \mathcal{V}M(G/M, MN/M) \\ &\to \pi_0(\ker(\alpha_V, \gamma_V)) \to N/[NV^*G] \to MN/M[NV^*G] \to 0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \cdots \quad &\to \mathcal{V}M(G,M,N) \to \mathcal{V}M(G,M) \to \mathcal{V}M(G/N, MN/N) \\ &\to \pi_0(\ker(\alpha_V, \gamma_V)) \to M/[MV^*G] \to MN/N[MV^*G] \to 0. \end{aligned}$$

Proof. Consider Diagram (1.1) corresponding to triple of groups (G, M, N). In [6] one can find the following isomorphisms

$$\begin{array}{llll} \pi_0(\ker \alpha_v) & \cong \frac{N}{[NV*G]} & \qquad \pi_0(\ker \beta_v) & \cong \frac{M}{[MV*G]} \\ \pi_0(\ker \gamma_v) & \cong \frac{MN}{N[MV*G]} & \qquad \pi_0(\ker \delta_v) & \cong \frac{MN}{M[NV*G]} \\ \pi_0(\frac{K_.}{V(K_.)}) & \cong \frac{G}{V(G)} & \qquad \pi_0(\frac{L_.}{V(L_.)}) & \cong \frac{G/N}{V(G/N)} \\ \pi_0(\frac{R_.}{V(R_.)}) & \cong \frac{G/M}{V(G/M)} & \qquad \pi_0(\frac{S_.}{V(S_.)}) & \cong \frac{G/MN}{V(G/MN)}. \end{array}$$

The left column of Diagram (1.1) and Theorem 2.1 (ii) yield the long exact sequence of homotopy groups, using the various isomorphisms, we can rewrite it in the group theory language as the exact sequence (3.1). Also the top row of Diagram (1.1) induces the other long exact sequence. □

Theorem 3.3. Let $G = MN$ and $\mathcal{V}$ be the variety of abelian group of exponent q. Then we have the following long exact sequence

$$\begin{aligned}\cdots \quad &\to \mathcal{V}M(G,M,N) \to \mathcal{V}M(G,N) \to \mathcal{V}M(G/M)\\ &\to M\cap N/M\#_q N \to N/[NV^*G] \to G/M/V(G/M) \to 1,\end{aligned}$$

where $M\#_q N$ is the subgroup of G generated by

$$[m,n]t^q;\quad m \in M,\ n \in N \ and\ t \in M\cap N.$$

Proof. Barja and Rodriguez in [1] obtained $\pi_0(\alpha_V, \gamma_V) \cong M\cap N/M\#_q N$ which gives the result. □

The long exact sequence of (3.1) implies the following theorem.

Theorem 3.4. The Baer invariant of a pair of groups is a special case of the Baer invariant of a triple of groups i.e. $\mathcal{V}M(G,M,M) \cong \mathcal{V}M(G,M) \cong \mathcal{V}M(G,G,M)$ for any group G and any normal subgroup M.

The following result shows that the Baer invariant of triples commutes with the direct limits of directed systems.

Theorem 3.5. Let $\{(G_i, M_i, N_i)\}_{i\in I}$ be a directed system of triples of groups. Then

$$\mathcal{V}M(\varinjlim G_i, \varinjlim M_i, \varinjlim N_i) \cong \varinjlim \mathcal{V}M(G_i, M_i, N_i).$$

Proof. For any $i \in I$, let $K_{i.}, L_{i.}, R_{i.}$ and $S_{i.}$ be corresponding free simplicial resolutions of $G_i, G_i/N_i, G_i/M_i$ and G_i/M_iN_i, respectively. Assume the following diagram of simplicial groups

$$\begin{array}{ccc}\frac{K_{i.}}{V(K_{i.})} & \overset{\alpha_{V_i}}{\to} & \frac{L_{i.}}{V(L_{i.})}\\ \beta_{V_i}\downarrow & & \downarrow\delta_{V_i}\\ \frac{R_{i.}}{V(R_{i.})} & \overset{\gamma_{V_i}}{\to} & \frac{S_{i.}}{V(S_{i.})}.\end{array}$$

