MULTIVARIATE SHEWHART QUALITY CONTROL FOR STANDARD DEVIATION

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ABSTRACT

In univariate quality control, the S Chart have been useful in determining whether the process dispersion is in-control or not. It would be very useful to have a similar chart applied to the multivariate case. The existing methods do not provide all the information that a quality control practitioner would like to possess such as the indication of which variables are causing the process to be out-of-control. In this paper, we propose a method which allows us to simultaneously control the overall process quality characteristics and to identify the responsible variables leading to an out-of-control condition. This method is based on the adequate selection of the symmetric square root of the correlation matrix.

The associated critical region is also discussed. The process considered is assumed to be multivariate normal with parameters known from historical data or estimated from a large sample. We call this method, "Multivariate Shewhart Chart (MS Chart)", because it reduces to the Shewhart Chart when the process involves only one variable. The procedure has been illustrated with the help of two examples.

Key Words:

Multivariate Quality Control, Multivariate Shewhart Chart, Symmetric Square Root, and Critical Region.

INTRODUCTION

Quality control problems in industry may involve more than a single quality characteristic, i.e. a vector of characteristics, and when these characteristics are correlated, a more appropriate approach would be to monitor them simultaneously. This has formed the basis of extensive work performed in the field of multivariate quality control. Shewhart, who is famous for the development of the statistical control chart (Shewhart Charts) first recognized the need to consider quality control problems as multivariate in character. The general multivariate statistical quality control problem considers a repetitive process where each item is characterized by p-quality characteristics, $X_1, X_2, ..., X_p$. Because of the chance causes inherent in the process, these quality characteristics are random variables. Because of the independency between the characteristics, the random variables, are correlated. The problem thus requires a multivariate approach. The underlying probability distribution of the p quality characteristics is assumed to be multivariate normal with mean vector μ and covariance matrix Σ. The multivariate approach to quality control was first widely publicized in 1947 and 1951 by Hotelling [16] in the testing of bombsights.

In a set of related papers, Jacson [17, 18] and Jackson and Morris [21] use an elliptical control region and extended Hotelling's procedure for use the principal components in monitoring a photographic process. Ghare and Torgersen [12], Alt [3, 5], and Alt et al. [6] examined the simultaneous control of several related variables when the data is in the form of rational subgroups. Some of the conclusions and results presented in the papers by Jackson [17, 18] and those discussed in Ghare and Torgersen [12] require certain alterations. Specific details are in Alt [4, 5]. Alt and Deutsch [7] determined the appropriate sample size and control chart constant by extending the univariate scheme of to multivariate data. Alt et al. [8] developed control charts for when there is correlation across the data vectors as well as within each vector. In the univariate case, the process dispersion is monitored by sigma charts or range charts. Alt [3] and Alt et al. [1] develop and present the multivariate counterparts.

There are two distinct phases of control chart practice. Phase I consists of using the charts for (1) retrospectively testing whether the process was in control when the first subgroups were being drawn and (2) testing whether the process remains in control when future subgroups are drawn. These are two separate and distinct stages of analysis.

Phase II consist of using the control chart to detect any departure of the underlying process from standard values (μ_0, Σ_0). In this paper, we consider Phase II.

Notations and Assumptions

μ-Mean vector, σ-standard deviation, Σ-covariance matrix, R-correlation matrix, $R^{\frac{1}{2}}$-square root of the covariance matrix, X'- transpose of X, $\chi^2_{n,1-\alpha}$-chi square value. The process considered, is assumed to be multivariate normal and process parameters, μ and Σ known or estimated from a large sample.

CONTROL SHEWHART CHARTS FOR DISPERSION (MS CHARTS)

It was assumed that the process dispersion remained constant at Σ_0. This assumption must be validated in practice; methods for investigating it are presented in this section. Several different control charts for process dispersion will be presented since different statistics can be used to describe variability. To lay the groundwork for the development of control charts for multivariate data, first consider the case of one quality characteristic.

Let S^2 denote the (unbiased) sample variance for a random sample size n from a process. When the process variance is σ_0^2, then $(n-1)S^2 / \sigma_0^2 \sim \chi_{n-1}^2$, and an S^2-chart is obtained by pivoting on this expression. The control limits are presented as Case1a in Table 1. For successive random samples of size n, this control chart can also be viewed as repeated tests of significance of the form $H_0 : \sigma^2 = \sigma_0^2$ vs. $H_1 : \sigma^2 \neq \sigma_0^2$. A control chart for S is obtained by taking the square root of these limits (Table 1, Case 1b). Both charts are presented in Guttman and Wilks [14].

Table 1: Univariate Dispersion Control Charts (Phase II)

Statistics	LCL	CL	UCL
1a. $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$	$\sigma_0^2 \chi^2_{n-1,1-(\alpha/2)} /(n-1)$	-	$\sigma_0^2 \chi^2_{n-1,\alpha/2} /(n-1)$
1b. $S = \sqrt{S^2}$	$\sigma_0 \sqrt{\chi^2_{n-1,1-(\alpha/2)} /(n-1)}$	-	$\sigma_0 \sqrt{\chi^2_{n-1,\alpha/2} /(n-1)}$
2. $S = \sqrt{S^2}$	$\max\{0, \sigma_0[c_2' - 3(1-c_2'^2)^{0.5}]\}$	$\sigma_0 c_2'$	$\sigma_0[c_2' + 3(1-c_2'^2)^{0.5}]$
3[1]. $S = \sqrt{S^2}$	$B_3\bar{S}$	$\bar{S}$	$B_4\bar{S}$
4. $V = \sqrt{n^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$	$\max\{0, \sigma_0[c_2 - 3n^{-0.5}(n-1-nc_2^2)^{0.5}]\}$ or $\max\{0, \sigma_0 B_1\}$	$\sigma_0 c_2$	$\sigma_0[c_2 + 3n^{-0.5}(n-1-nc_2^2)^{0.5}]$ or $\sigma_0 B_2$

[1]. Refer to Appendix.

