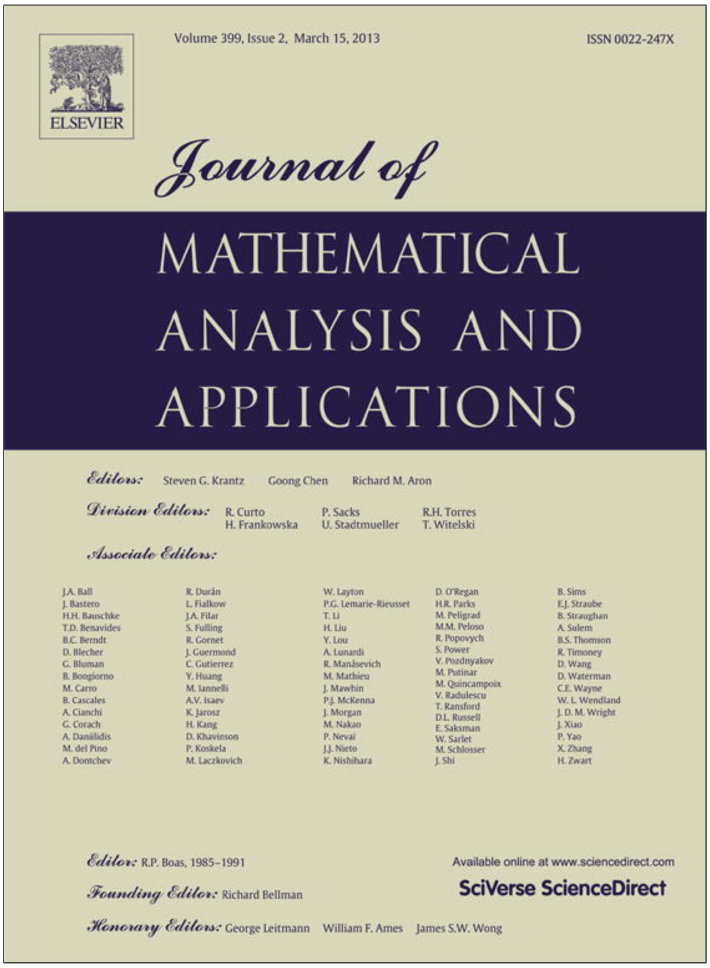
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Discontinuous piecewise quadratic Lyapunov functions for planar piecewise affine systems

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ABSTRACT

For planar piecewise affine systems, this paper proposes sufficient stability conditions based on discontinuous Lyapunov functions. The monotonicity condition for discontinuous functions at switching instants is presented based on the behavior of state trajectories on the switching surfaces. First, the stability conditions are derived for a typical multiple Lyapunov function and then these conditions are formulated as a set of linear matrix inequalities for piecewise quadratic Lyapunov functions. The implementation of the proposed method is illustrated by an example.

1. Introduction

Hybrid systems characterize classes of dynamical systems which consist of both continuous and discrete dynamics. Since hybrid models can adequately describe the behavior of many physical systems, there is a great interest in studying hybrid systems. The class of piecewise affine (PWA) systems is a general and well-studied class of hybrid systems. It consists of a set of affine subsystems and a switching law that selects the active subsystem based on the sub-region that the states of the system belong to. A large class of nonlinear systems in engineering applications can be approximated by PWA systems [1]. Also, PWA systems are equivalent to several classes of hybrid systems [2,3], whereas PWA systems allow using tractable mathematical tools for analysis and synthesis. Thus, piecewise affine systems provide a powerful means for analysis and synthesis of many nonlinear systems. A wide range of PWA systems is continuous. To name a few, the approximated PWA systems obtained from modeling a nonlinear system are continuous [4–6]. Also some PWA systems which describe physical nonlinearities like dead-zone, saturation and hysteresis are continuous. In the recent decade, the stability issues of PWA systems have drawn a lot of attention [7–10]. Due to hybrid behavior of PWA systems, the analysis of even simple PWA systems can lead to an NP hard problem [11]. The existence of a single quadratic Lyapunov function for all subsystems of PWA system can ensure the quadratic stability of the switched system. In order to find less conservative stability conditions for hybrid systems, the theorem of multiple Lyapunov functions is presented in [12]. As the behavior of the system at switching instants must be known priori, the application of this theorem is difficult. However, for PWA systems, this theorem is relaxed by posing the continuity condition of the Lyapunov function on the boundaries of sub-regions. In the last decades, several multiple Lyapunov functions have been proposed based on the mentioned relaxation method. To name a few, in [13], piecewise quadratic (PWQ) Lyapunov functions are introduced for continuous-time PWA systems. [14–16] present piecewise affine Lyapunov functions and in [17], sufficient conditions for the stability of piecewise linear (PWL) systems are proposed using homogeneous polynomial Lyapunov functions. Also an extension for discontinuous PWQ Lyapunov functions is presented in [18], however it is only for PWA systems in which the switching surfaces are traversed by trajectories of the system in the known directions.