The fact that homology groups commute with direct limit of chain complexes and Theorem 2.1 (iii) imply $\varinjlim \pi_n(K_{i.}) \cong \pi_n(\varinjlim K_{i.})$, where $K_{i.}$ are simplicial groups (for more details see [13]). Hence $\varinjlim K_{i.}$, $\varinjlim L_{i.}$, $\varinjlim R_{i.}$ and $\varinjlim S_{i.}$ are simplicial groups corresponding to $\varinjlim G_i$, $\varinjlim G_i/N_i$, $\varinjlim G_i/M_i$ and $\varinjlim G_i/M_iN_i$, respectively. Since direct limit preserve the exact sequence, we have $\varinjlim(\ker\alpha_{V_i}) \cong \ker\varinjlim(\alpha_{V_i})$ and similarly $\varinjlim(\ker\gamma_{V_i}) \cong \ker\varinjlim(\gamma_{V_i})$ (see [14, Theorem 2.7] for further details). Now consider the following commutative diagram:

$$\begin{array}{ccccccc}1\to & \ker((\varinjlim\alpha_{V_i}, \varinjlim\gamma_{V_i})) & \to & \ker\varinjlim(\gamma_{V_i}) & \to & \ker\varinjlim(\alpha_{V_i}) & \to 1\\ & \downarrow & & \downarrow & & \downarrow & \\ 1\to & \varinjlim(\ker(\alpha_{V_i}, \gamma_{V_i})) & \to & \varinjlim(\ker\gamma_{V_i}) & \to & \varinjlim(\ker\alpha_{V_i}) & \to 1.\end{array}$$

The Five Lemma [12, Theorem 9.1.5] implies that

$$\varinjlim(\ker(\alpha_{V_i}, \gamma_{V_i})) \cong \ker((\varinjlim\alpha_{V_i}, \varinjlim\gamma_{V_i})).$$

Since homotopy groups of simplicial groups commute with direct limits, the following isomorphisms hold

$$\begin{array}{ll} \mathcal{V}M(\varinjlim G_i, \varinjlim M_i, \varinjlim N_i) & \cong \pi_1\big(\ker(\varinjlim \alpha_{V_i}, \varinjlim \gamma_{V_i})\big) \\ \cong \pi_1\Big(\varinjlim\big(\ker(\alpha_{V_i}, \gamma_{V_i})\big)\Big) & \cong \varinjlim \mathcal{V}M(G_i, M_i, N_i). \end{array}$$

$\square$

Lemma 3.1. Let G_1 and G_2 be two finite groups with $(|G_1|, |G_2|) = 1$. Then for all $c \geq 1$ the following isomorphism holds

$$\pi_n\left(\frac{K_{1.} * K_{2.}}{\gamma_c(K_{1.} * K_{2.})}\right) \cong \pi_n\left(\frac{K_{1.}}{\gamma_c(K_{1.})}\right) \oplus \pi_n\left(\frac{K_{2.}}{\gamma_c(K_{2.})}\right),$$

where $K_{1.}$ and $K_{2.}$ are free simplicial resolutions of G_1 and G_2, respectively.