The next two process dispersion charts do not utilize the complete distributional properties of the sample statistic but only the first two moments. It can be shown that $E(S) = \sigma_0 c_2'$ and $Var(S) = \sigma_0^2(1 - c_2'^2)$, where tables of c_2' for n=2,…,25 are given in Johnson and Leone [21]. Since most of the probability distribution of S is contained in the interval $E(S) \pm 3\sqrt{Var(S)}$, it seems reasonable that a control chart for S would have control limits corresponding to this interval (Table 1, Case 2). The lower control limit is replaced by 0 for $n < 6$. Since S is not normally distributed, these control limits can not be thought of as probability limits [11]. The same rationale applies in development of what is traditionally referred to as the sigma chart, except that V is used in place of S, where $V = \left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}\right]^{1/2}$ (Table 1, Case 3). Tables of c_2 and the B_1 and B_2 factors are available in Duncan [11].

Although the range chart and the standardized range chart are used widely to monitor univariate process dispersion, they will not discuss here since the multivariate analog is intractable.

For multivariate data, the first chart to be considered is the along of the S^2-chart (Table 1, Case a1), which is equivalent to repeated tests of significance of the form $H_0 : \sigma^2 = \sigma_0^2$ vs. $H_1 : \sigma^2 \neq \sigma_0^2$. Here $H_0 : \Sigma = \Sigma_0$ vs. $H_1 : \Sigma \neq \Sigma_0$. Using the asymptotic likelihood ratio test result [9], one would compute the following statistic for each random sample:

$$W = -pn + pn \ln n - n \ln(|\mathrm{A}|/|\Sigma_0|) + tr(\Sigma_0^{-1}\mathrm{A})$$

Where A denotes the sum of squares and cross-products matrix and tr is the trace operator. Note that $\mathrm{A} = (n-1)S$, where S is the ($p \times p$) sample variance-covariance matrix. If the value of this test statistic plots above the UCL=$\chi^2_{p(p+1)/2,\alpha}$, the process is deemed to be out of control. Refer to Table 2, Case 1. Korin [24] found that the asymptotic chi-square approximation, slightly modified, is quite good even for moderate n. he proposes an F approximation that appears to be better.

Table 2: Multivariate Dispersion Control Charts (Phase II)

Statistic	LCL	CL	UCL
1. W	-	-	$\chi^2_{p(p+1)/2,\alpha}$ (Approximate)
2. $\lvert S\rvert$	$\lvert\Sigma_0\rvert\left(\chi^2_{2n-4,1-(\alpha/2)}\right)^2 / [4(n-1)^2]$	-	$\lvert\Sigma_0\rvert\left(\chi^2_{2n-4,\alpha/2}\right)^2 / [4(n-1)^2]$
3[a]. $\lvert\Sigma\rvert$	$\lvert\Sigma_0\rvert\left(b_1 - 3b_2^{0.5}\right)$	$\lvert\Sigma_0\rvert b_1$	$\lvert\Sigma_0\rvert\left(b_1 + 3b_2^{0.5}\right)$

a. Refer to Multivariate Quality Control [2]

A widely used measure of multivariate dispersion is the sample generalized variance, denoted by $|S|$, where S is the ($p \times p$) sample variance-covariance matrix. The sample-generalized variance is the basis for the other multivariate dispersion charts to be considered. However, Johnson and Wichern [23] present three sample covariance matrices for bivariate data that all have the same generalized variance and yet have distinctly different correlation, r=0.8, 0.0, and -0.8. "Consequently, it is often desirable to provide more than the single number $|S|$ as a summary of S." Thus a control chart for $|S|$ should be used in conjunction with the univariate dispersion charts. Suppose there are two quality characteristics. Since $2(n-1)|S|^{1/2}/|\Sigma_0|^{1/2}$ is distributed as χ^2_{2n-4}. This yields the control chart limits in Table 2, Case 2.

When there are more than two quality characteristics, one may employ Anderson's asymptotic normal approximation [9] or the approximation suggested by Gnanadesikan and Gupta [13].

Another $|S|$-control chart can be constructed using only the first two moments of $|S|$ and the property that most of the probability distribution of $|S|$ is contained in the interval $E(|S|) \pm 3\sqrt{Var(|S|)}$. The control chart limits are presented in Table 2, Case 3.

The Multivariate Shewhart Charts is presented beginning with the simplest case of a multivariate quality control problem as follows: Consider a process that depends upon two characteristics X_1 and X_2 that are completely independent of each other. Suppose these two variables are normally distributed with mean zero and variance $\sigma_{11} = \sigma_{12} = 1$. If one sets an overall type I error α, then both of the critical regions depicted in Figure 1-a & 1-b can be used to monitor the process mean.

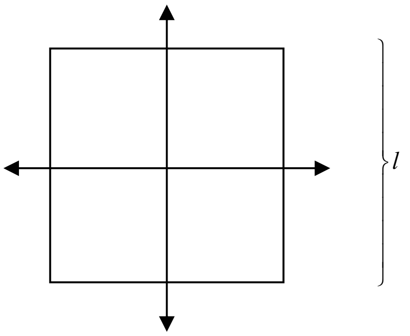


Fig 1-a: Square Critical Region

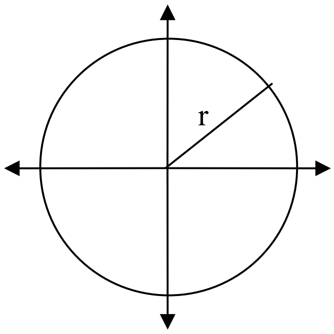


Fig 1-b: Circular Critical Region

Therefore, we can find a radius r and a length l such that

$$\int_{-l}^{l}\int_{-l}^{l} \tfrac{1}{2\pi} \exp(-\tfrac{1}{2}(x_1^2 + x_2^2))dx_2dx_1 = \int_{-r}^{r}\int_{-\sqrt{-r^2-x_1^2}}^{\sqrt{-r^2-x_1^2}} \tfrac{1}{2\pi} \exp(-\tfrac{1}{2}(x_1^2 + x_2^2))dx_2dx_1 = 1-\alpha \qquad (1)$$

For example, when $\alpha = 0.05$, the radius and the length will be $r = \sqrt{5.991} = 2.4477$ and $l = 2.237$ (See Table 3 for different values of α and p.) The value 5.991 is precisely $\chi^2_{2,0.95}$.