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This paper presents stability conditions based on discontinuous multiple Lyapunov functions, for planar continuous PWA systems. As the set of continuous functions is the subset of the set of discontinuous functions, it is clear that by relaxing the continuity of Lyapunov functions on switching surfaces, the search for Lyapunov functions is done in a bigger set of functions and so the conservativeness in stability analysis decreases considerably. In the proposed method, for monotonicity of the Lyapunov function at switching instants, one does not need to have any information on the trajectories of the system at switching surfaces and the monotonicity condition is only presented based on the vector field of the system. For discontinuous PWQ Lyapunov functions, the sufficient conditions for the stability of PWA systems are formulated as linear matrix inequalities (LMIs) which can be solved using a standard LMI solver.

The remainder of this paper is organized as follows. The notation used throughout the text and some preliminary results are presented in Section 2. Section 3 deals with the discontinuous functions and monotonicity of these functions at switching surfaces. The conditions for the stability of hybrid systems via discontinuous Lyapunov functions are given in Section 4. These conditions are formulated for discontinuous PWQ Lyapunov functions in Section 5. Section 6 is dedicated to Simulation results and finally, some concluding remarks are drawn in Section 7.

2. Notation and preliminaries

Definition 1 (*Polyhedron*)**.** A convex set X in the d-dimensional space which defined as $X = \left\{x \in R^d | a^T x \geq b\right\}$ is called a polyhedron with $a \in R^{d\times n}$ and $b \in R^n$. The mentioned inequality means the element-wise inequality.

Definition 2 (*Polyhedral Partition*)**.** A collection of polyhedron $X_i \subseteq X, i \in I \subset N$, is the polyhedral partition of the polyhedron X, if and only if $\bigcup_{i\in I} \bar{X}_i = X$ and $X_i \bigcap X_j = \phi,\ \forall i, j \in I,\ i \neq j$.

The state space equations describing a planar PWA system in $X \subset R^2$ are

$$\dot{x}(t) = A_i x(t) + a_i \quad x \in \bar{X}_i,\ i \in I \tag{1}$$

where $x(t) \in X$ is the state vector and A_i and a_i are constant matrix/vector of suitable dimensions. Let I be the set of mode indices and $\{X_i\}_{i\in I}$ be a polyhedral partition of X into a number of cells with $\bigcup_{i\in I} \bar{X}_i = X$ and $X_i \bigcap X_j = \phi,\ \forall i,\ j \in I,\ i \neq j$ where $\bar{X}_i$ denotes the closure of X_i. Suppose the countable set I, *card* $\{I\}$ denotes the cardinality of I. It is assumed that for $i \in I_0, I_0 = \left\{i \in I : 0 \in \bar{X}_i\right\}$, the origin is the only equilibrium point for the corresponding subsystem and for $i \notin I_0$, the corresponding subsystem has not any equilibrium point in X_i.