Proof. First, we consider the following exact sequence

$$1 \to \ker\phi_c \to \frac{K_{1.} * K_{2.}}{\gamma_{c+1}(K_{1.} * K_{2.})} \to \frac{K_{1.} \times K_{2.}}{\gamma_{c+1}(K_{1.} \times K_{2.})} \to 1. \tag{3.2}$$

Note that $\ker\phi_c \cong \frac{[K_{1.}, K_{2.}]^{F.} \gamma_{c+1}(F.)}{\gamma_{c+1}(F.)} \cong \frac{[K_{1.}, K_{2.}]^{F.}}{[K_{1.}, K_{2.}]^{F.} \cap \gamma_{c+1}(F.)} \cong \frac{[K_{1.}, K_{2.}]^{F.}}{[K_{1.}, K_{2.}, {}_{c-1}F.]^{F.}}$ which satisfies in the following exact sequence

$$1 \to \frac{[L_{1.}, K_{2.}, {}_{c-2}F.]^{F.}}{[K_{1.}, K_{2.}, {}_{c-1}F.]^{F.}} \to \ker\phi_c \to \ker\phi_{c-1} \to 1, \tag{3.3}$$

where $F. = K_{1.} * K_{2.}$. Let K_1 and K_2 be free groups freely generated by $\{x_1, \cdots, x_m\}$ and $\{x_{m+1}, \cdots, x_{m+n}\}$, respectively, and $F = K_1 * K_2$. Then by a theorem of Hall [8, Theorem 11.2.4], it is easy to show that $\frac{[K_1, K_2, {}_{c-2}F]^F}{[K_1, K_2, {}_{c-1}F]^F}$ is a free abelian group with the basis $\bar{B} = \{b[K_1, K_2, {}_{c-1}F]^F | b \in B\}$, where $B = B_1 - B_2 - B_3$ in which B_1, B_2 and B_3 are the set of all basic commutators of weight c on $\{x_1, \cdots, x_m, \cdots, x_{m+n}\}$, $\{x_1, \cdots, x_m\}$ and $\{x_{m+1}, \cdots, x_{m+n}\}$, respectively. Therefore

$$\frac{[K_{1.}, K_{2.}, {}_{c-2}F.]^{F.}}{[K_{1.}, K_{2.}, {}_{c-1}F.]^{F.}} \cong \oplus \sum_{for\ some\ i+j=c} \underbrace{K_{1.}^{ab} \otimes ... \otimes K_{1.}^{ab}}_{i-times} \otimes \underbrace{K_{2.}^{ab} \otimes ... \otimes K_{2.}^{ab}}_{j-times}.$$

Note that the number of copies in the above direct sum is the number of all basic commutators subgroups of weight c on K_1 and K_2 (see [13] for further details).

By Fixing n and using induction on c, we claim that $\pi_n(\ker\phi_c)$ is trivial. For $c = 2$, since $K_{1.}$ and $K_{2.}$ are free groups, we have $(K_{1.} * K_{2.})^{ab} \cong K_{1.}^{ab} \oplus K_{2.}^{ab}$. Thus $\ker\phi_c = 0$.

Let $c > 2$, using Theorem 2.1 (iii), (iv) and the fact that the order of $H_i(G)$ divides a power of the order of G (see [11, Theorem 10.26]) and the hypothesis $(|G_1|, |G_2|) = 1$, we have

$$\begin{aligned} \pi_n(K_{1.}^{ab} \otimes K_{2.}^{ab}) & \cong \oplus \textstyle\sum_{i+j=n} \pi_i(K_{1.}^{ab}) \otimes \pi_j(K_{2.}^{ab}) \\ & \oplus \textstyle\sum_{i+j=n-1} Tor(\pi_i(K_{1.}^{ab}), \pi_j(K_{2.}^{ab})) \\ & \cong \oplus \textstyle\sum_{i+j=n} H_{i+1}(G_1) \otimes H_{j+1}(G_2) \\ & \oplus \textstyle\sum_{i+j=n-1} Tor(H_{i+1}(G_1), H_{j+1}(G_2)) = 0, \end{aligned}$$