Table 3: Table of critical values for different values of P and α

$1-\alpha$	1	2	3	4	5
0.90	1.645	1.949	2.114	2.227	2.311
0.95	1.960	2.237	2.388	2.491	2.570
0.99	2.577	2.804	2.934	3.028	3.089
0.995	2.808	3.023	3.114	3.227	3.290
0.999	3.298	3.488	3.590	3.668	3.720

To control the process using the square as the critical region, we calculate the sample mean $(\sqrt{n}\overline{X}_1, \sqrt{n}\overline{X}_2)$. If the point lies inside the box with corners at $(\pm l, \pm l)$ then the process is in control. If not, either $\sqrt{n}\left|\overline{X}_1\right| > l$ which makes the first characteristic to be out-of-control and/or $\sqrt{n}\left|\overline{X}_2\right| > l$ which, makes the second characteristic out-of-control. In either case, the process would be considered to be out-of-control. However, since $\overline{X}_i > 0, i = 0,1$ then, the lower limit for either case is zero.
Now consider the matrix

$$R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad , \quad |\rho| < 1$$

and, let C be such that $CC' = R$. The random variable $Y = CX$ has a bivariate normal distribution with mean zero and correlation matrix R. If we let $\rho = Sin(2\theta)$, $|\theta| < \frac{\Pi}{4}$, some of the choices for matrix C are

$$C_l = \begin{bmatrix} 1 & 0 \\ Sin(2\theta) & Cos(2\theta) \end{bmatrix} \qquad C_u = \begin{bmatrix} Cos(2\theta) & Sin(2\theta) \\ 0 & 1 \end{bmatrix}$$

$$C_p = \begin{bmatrix} (Cos(\theta) + Sin(\theta))/\sqrt{2} & (-Cos(\theta) + Sin(\theta))/\sqrt{2} \\ (Cos(\theta) + Sin(\theta))/\sqrt{2} & (Cos(\theta) - Sin(\theta))/\sqrt{2} \end{bmatrix}$$

$$C_o = \begin{bmatrix} (\sqrt{1 - Sin(2\theta)} + \sqrt{1 + Sin(2\theta)})/2 & (-\sqrt{1 - Sin(2\theta)} + \sqrt{1 + Sin(2\theta)})/2 \\ (-\sqrt{1 - Sin(2\theta)} + \sqrt{1 + Sin(2\theta)})/2 & (\sqrt{1 - Sin(2\theta)} + \sqrt{1 + Sin(2\theta)})/2 \end{bmatrix}$$

According to relation between Sinus and Cosinus, we have,

$$C_o = \begin{bmatrix} Cos(\theta) & Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

Note that among these, the matrix C_o is symmetric. The decompositions C_l and C_u are called the lower and upper triangular decomposition of R (Chelovsky decomposition) and C_p is the one used in principal component decomposition. In general any choice of C for the decomposition of R is of the form $C = R^{1/2}T$ where,

$$T = \begin{bmatrix} Cos(t) & -Sin(t) \\ Sin(t) & Cos(t) \end{bmatrix}, -\pi \le t \le \pi$$

i.e. the general form of C is of the forms

$$C_t = \begin{bmatrix} Cos(\theta) & Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix} . \begin{bmatrix} Cos(t) & -Sin(t) \\ Sin(t) & Cos(t) \end{bmatrix} = \begin{bmatrix} Cos(\theta - t) & Sin(\theta - t) \\ Sin(\theta + t) & Cos(\theta + t) \end{bmatrix}$$

We denote this general form of the decomposition of the correlation matrix R as C_t. Note that when t = 0 we obtain the symmetric decompositions $C_o = R^{1/2}$ and when $t = \pm\theta$ we get the triangular decompositions C_l and C_u. When $t = \pi/4$, we get the decomposition $C_{\pi/4} = C_p$. The transformations $C'X$ (for any choice of C such that $CC' = R$), transforms the circle with radius r to the ellipse

$$X_1^2 + X_2^2 + 2\rho X_1 X_2 = r^2$$

The transformations $C_l X$, $C_u X$, $C_p X$, and $C_o X = R^{1/2} X$ will transform the square with corners $(\pm l, \pm l)$ to the regions $\Re_l$, $\Re_u$, $\Re_p$ and $\Re_o$ respectively. Note that in general, the transformation C_t is a composition of a rotation T and a transformation under $C_o = R^{1/2}$.

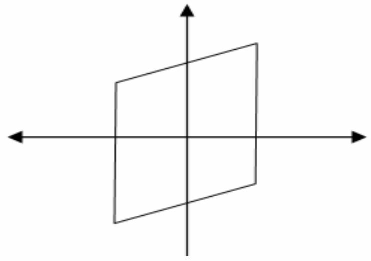


Fig 2-a: Critical Region $\Re_l$

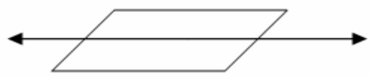


Fig 2-b: Critical Region $\Re_u$

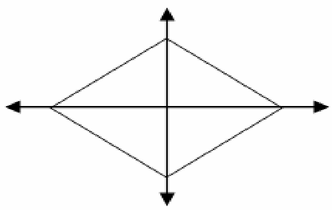


Fig 2-c: Critical Region $\Re_p$

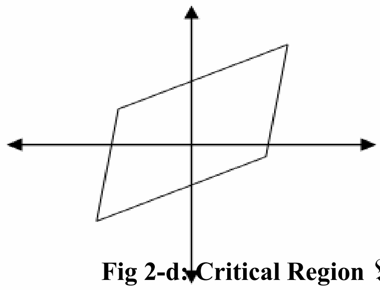


Fig 2-d: Critical Region $\Re_o$

If X has a bivariate normal distribution with correlation matrix R , the variable $Z = C_t^{-1} X$ has a standardized normal distribution with correlation matrix I. If we are only interested in the overall control of the process mean, that is, we are not concerned with the variables that cause the out-of-control condition, then for any given type I error α choose a region $\Re$ such that

$$\iint_{\Re} f(x_1, x_2) dx_1 dx_2 = 1 - \alpha$$

Where, $f(x_1, x_2)$ is the density function of a bivariate normal distribution with correlation matrix R .

Then, S_1, S_2 (With given n sample (Large sample) of size m) has a approximately bivariate normal distribution with:

$\mathrm{E}S_1 = C_4 \sigma_1$, $Var(S_1) = \sigma_1^2 (1 - C_4^2)$, $\mathrm{E}S_2 = C_4 \sigma_2$, $Var(S_2) = \sigma_2^2 (1 - C_4^2)$, and $Cov(S_1, S_2) = \sigma_{12} (1 - C_4^2)$, or has a approximately bivariate normal distribution with correlation matrix $R = Corr(S_1, S_2)$ such that

$$Var(X_1) = \sigma_1^2 ,\ Var(X_2) = \sigma_2^2 ,\ Cov(X_1, X_2) = \sigma_{12}$$

then, choose a decomposition C_t of R and take a sample of size n and calculate the statistic:

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = (1-c_4^2)^{-\frac{1}{2}} R^{-\frac{1}{2}} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

If the point $(k_1, k_2)'$ lies outside the region, $\Re$ we may conclude that the process is out-of-control. An equivalent procedure would be to find the image of the region $\Re$ under the transformation made by C_t to get $\Re_t$ and plot the point (S_1, S_2). If this point lies outside the region, $\Re_t$ we may conclude that the process is out-of-control.