Since the cells are polyhedron, we have,

$$\bar{X}_i = \left\{x \in R^2 : E_i x \geq e_i\right\}, \quad i \in I \tag{2}$$

where E_i and e_i are constant matrix/vector. A parametric description of the boundary between two regions X_i and X_j where $\bar{X}_i \bigcap \bar{X}_j \neq \phi$, can be described as

$$\bar{X}_i \bigcap \bar{X}_j \subseteq \left\{x | x = F_{ij} s + f_{ij},\ s \in R\right\}. \tag{3}$$

By setting $\bar{x} = \begin{bmatrix} x^T & 1\end{bmatrix}^T$, (1)–(3) can be written in the more compact form

$$\dot{\bar{x}} = \bar{A}_i \bar{x} \quad x \in \bar{X}_i \tag{4}$$

$$\bar{X}_i = \left\{x \in R^2 : \bar{E}_i \bar{x} \geq 0\right\} \tag{5}$$

$$\bar{X}_i \bigcap \bar{X}_j \subseteq \left\{x | \bar{x} = \bar{F}_{ij}\bar{s},\ \bar{s} = [s^T \quad 1]^T,\ s \in R\right\} \tag{6}$$

where

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \qquad \bar{E}_i = \begin{bmatrix} E_i & -e_i \end{bmatrix}$$

$$\bar{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{bmatrix}.$$

For boundaries defined in (3), if $F_{ij} \neq 0$, then the boundary is a part of a line and if $F_{ij} = 0$, then the boundary is a point. For $\bar{X}_i \bigcap \bar{X}_j \neq \phi$ with $F_{ij} \neq 0$ (linear boundary), we can define $\bar{C}_{ij} = \begin{bmatrix} C_{ij} & c_{ij} \end{bmatrix}$and $S_{ij} = \left\{x | \bar{C}_{ij}\bar{x} = 0\right\}$ in which C_{ij} is the normal vector of S_{ij} (a vector perpendicular to S_{ij}) with the direction from X_i to X_j such that $\bar{X}_i \bigcap \bar{X}_j \subseteq S_{ij}$.

Consider the hybrid system:

$$\dot{x}(t) = f_i(x) \quad x \in \bar{X}_i,\ i \in I \tag{7}$$

where f_i is a continuous function and X_i is a polyhedral region which belongs to R^2. In the following, at first, discontinuous Lyapunov functions and the monotonicity condition of discontinuous functions on boundaries are discussed for system (7) and then these results are regenerated for PWA system (4). Note that the hybrid system (7) is assumed to be continuous, so based on the continuity of the system, the vector fields of the neighbor subsystems on the common boundary are the same or $f_i(x) = f_j(x)$ for $x \in \bar{X}_i \bigcap \bar{X}_j$.

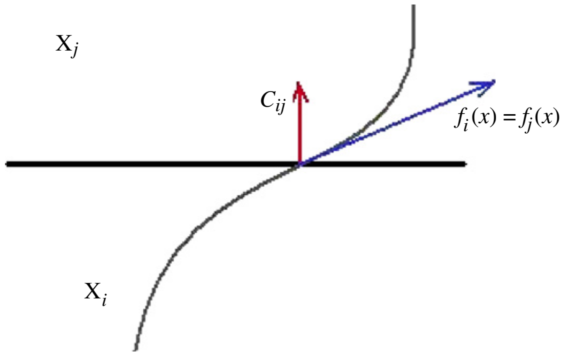

Fig. 1. The behavior of state trajectory on linear boundary.

3. Discontinuous functions

In this paper we introduce a special class of discontinuous functions and in the rest of the paper we refer to these functions as the candidate Lyapunov functions. Then, by using the behavior of the vector field (7) on the boundaries of the sub-regions, the monotonicity condition of this class of functions at switching instants is presented.

Consider continuous function $V_i(x) : \bar{X}_i \to R,\ i \in I$. A discontinuous function $V(x)$ is defined as

$$V(x) = \begin{cases} V_i(x) & x \in X_i \\ \min\ \{V_i(x)\}_{i\in I_x} & x \in \bigcap\limits_{i\in I_x} \bar{X}_i \end{cases} \tag{8}$$

where $I_x = \left\{i | \, x \in \bar{X}_i\right\}$.

Suppose for $x \in \bar{X}_i \bigcap \bar{X}_j \subseteq S_{ij},\ \forall i,\ j \in I_x$ where $F_{ij} \neq 0$,

$$V_i(x) - V_j(x) = \omega_{ij}^1 C_{ij} f_i(x) + \omega_{ij}^2 C_{ij} f_j(x) \tag{9}$$

where $\omega_{ij}^1,\ \omega_{ij}^2,\ i, j \in I_x$ are nonnegative scalars. The properties that (9) dedicates to $V(x)$ will be described in Lemmas 1–4.