and

$$\pi_n(\underbrace{K_{1.}^{ab}\otimes ...\otimes K_{1.}^{ab}}_{i-times}\otimes \underbrace{K_{2.}^{ab}\otimes ...\otimes K_{2.}^{ab}}_{j-times})\cong$$
$$\oplus \textstyle\sum_{r+s=n}\pi_r(K_{1.}^{ab}\otimes K_{2.}^{ab})\otimes \pi_s(\underbrace{K_{1.}^{ab}\otimes ...\otimes K_{1.}^{ab}}_{i-1-times}\otimes \underbrace{K_{2.}^{ab}\otimes ...\otimes K_{2.}^{ab}}_{j-1-times})$$
$$\oplus \textstyle\sum_{r+s=n-1} Tor(\pi_r(K_{1.}^{ab}\otimes K_{2.}^{ab}),\pi_s(\underbrace{K_{1.}^{ab}\otimes ...\otimes K_{1.}^{ab}}_{i-1-times}\otimes \underbrace{K_{2.}^{ab}\otimes ...\otimes K_{2.}^{ab}}_{j-1-times}))$$
$$=0.$$

Therefore the induction hypothesis and the exact sequence (3.3) imply that $\pi_n(\ker\phi_c)$ is trivial for all $n\geq 1$.

The exact sequence (3.2) and Theorem 2.1 (ii) give rise the following long exact sequence of homotopy groups

$$\pi_n(\ker\phi_c)\to \pi_n(\frac{K_{1.}*K_{2.}}{\gamma_c(K_{1.}*K_{2.})})\to \pi_n(\frac{K_{1.}\times K_{2.}}{\gamma_c(K_{1.}\times K_{2.})})\to \pi_{n-1}(\ker\phi_c).$$

Hence the result holds. □

Now we study the behavior of Baer invariants of triples of free products.

Theorem 3.6. Let (G_i, M_i, N_i) be triples of groups, for $i=1,2$. Then the following isomorphism holds

1. $M(G_1*G_2,\langle M_1*M_2\rangle^{G_1*G_2},\langle N_1*N_2\rangle^{G_1*G_2})\cong M(G_1,M_1,N_1)\oplus M(G_2,M_2,N_2)$.
2. If G_1 and G_2 are two finite groups with $(|G_1|,|G_2|)=1$, then for all $c\geq 1$

$$M^{(c)}(G_1*G_2,\langle M_1*M_2\rangle^{G_1*G_2},\langle N_1*N_2\rangle^{G_1*G_2})\cong M^{(c)}(G_1,M_1,N_1)\oplus M^{(c)}(G_2,M_2,N_2).$$

Proof. (i) Van-Kampen theorem for simplicial groups implies that $K_{1.}*K_{2.}$, $L_{1.}*L_{2.}$, $R_{1.}*R_{2.}$ and $S_{1.}*S_{2.}$ are free simplicial resolutions corresponding to G_1*G_2, $(G_1/N_1*G_2/N_2)$, $(G_1/M_1*G_2/M_2)$ and $(G_1/M_1N_1*G_2/M_2N_2)$, respectively [2]. Since $K_{1.}$ and $K_{2.}$ are free groups, we have $(K_{1.}*K_{2.})^{ab}\cong K_{1.}^{ab}\oplus K_{2.}^{ab}$. Hence we have the following commutative diagram

$$\begin{array}{ccc} K_{1.}^{ab}\oplus K_{2.}^{ab} & \overset{\alpha_1\oplus\alpha_2}{\to} & L_{1.}^{ab}\oplus L_{2.}^{ab}\\ \downarrow & & \downarrow\\ R_{1.}^{ab}\oplus R_{2.}^{ab} & \overset{\gamma_1\oplus\gamma_2}{\to} & S_{1.}^{ab}\oplus S_{2.}^{ab}. \end{array}$$

Consequently, $\ker(\alpha_1\oplus\alpha_2,\gamma_1\oplus\gamma_2)\cong\ker(\alpha_1,\gamma_1)\oplus\ker(\alpha_2,\gamma_2)$ which completes the proof of (i).
(ii) Consider the same notation as the above and a fix c. Applying the homomorphism