But if we are concerned about which variable causes the out-of-control condition, we must choose a region $\Re_t$ whose image under some transformation C_t^{-1} is the square with corners at $(\pm l, \pm l)$. The adequate choice of $\Re_t$ and the adequate choice of C_t is needed in order to draw the right conclusions about the state of the process standard deviation. As far as Multivariate Statistical Quality Control (MSQC) is concerned, any transformation C_t can be used for the purpose of statistical control.

The above ideas can be extended to the case of $p > 2$. The following computational procedure can be used to, identify the errant variable(s).

PROCEDURE

Consider a process involving p characteristics $x_1, x_2, ..., x_p$ that we wish to control. Suppose the mean $\mu = (\mu_1, \mu_2, ..., \mu_p)$ variances $\sigma_1^2, \sigma_2^2, ..., \sigma_p^2$ and the correlation matrix R of this process, are known. Given n samples of size m taken from a multivariate normal population, we can normalize the data by subtracting the mean of each characteristic and dividing by the corresponding standard deviation. Therefore, we can assume that the sample is taken from a multivariate normal population with mean 0 and correlation, matrix R. To control the process standard deviation, first set an overall type I error α with individual α_i's satisfying the condition

$$1-\alpha = (1-\alpha_1)(1-\alpha_2)...(1-\alpha_p)$$

Set all α_i's to be equal if all variables are equally important. In this case the values for the individual type I errors is given by

$$\alpha_i = 1 - \sqrt[p]{1-\alpha}$$

Find the values $b_i = z_{\alpha_i}$. Find the $S' = (S_1, S_2, ..., S_p)$. calculate the value of the statistic:

$$K = (K_1, K_2, ..., K_p)' = (1-C_4^2)^{-\frac{1}{2}} R^{-\frac{1}{2}} (S)'$$

where, $S' = (S_1, S_2, ..., S_p)$ is the sample standard deviation of the n observations.

If $|Z_i| < z_{\alpha_i}$ for all i's, the process will be in-control. If on the other hand $|Z_i| > z_{\alpha_i}$ the process, will be out-of-control and the variable x_i is responsible for this out-of-control condition.

The above procedure can be displayed in a Chart, supplying information about the behavior of the individual characteristics in a multivariate process. Note that if p=1, we have the Shewhart S chart. To illustrate the above procedure, consider the following example.

EXAMPLE I

Consider the data for a Carton industrial, Rctcd and Cctcd (two characteristics for stiffness of carton) the standard values, either derived from a large amount of past data or selected by management to attain certain objectives, are

$$\sigma_{0x} = 2.838,\ \sigma_{0y} = 4.345,\ \rho = 0.84$$

In matrix notation, we have

$$R = \begin{bmatrix} 1 & 0.84 \\ 0.84 & 1 \end{bmatrix}$$

Where, R is the correlation matrix.
Let the overall type I error be α=0.05 with $z_{\alpha_1} = z_{\alpha_2}$ =2.237.
Setting $Sin(2\theta) = 0.84$, $\theta = 0.498642$ and

$$R^{½} = \begin{bmatrix} Cos(\theta) & Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

Therefore, square root of correlation matrix will be

$$R^{½} = \begin{bmatrix} 0.8782 & 0.4782 \\ 0.4782 & 0.8782 \end{bmatrix}$$

$$R^{-½} = \begin{bmatrix} 1.61860 & -0.88140 \\ -0.88140 & 1.61860 \end{bmatrix}$$

The values (K_1, K_2), are obtained by the formula

$$2.93105 R^{-½} \begin{bmatrix} \dfrac{S_x - C_4\sigma_x}{\sigma_x} \\ \dfrac{S_y - C_4\sigma_y}{\sigma_y} \end{bmatrix}$$

Given 40 samples of size 5, the values (S_{xh}, S_{yh}), and h=1, 2... 40 along with (S_{xh}, S_{yh}), (K_{1h}, K_{2h}) (for h = 1, 2… 40) are presented in the Table 4.

Another multivariate control chart for despersion is $|S|$-chart [2], that upper and lower limits are

$$\text{LCL}=|\Sigma_0|\frac{(\chi^2_{2n-4,1-\alpha/2})^2}{4(n-1)^2} \qquad \text{UCL}=|\Sigma_0|\frac{(\chi^2_{2n-4,\alpha/2})^2}{4(n-1)^2}$$

$$\Sigma_0 = \begin{bmatrix} 8.1 & 10.36 \\ 10.36 & 18.88 \end{bmatrix}, \ \chi^2_{6,0.975} = 14.45$$

Then,

$$\text{LCL}=1.07089 \qquad \text{UCL}=146.036$$

And,

$$|S|=3.09020$$

According to multivariate charts for standard deviation $|S|$-chart and recent approach, standard deviation of the process is in control.

Table 4: Sample data for Example I, along with the correspondence K_{ph} values

n	S_{xh}	S_{yh}	K_{1h}	K_{2h}
1	1.35839	2.23284	-0.56803	-1.78449
2	0.87563	1.12722	-1.08801	-1.87215
3	1.50713	1.48628	-0.40782	-1.75748
4	2.64747	2.11528	0.82045	-1.55041
5	1.11596	1.23241	-0.82915	-1.82851
6	1.08588	0.95356	-0.86155	-1.83397
7	1.31293	0.96703	-0.61699	-1.79274
8	2.06162	2.09190	0.18943	-1.65679
9	0.90037	1.18799	-1.06136	-1.86766
10	1.63316	1.29558	-0.27207	-1.73459
11	0.80697	0.63251	-1.16196	-1.88462
12	2.62240	2.79263	0.79344	-1.55496
13	1.94903	2.17228	0.06816	-1.67724
14	2.60508	1.71951	0.77479	-1.55811
15	1.73163	1.56012	-0.16601	-1.71671
16	1.16484	2.42740	-0.77650	-1.81963
17	2.27074	3.15219	0.41467	-1.61882
18	2.53400	2.82669	0.69823	-1.57102
19	1.22268	0.84206	-0.71420	-1.80913
20	1.26810	1.54573	-0.66528	-1.80088
21	1.51699	1.12490	-0.39720	-1.75569