Lemma 1. *Suppose at $t = t_0$, the state trajectory of system (7) crosses the boundary at $x(t_0) \in \bigcap_{i\in I_x} \bar{X}_i$ and goes to X_j. If card $\left\{I_{x(t_0)}\right\} = 2$ and (9) holds, then*

$$\lim_{t\to t_0^+} V(x(t)) \leq \lim_{t\to t_0^-} V(x(t)). \tag{10}$$

Proof. If $x \in \bigcap_{i\in I_x} \bar{X}_i$ and *card* $\{I_x\} = 2$, it means that the boundary is a part of a line. This situation is drawn in Fig. 1.

Assume that the trajectory goes from X_i to $X_j,\ i, j \in I_x$ and $i \neq j$, so we have

$$\lim_{t\to t_0^-} V(x(t)) = V_i(x(t_0))$$
$$\lim_{t\to t_0^+} V(x(t)) = V_j(x(t_0)).$$

From Fig. 1, it is concluded that

$$C_{ij} f_i(x(t_0)) = C_{ij} f_j(x(t_0)) > 0.$$

By (9),

$$V_i(x(t_0)) \geq V_j(x(t_0)).$$

So (10) is verified. $\square$

This lemma implies that $V(x)$ decreases at linear boundary along state trajectory.

Lemma 2. *Suppose at $t = t_0$, the state trajectory of system (7) is tangent to the boundary at $x(t_0) \in \bigcap_{i\in I_x} \bar{X}_i$ which card $\left\{I_{x(t_0)}\right\} = 2$. If (9) holds, then*

$$V_i(x(t_0)) = V_j(x(t_0)), \quad i, j \in I_x,\ i \neq j. \tag{11}$$

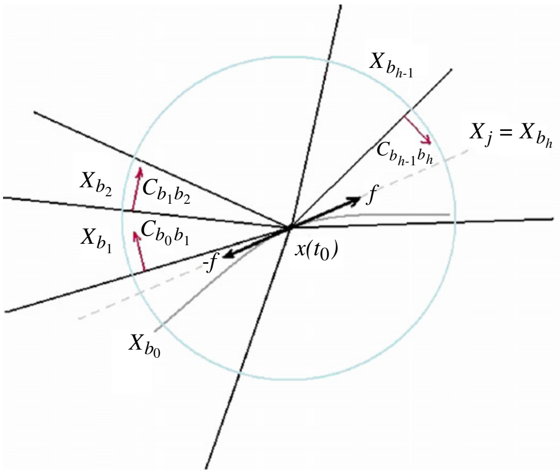

Fig. 2. The behavior of state trajectory on boundary point.

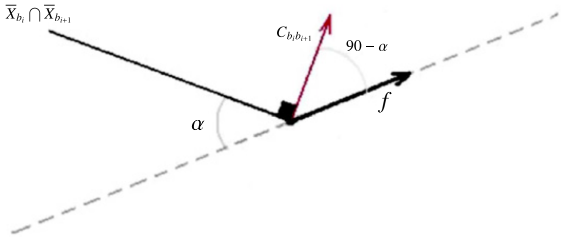

Fig. 3. The angle between vector field and normal vectors of boundaries.

Proof. Since the trajectory is tangent to the boundary at $x(t_0)$, so the vector field coincides the boundary at $x(t_0)$ and we have

$$C_{ij} f_i(x(t_0)) = C_{ij} f_j(x(t_0)) = 0.$$

Using (9), (11) is verified. □

This lemma implies that $V(x)$ is continuous where the state trajectory coincides to the boundary.

Lemma 3. *Suppose at $t = t_0$, the state trajectory of system (7) crosses the boundary at $x(t_0) \in \bigcap_{i \in I_x} \bar{X}_i$ and goes to X_j. If card $\{I_{x(t_0)}\} \succ 2$ and (9) holds, then*

$$\lim_{t \to t_0^+} V(x(t)) \le \lim_{t \to t_0^-} V(x(t)). \quad (12)$$

Proof. If $x \in \bigcap_{i \in I_x} \bar{X}_i$ and *card* $\{I_x\} \succ 2$, it means that the boundary is a point (boundary point). This situation is drawn in Fig. 2.