$\frac{K_{1.}*K_{2.}}{\gamma_c(K_{1.}*K_{2.})} \to \frac{K_{1.}\times K_{2.}}{\gamma_c(K_{1.}\times K_{2.})}$ to the simplicial resolutions of pairs of groups (G_i, N_i), for $i = 1, 2$, Theorem 2.1 (ii) gives the following commutative diagram

$$\begin{array}{ccccccc}
\pi_{n+1}(\frac{L_{1.}*L_{2.}}{\gamma_c(L_{1.}*L_{2.})}) & \to & \pi_n(\ker(\alpha_1 * \alpha_2)) & \to & \pi_n(\frac{K_{1.}*K_{2.}}{\gamma_c(K_{1.}*K_{2.})}) & \to & \pi_n(\frac{L_{1.}*L_{2.}}{\gamma_c(L_{1.}*L_{2.})}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{n+1}(\frac{L_{1.}\times L_{2.}}{\gamma_c(L_{1.}\times L_{2.})}) & \to & \pi_n(\ker(\alpha_1 \times \alpha_2)) & \to & \pi_n(\frac{K_{1.}\times K_{2.}}{\gamma_c(K_{1.}\times K_{2.})}) & \to & \pi_n(\frac{L_{1.}\times L_{2.}}{\gamma_c(L_{1.}\times L_{2.})}).
\end{array}$$

Lemma 3.1 and the Five Lemma imply that $\pi_n(\ker(\alpha_1*\alpha_2)) \cong \pi_n(\ker(\alpha_1))\oplus\pi_n(\ker(\alpha_2))$. Similarly, for the pair of groups $(G/M, MN/M)$, we have $\pi_n(\ker(\gamma_1*\gamma_2)) \cong \pi_n(\ker(\gamma_1))\oplus \pi_n(\ker(\gamma_2))$. With a similar argument for the exact sequence (3.1) we have the following commutative diagram

$$\begin{array}{ccccccc}
\to & \pi_n(\ker(\alpha_1 * \alpha_2), \ker(\gamma_1 * \gamma_2)) & \to & \pi_n(\ker(\alpha_1 * \alpha_2)) & \to & \pi_n(\ker(\gamma_1 * \gamma_2)) \\
 & \downarrow & & \downarrow & & \downarrow \\
\to & \pi_n(\ker(\alpha_1, \gamma_1), \ker(\alpha_2, \gamma_2)) & \to & \pi_n(\ker(\alpha_1, \alpha_2)) & \to & \pi_n(\ker(\gamma_1, \gamma_2)).
\end{array}$$

Therefore we can prove that $\pi_n(\ker(\alpha_1 * \alpha_2, \gamma_1 * \gamma_2)) \cong \pi_n(\ker(\alpha_1, \gamma_1)) \oplus \pi_n(\ker(\alpha_2, \gamma_2))$. Hence for $n = 1$, we have

$$M^{(c)}(G_1 * G_2, \langle M_1 * M_2\rangle^{G_1*G_2}, \langle N_1 * N_2\rangle^{G_1*G_2}) \cong M^{(c)}(G_1, M_1, N_1) \oplus M^{(c)}(G_2, M_2, N_2).$$

□

Note that since the direct product of two free groups is not free, we can not apply the above method to get a similar result for the Baer invariant of triples of direct products.

Theorem 3.7. Let $M^{(1)}(G, N)$, $M^{(1)}(G)$ and $H_3(G/N)$ be torsion groups. Then $M^{(c)}(G, N)$ is also a torsion group for all $c \geq 2$.