22	2.20065	1.79217	0.33918	-1.63155
23	1.99978	1.11309	0.12282	-1.66802
24	1.21487	0.70610	-0.72261	-1.81055
25	2.65055	3.70319	0.82376	-1.54985
26	2.21610	1.53397	0.35582	-1.62874
27	3.52817	2.66840	1.76905	-1.39049
28	2.10705	1.80385	0.23836	-1.64854
29	1.97382	0.64751	0.09486	-1.67274
30	2.37516	1.68241	0.52714	-1.59986
31	1.74485	2.02604	-0.15177	-1.71431
32	1.60755	1.19873	-0.29965	-1.73925
33	1.26172	2.18296	-0.67215	-1.80204
34	1.83433	0.76881	-0.05539	-1.69807
35	1.35289	1.03336	-0.57395	-1.78549
36	3.28812	3.69191	1.51049	-1.43408
37	2.05870	2.45654	0.18628	-1.65732
38	2.99180	3.10227	1.19133	-1.48789
39	0.98688	0.46198	-0.96818	-1.85195
40	2.37587	2.59248	0.52791	-1.59973

EXAMPLE II

As an example of a higher dimensional problem, consider the data for a carton industrial, grammage(X_1), cobb(X_2), burst(X_3) and density(X_4) (four characteristics for stiffness of carton) The standard values, either derived from a large amount of past data or selected by management to attain certain objectives, are. In this example p=4, and the variances, and the correlation matrix are given as follows:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{44}\end{bmatrix} = \begin{bmatrix}319.436\\ 20286.69\\ 3616.170\\ 12425.384\end{bmatrix}, \quad R = \begin{bmatrix}1 & -0.234 & 0.315 & 0.155\\ -0.234 & 1 & -0.647 & -0.518\\ 0.315 & -0.647 & 1 & 0.480\\ 0.155 & -0.518 & 0.480 & 1\end{bmatrix}$$

The inverse of the square root of the matrix R is given by

$$R^{-1/2} = \begin{bmatrix}1.04331 & 0.05176 & -0.14440 & -0.01270\\ 0.05176 & 1.30131 & 0.39181 & 0.23844\\ -0.14440 & 0.39181 & 1.30003 & -0.18559\\ -0.01270 & 0.23844 & -0.18559 & 1.16030\end{bmatrix}$$

From Table 3, for p=4 and α=0.05, the critical value is 2.491.

Given 40 samples of size 5, the values $(S_{1h}, S_{2h}, S_{3h}, S_{4h})$, such that h=1,2...40 along with are presented in the Table 5.

Table 5: Sample data for Example II

h	S_{1h}	S_{2h}	S_{3h}	S_{4h}
1	3.335	11.491	54.245	27.743
2	5.584	137.058	13.229	37.634
3	6.314	7.508	13.874	178.718
4	2.550	13.736	23.875	167.471
5	5.587	11.802	38.308	22.874
6	4.329	4.367	17.103	57.839
7	6.753	20.799	4.183	29.254
8	5.605	67.409	10.954	63.581
9	21.730	47.574	26.315	33.856
10	6.489	36.578	42.220	82.597
11	6.908	16.534	20.187	33.294
12	4.143	12.213	34.351	71.123
13	8.220	26.556	15.572	84.118
14	4.456	14.589	42.778	47.741
15	3.407	5.399	9.618	37.043
16	3.948	11.685	38.987	39.004
17	2.123	20.916	15.411	27.770
18	5.869	3.262	24.884	27.763
19	15.989	21.446	9.354	33.038
20	2.754	30.229	6.124	30.080
21	3.687	16.051	12.042	42.753
22	8.446	37.545	8.944	47.366
23	25.904	40.077	23.822	56.679
24	1.354	10.014	6.708	31.856
25	2.478	162.822	17.464	22.753
26	3.210	75.678	29.283	28.780
27	10.561	6.965	25.884	12.915
28	0.766	2.958	16.583	16.832
29	6.205	23.524	11.180	49.835
30	5.670	18.677	21.909	32.384
31	4.123	18.358	23.076	14.387
32	11.063	1.534	25.243	44.505
33	3.335	10.544	22.361	48.073
34	5.584	3.780	6.519	55.908
35	6.314	42.073	10.368	17.401
36	2.550	5.745	34.946	66.651
37	5.587	2.751	12.550	16.483
38	4.329	36.652	33.466	17.421
39	6.753	51.752	19.243	30.369
40	5.605	1.117	14.405	20.376

The values of (K_1, K_2, K_3, K_4) are computed and presented in Table 6.

Table 6: The correspondence K_{ph} values for Example II

h	K_{1h}	K_{2h}	K_{3h}	K_{4h}
1	-2.53993	-4.23017	-0.65044	-3.09095
2	-1.736	-1.56984	-2.33872	-1.80666
3	-1.80074	-4.13596	-4.04825	1.8551
4	-2.50476	-3.88064	-3.22024	1.45986
5	-2.04049	-4.53765	-1.68744	-3.09851
6	-2.12607	-4.93318	-3.232	-1.87367
7	-1.59334	-4.89857	-3.83615	-2.5534
8	-1.79924	-3.31556	-3.1716	-1.33617
9	0.84038	-3.60287	-2.59488	-2.51302
10	-1.90726	-3.41731	-1.55259	-1.19192
11	-1.68536	-4.68048	-2.87975	-2.59617
12	-2.27538	-4.31194	-2.13616	-1.5855
13	-1.4347	-4.17041	-3.37051	-0.95722
14	-2.2708	-4.23131	-1.47628	-2.36416
15	-2.22309	-5.18672	-3.57468	-2.4335
16	-2.33121	-4.44059	-1.68526	-2.60965
17	-2.46393	-4.72958	-3.0068	-2.69004
18	-1.90848	-4.98968	-2.63752	-2.87038
19	-0.05006	-4.68037	-3.74044	-2.50079
20	-2.28145	-4.63772	-3.54645	-2.49116
21	-2.18281	-4.817	-3.36968	-2.22953
22	-1.3254	-4.23125	-3.52791	-1.96519
23	1.55648	-3.67271	-3.02354	-1.83957
24	-2.54722	-5.16865	-3.64794	-2.53852
25	-2.26483	-0.9187	-1.71644	-2.16614
26	-2.3176	-2.9826	-1.71685	-2.51828
27	-1.10383	-4.92469	-2.58293	-3.32407
28	-2.71983	-5.2682	-2.99181	-3.11967
29	-1.74032	-4.56757	-3.45825	-1.97422
30	-1.90671	-4.60639	-2.71959	-2.62641
31	-2.17394	-4.71864	-2.52374	-3.18445
32	-1.02976	-4.88004	-2.83341	-2.38209
33	-2.32331	-4.73703	-2.7778	-2.18684
34	-1.83682	-5.1525	-3.92774	-1.84234
35	-1.68537	-4.28877	-3.20443	-2.86572
36	-2.55752	-4.51534	-2.09105	-1.75575
37	-1.86669	-5.31205	-3.36152	-3.1049
38	-2.19336	-4.00952	-1.7375	-3.09653
39	-1.66675	-3.77504	-2.63766	-2.50374
40	-1.87971	-5.29581	-3.27656	-3.01095

If, $|K_{1h}|, |K_{2h}|, |K_{3h}|$ and $|K_{4h}|$ are greater than 2.491, the first, second, third, and fourth variables are out-of control for h-observations, respectively.

But, another multivariate control chart for despersion is $|S|$-chart [2], that upper and lower limits are

$$\text{LCL}=|\Sigma_0|\frac{(\chi^2_{2n-4,1-\alpha/2})^2}{4(n-1)^2} \qquad \text{UCL}=|\Sigma_0|\frac{(\chi^2_{2n-4,\alpha/2})^2}{4(n-1)^2}$$

$$\chi^2_{6,0.995}=18.5476$$

Then,

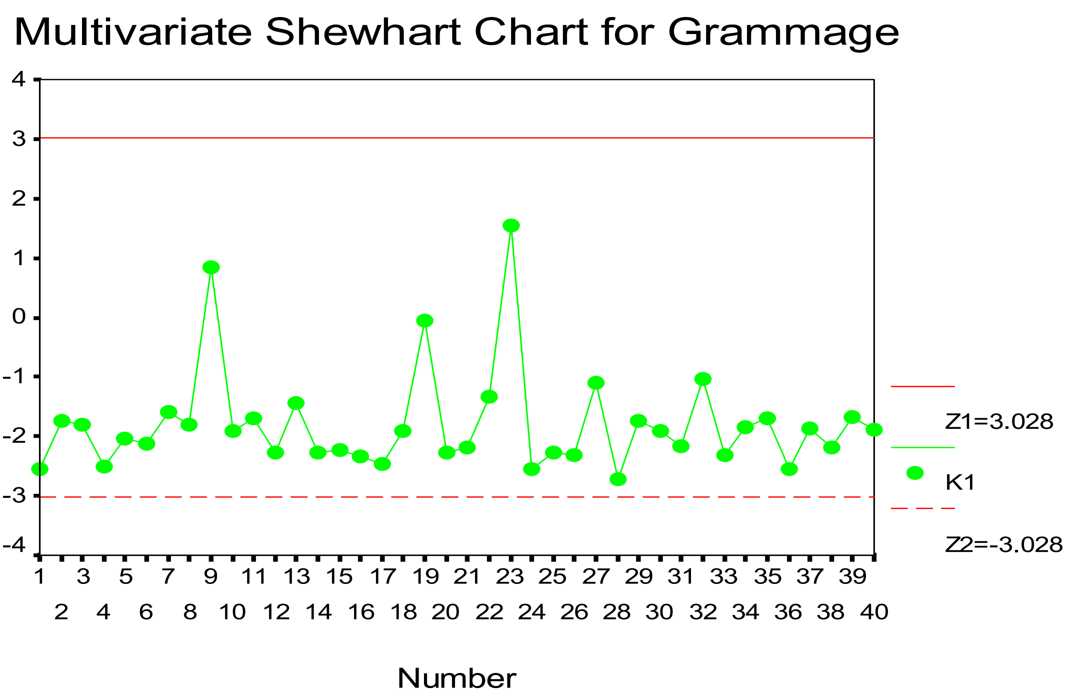
$$\text{LCL}=7.5299287\times10^7 \qquad \text{UCL}=5.673138\times10^{14}$$

And,

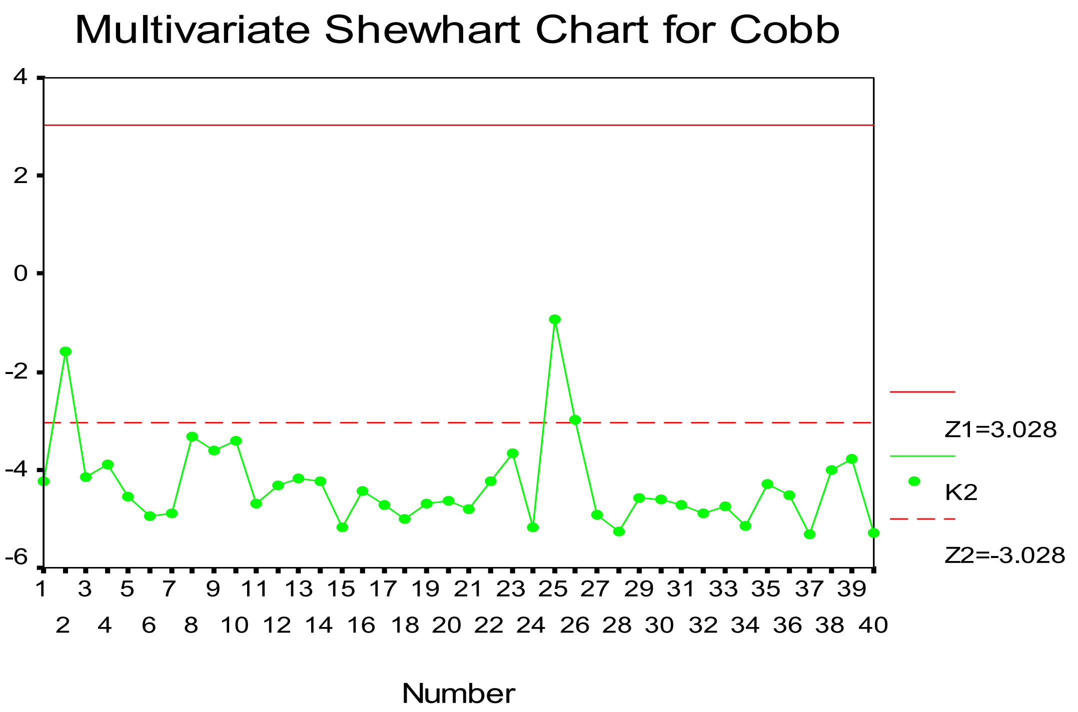
$$|S|=4.319343\times10^{13}$$

According to multivariate charts for standard deviation $|S|$-chart standard deviation of the process is in control, and recent approach, shows the standard deviation of Cobb, Burst and Density is out of control for some cases that present in following graphs (Graph1-4).