Proof. Let $K_.$ and $L_.$ be free simplicial resolutions of G and G/N, respectively. Consider the following commutative exact diagram (see [14])

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \ker(\alpha_n/\alpha_{n+1}) & \longrightarrow & \frac{\gamma_n(K_.)}{\gamma_{n+1}(K_.)} & \xrightarrow{\alpha_n/\alpha_{n+1}} & \frac{\gamma_n(L_.)}{\gamma_{n+1}(L_.)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \ker(\alpha_{n+1}) & \longrightarrow & \frac{K_.}{\gamma_{n+1}(K_.)} & \xrightarrow{\alpha_{n+1}} & \frac{L_.}{\gamma_{n+1}(L_.)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \ker(\alpha_n) & \longrightarrow & \frac{K_.}{\gamma_n(K_.)} & \xrightarrow{\alpha_n} & \frac{L_.}{\gamma_n(L_.)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

The left column exact sequence and Theorem 2.1 (ii) imply the following exact sequence

$$\pi_1(\ker \alpha_n/\alpha_{n+1}) \to M^{(n+1)}(G, N) \to M^{(n)}(G, N). \quad (3.4)$$

Note that $\pi_1(\ker\alpha_n/\alpha_{n+1})$ satisfies in the following exact sequence

$$\pi_2(\frac{\gamma_n(L_.)}{\gamma_{n+1}(L_.)}) \to \pi_1(\ker\alpha_n/\alpha_{n+1}) \to \pi_1(\frac{\gamma_n(K_.)}{\gamma_{n+1}(K_.)}). \tag{3.5}$$

Since $M^{(1)}(G)$ is a torsion group, by [4, Claim 2] $\pi_1(\gamma_n(K_.)/\gamma_{n+1}(K_.))$ is a torsion group.

An abelian group is torsion if and only if it is trivial when tensored by rationals. Since $H_3(G/N)$ is a torsion group, we can get the isomorphism $\pi_2(L_.^{ab}\otimes Q) \cong H_3(G/N)\otimes Q = 0$. With a similar argument of [4, Claim 2] $\pi_2(L_.^{ab}\otimes Q) \cong 0$ leads to $\pi_2(\gamma_n(L_.)/\gamma_{n+1}(L_.))\otimes Q \cong 0$. Therefore $\pi_2(\gamma_n(L_.)/\gamma_{n+1}(L_.))$ is a torsion group. Thus (3.5) implies that $\pi_1(\ker\alpha_n/\alpha_{n+1})$ is a torsion group. Since $M^{(1)}(G,N)$ is a torsion group, the result follows by induction from (3.4). $\square$

Theorem 3.7 can be extended to triples of the groups with a similar argument.

Theorem 3.8. Let $M^{(1)}(G,M,N)$, $M^{(1)}(G)$, $H_3(G/N)$, $H_3(G/M)$ and $H_4(G/MN)$ be torsion groups. Then $M^{(c)}(G,M,N)$ is a torsion group for all $c \geq 1$.

Proof. We proceed by induction on c. Since $M^{(1)}(G,M,N)$ is a torsion group, the first step of the induction holds. A detailed proof is quite lengthy, so we give a sketch of the proof. Consider the following commutative three dimensional diagram which induce from the diagram of the free simplicial resolutions of triple of groups (G,M,N) (1.1)