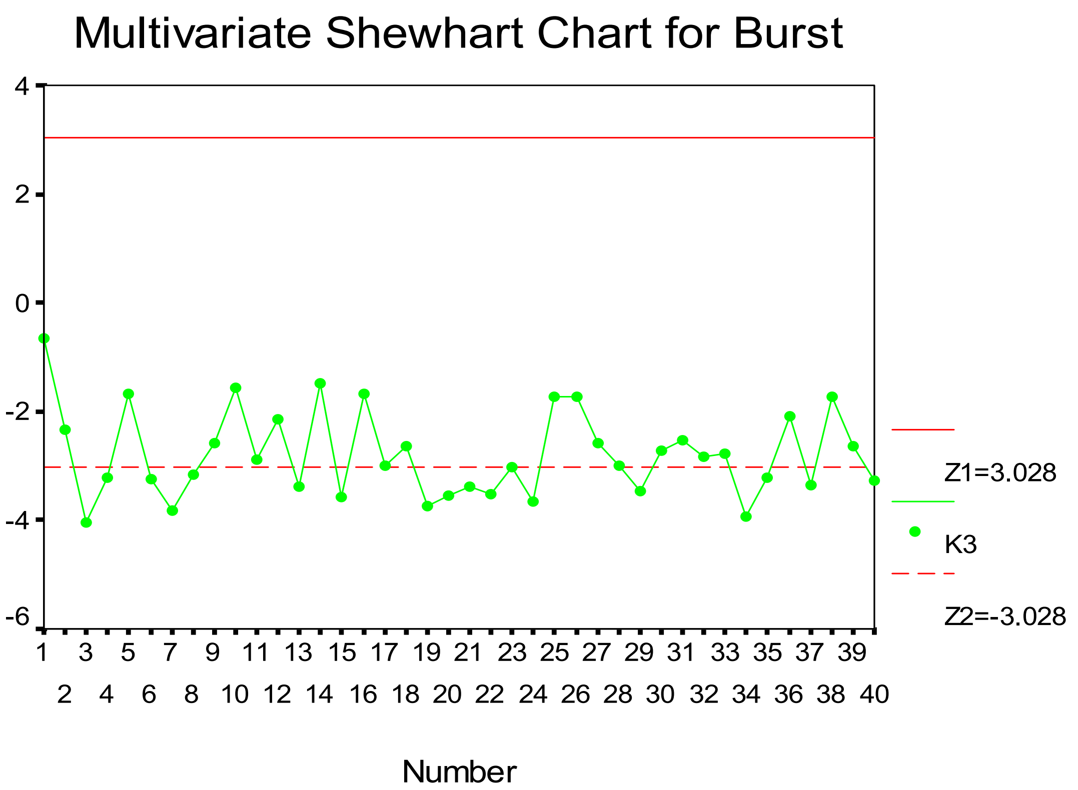
Graph 1



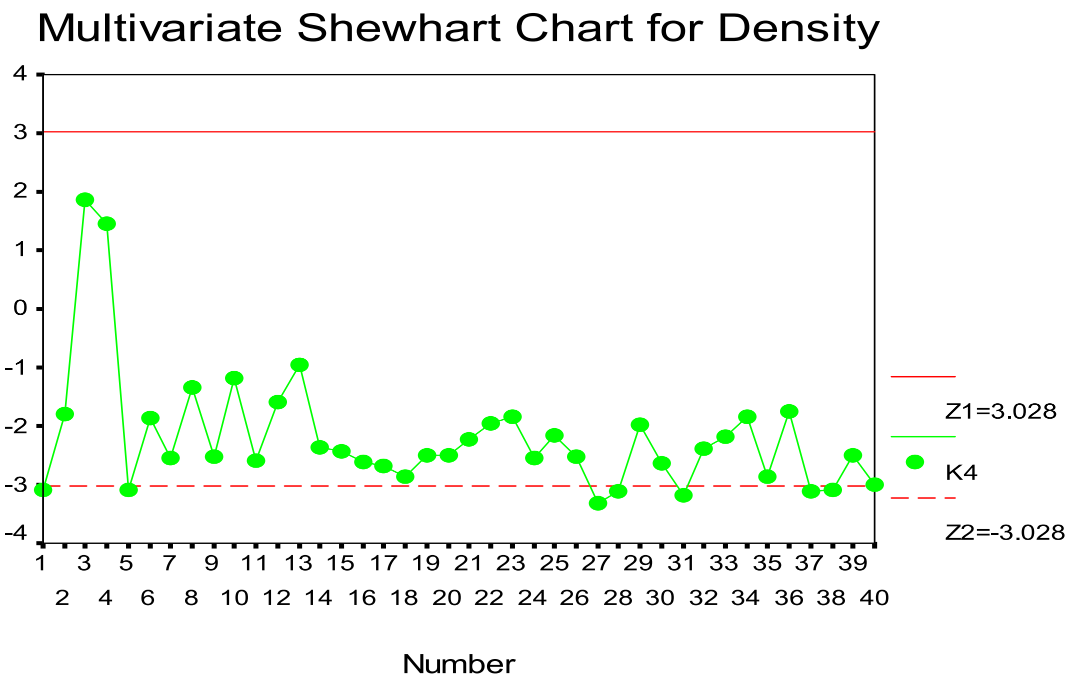
Graph 2



Graph 3



Graph 4



CONCLUSIONS

Many quality control related problems are multivariate in nature. Treating such problems as a series of independent univariate problems for monitoring purposes may lead to incorrect conclusions. In this paper, we have suggested a multivariate chart for standard deviation of process, MS Chart, which is based on the extension of the univariate Shewhart S Chart. This charting technique has the advantage of directing the investigators to the possible cause(s) of an out-of-control signal. We believe that a good extension of this research would be in developing corresponding charting techniques for the Range structure of multivariate processes.

Appendix

We can easily proof that:

$$C_4 = \left(\frac{2}{n-1}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

Where, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$,

$$B_3 = 1 - \frac{3}{C_4}\sqrt{1-C_4^2}\ ,\ B_4 = 1 + \frac{3}{C_4}\sqrt{1-C_4^2}$$

If, given m samples of size n then:

$LCL_S = B_3\overline{S}$, $CL_S = \overline{S}$, $UCL_S = B_4\overline{S}$ (Shewhart Chart for Standard Deviation)

Where, $\overline{S} = \dfrac{S_1 + S_2 + ... + S_m}{m}$.

Factors for Constructing Variables Control Charts

Observations in Sample, n	**Factors for Center Line**		**Factors for Control Limits**	
	C_4	$1/C_4$	B_3	B_4
2	0.7979	1.2533	0	3.267
3	0.8862	1.1248	0	2.568
4	0.9213	1.0854	0	2.266
5	0.9400	1.0638	0	2.089
6	0.9515	1.0510	0.030	1.970
7	0.9594	1.0423	0.118	1.882
8	0.9650	1.0363	0.185	1.815
9	0.9693	1.0317	0.239	1.761
10	0.9727	1.0281	0.284	1.716
11	0.9754	1.0252	0.321	1.679
12	0.9776	1.0229	0.354	1.646
13	0.9794	1.0210	0.382	1.618
14	0.9810	1.0194	0.406	1.594
15	0.9823	1.0180	0.428	1.572
16	0.9835	1.0168	0.448	1.552
17	0.9845	1.0157	0.466	1.534
18	0.9854	1.0148	0.482	1.518
19	0.9862	1.0140	0.497	1.503
20	0.9869	1.0133	0.510	1.490
21	0.9876	1.0126	0.523	1.477
22	0.9882	1.0119	0.534	1.466
23	0.9887	1.0114	0.545	1.455
24	0.9892	1.0109	0.555	1.445
25	0.9896	1.0105	0.565	1.435

For $n > 25$: $C_4 = \dfrac{4(n-1)}{4n-3}$, $B_3 = 1 - \dfrac{3}{C_4\sqrt{2(n-1)}}$, $B_4 = 1 + \dfrac{3}{C_4\sqrt{2(n-1)}}$

REFERENCES

1. ALT, F. B., Goode, J. J., Montgomery, D. C., and Wadsworth, H. M. (1973). "Variables Control Charts for Multivariate Data", Proceedings of the 133rd Annual Meeting of the American Statistical Association.
2. ALT, F. B. (1985). "Multivariate Quality Control", in Encyclopedia of Statistical Sciences 6, edited by S. Kotz and N. L. Johnson. John Wiley & Sons, New York, NY.
3. ALT, F. B. (1973). "Aspects of Multivariate Control Charts." M.S. thesis, Georgia Institute of Techonology, Atlanta.
4. ALT, F. B. (1976). Manag. Sci., 22, 1167-1168.
5. ALT, F. B. (1982). Ann. Tech. Conf. Trans. ASQC, 886-893.
6. ALT, F. B., Goode, J. J., and Wadsworth. H. M. (1976). Ann. Tech. Conf. Trans., ASQC, 170-176.
7. ALT, F. B. and Deutsch, S. J. (1978). Proc. Seventh Ann. Meeting, Northeast Regional Conf., Amer. Inst. Decision Sci., 109-112.
8. ALT, F. B. and Deutsch, S. J., and Walker, J. W. (1977). Ann. Tech. Conf. Trans., ASQC, 360-369.
9. Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis. Wiley, New York.
10. Doganaksoy, N., Faltin, F. W., and Tucker, W. T. (1991). "Identification of Out of Control Quality Characteristics in a Multivariate Manufacturing Environment" Communications in Statistics - Theory and Methods 20 , pp. 2775-2790.
11. Duncan, A. J. (1974). Quality Control and Industrial Statistics, 4th ed. Richard D. Irwin, Homewood, IL.
12. Ghare, P. M., and Torgersen, P. E. (1968) . J. Ind. Eng., 19, 269-272.
13. Gnanadesikan, M. and Gupta, S. S. (1970). Technometrics, 12, 103-117.
14. Guttman, I. and Wilks,S. S. (1965). Introductory Engineering Statistics, Wiley, New York.
15. Hawkins, D. M. (1991). "Regression Adjustment for Variables in Multivariate Quality ControL", Journal of Quality Technology 25, pp. 175-182.

16. Hotelling, H. (1947). "Multivariate Quality Control", in Techniques of Statistical Analysis, edited by Eisenhart, Hastay, and Wallis. McGraw-Hill, New York, NY.
17. Jackson, J. E. (1956). "Ind. Qual. Control", 12(7) 2-6.
18. Jackson, J. E. (1959). Technometrics, 1, 359-377.
19. Jackson. J. E. (1980). "Principle Components and Factor Analysis: Part 1- Principal Components", Journal of Qualitv Technology 12 , pp. 201-213.
20. Jackson. J. E. (1985). "Multivariate Quality Control", Communications in Statistics-Theory and Methods 14, pp. 2657-2688.
21. Jackson. J. E., and Morris. R. H. (1957). J. Amer. Statist. Ass., 52, 186-199.
22. Johnson, N. L. and Leeone, F. C. (1977). Statistics and Experimental Design in Engineering and the Physical Sciences, Vol. 1, 2nd ed., Wiley, New York.
23. Johnson, R. A. and Wichern, D. W. (1982). Aplied Multivariate Statistical Analysis. Prentice Hall, Englewood Cliffs, NJ.
24. Korin, B. P. (1968). Biometrika, 55, 171-178.
25. Murphy. B. J. (1987). "Selecting out of control variables with the T^2 multivariate quality control procedure", The Statistician 36, pp. 571-583.
26. Hayter. A. J., and Kwok-Leung. Tsui. (1994). "Identification and Quantification in Multivariate Quality Control Problems", Journal of Quality Technology 26, pp. 197-208.
27. Runger. G. C. (1996). "Projections and the U^2 Multivariate Control Chart", Journal of Quality Technology 2, pp. 313- 319.