$$\begin{tikzcd}
\ker(\alpha_n/\alpha_{n+1},\beta_n/\beta_{n+1}) \arrow[ddd, hook] \arrow[dr, hook] & & & & & & & & \\
& \ker(\alpha_{n+1},\beta_{n+1}) \arrow[ddd, hook] \arrow[dr, hook] & & & & & & & \\
& & \ker(\alpha_n,\beta_n) \arrow[ddd, hook] & & & & & & \\
\ker(\alpha_n/\alpha_{n+1}) \arrow[rrr, hook, dotted] \arrow[dr, hook] \arrow[ddd, dotted] & & & \frac{\gamma_n(K_.)}{\gamma_{n+1}(K_.)} \arrow[rrr, dotted, two heads, "\alpha_n/\alpha_{n+1}"] \arrow[dr, hook] \arrow[ddd, dotted, two heads] & & & \frac{\gamma_n(L_.)}{\gamma_{n+1}(L_.)} \arrow[dr, hook] \arrow[ddd, dotted, two heads] & & \\
& \ker(\alpha_{n+1}) \arrow[rrr, hook, dashed] \arrow[dr, hook] \arrow[ddd, dashed] & & & \frac{K_.}{\gamma_{n+1}(K_.)} \arrow[rrr, dashed, two heads, "\alpha_{n+1}"] \arrow[dr, two heads] \arrow[ddd, dashed, two heads] & & & \frac{L_.}{\gamma_{n+1}(L_.)} \arrow[dr, two heads] \arrow[ddd, dashed, two heads] & \\
& & \ker(\alpha_n) \arrow[rrr, hook] \arrow[ddd] & & & \frac{K_.}{\gamma_n(K_.)} \arrow[rrr, two heads, "\alpha_n"] \arrow[ddd, two heads] & & & \frac{L_.}{\gamma_n(L_.)} \arrow[ddd, two heads] \\
\ker(\beta_n/\beta_{n+1}) \arrow[rrr, hook, dotted] \arrow[dr, hook] & & & \frac{\gamma_n(R_.)}{\gamma_{n+1}(R_.)} \arrow[rrr, dotted, two heads, "\beta_n/\beta_{n+1}"] \arrow[dr, hook] & & & \frac{\gamma_n(S_.)}{\gamma_{n+1}(S_.)} \arrow[dr, hook] & & \\
& \ker(\beta_{n+1}) \arrow[rrr, hook, dashed] \arrow[dr, hook] & & & \frac{R_.}{\gamma_{n+1}(R_.)} \arrow[rrr, dashed, two heads, "\beta_{n+1}"] \arrow[dr, two heads] & & & \frac{S_.}{\gamma_{n+1}(S_.)} \arrow[dr, two heads] & \\
& & \ker(\beta_n) \arrow[rrr, hook] & & & \frac{R_.}{\gamma_n(R_.)} \arrow[rrr, two heads, "\beta_n"] & & & \frac{S_.}{\gamma_{n+1}(S_.)}
\end{tikzcd}$$

Consider exact sequences which contain $\ker(\alpha_n/\alpha_{n+1}, \beta_n/\beta_{n+1})$. By Theorem 2.1 (ii), we can obtain the following exact sequence

$$\pi_1(\ker(\alpha_n/\alpha_{n+1}, \beta_n/\beta_{n+1})) \to M^{(n+1)}(G, M, N) \to M^{(n)}(G, M, N)$$

which satisfies in the following exact sequence

$$\pi_2(\ker \beta_n/\beta_{n+1}) \to \pi_1 \ker((\alpha_n/\alpha_{n+1}, \beta_n/\beta_{n+1})) \to \pi_1(\ker \alpha_n/\alpha_{n+1}).$$

Since $M^{(1)}(G)$ and $H_3(G/N)$ are torsion groups, the proof of Theorem 3.7 implies that $\pi_1(\ker \alpha_n/\alpha_{n+1})$ is a torsion group. Moreover, $\pi_2(\ker \beta_n/\beta_{n+1})$ fits in the following exact sequence

$$\pi_3(\frac{\gamma_n(S.)}{\gamma_{n+1}(S.)}) \to \pi_2(\ker \beta_n/\beta_{n+1}) \to \pi_2(\frac{\gamma_n(R.)}{\gamma_{n+1}(R.)}).$$

Since $H_3(G/M)$ is a torsion group, by the proof of Theorem 3.7, we have $\pi_2(\frac{\gamma_n(R.)}{\gamma_{n+1}(R.)})$ is a torsion group. Hence the result is proved if we show that $\pi_3(\frac{\gamma_n(S.)}{\gamma_{n+1}(S.)})$ is a torsion group.

With a similar argument of the proof of Theorem 3.7, we can prove that if $H_4(G/MN)$ is a torsion group, then so is $\pi_3(\frac{\gamma_n(S.)}{\gamma_{n+1}(S.)})$. □

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