Assume that the trajectory goes from X_k to X_j, $k, j \in I_x$ and $k \neq j$, so we have

$$\lim_{t \to t_0^-} V(x(t)) = V_k(x(t_0))$$
$$\lim_{t \to t_0^+} V(x(t)) = V_j(x(t_0)).$$

As shown in Fig. 2, if the trajectory enters X_j from $x(t_0)$, then the vector field at $x(t_0)$ or $f = f_i(x(t_0))$, $\forall i \in I_x$, lies in the regionX_j. The vector $-f$ which starts at $x(t_0)$ is drawn in Fig. 2. This new vector lies in a region that we name it X_{b_0}. An auxiliary circle is also drawn with center $x(t_0)$ and radius ε (ε must be small enough such that there is no boundary point in the circle except $x(t_0)$). By moving along the perimeter of the circle from X_{b_0} to X_j in the clockwise or the counter-clockwise direction, a sequence of regions is observed. The names of these regions are put in an ordered set $X_b = \{X_{b_0}, X_{b_1}, \ldots, X_{b_{h-1}}, X_{b_h}\}$ from X_{b_0} to X_{b_h} respectively such that $X_{b_h} = X_j$.

For $\bar{X}_{b_i} \bigcap_{i=1,\ldots,h} \bar{X} b_{i+1}$ which surrounded with the auxiliary circle, consider the normal vectors $C_{b_i b_{i+1}}$. As shown in Fig. 3, since for each $\bar{X}_{b_i} \bigcap \bar{X}_{b_{i+1}}$, we have $0 \le \alpha \le 180$, so the absolute value of the angle between $C_{b_i b_{i+1}}$ and f does not exceed 90°.

This results in $C_{b_i b_{i+1}} f \geq 0$. By (9) we have,

$$V_{b_{i+1}}(x(t_0)) \leq V_{b_i}(x(t_0)), \quad i = 0, \ldots, h-1.$$

So,

$$V_{b_h}(x(t_0)) \leq V_{b_i}(x(t_0)), \quad i = 0, \ldots, h-1 \tag{13}$$

(13) is independent of the direction of moving on the auxiliary circle. So

$$V_j(x(t_0)) \leq V_i(x(t_0)), \quad \forall i \in I_x,\ i \neq j.$$

This completes the proof. □

Lemma 4. *If the state trajectory crosses the boundary at $x(t_0) \in \bigcap_{i \in I_x} \bar{X}_i$, where card $\{I_{x(t_0)}\} \succ 2$, and lies on the boundary $\bar{X}_j \bigcap \bar{X}_{j'}$, where $j,\ j' \in I_{x(t_0)}$ and $F_{jj'} \neq 0$, then the inequality (12) holds.*

Proof. Consider X_j or $X_{j'}$ as X_{b_h}. The proof is similar to the proof of Lemma 3. □

4. Stability analysis via discontinuous Lyapunov functions

Stability analysis based on multiple Lyapunov functions represents a Lyapunov function $V_i,\ i \in I$ for each vector field $f_i,\ i \in I$. Positive definiteness and monotonicity conditions of these Lyapunov functions are necessary but they cannot guarantee the stability of switched systems and it is needed to impose some restrictions on switching conditions. As mentioned in [12,18], if the value of the Lyapunov function is decreasing at the switching instants, the switched system is asymptotically stable.

For system (7), when a state trajectory passes a boundary, switching occurs and (3) defines the switching surfaces for the system. Theorem 1 represents the sufficient conditions for the stability of continuous hybrid system (7) via discontinuous Lyapunov functions.

Theorem 1. *Consider continuous system (7) defined on polyhedral partitions. It has local asymptotic stability if a selection of functions $V_i(x) : \bar{X}_i \to R,\ i \in I$ and nonnegative scalars, $\omega_{ij}^1,\ \omega_{ij}^2,\ i, j \in I$, exists which satisfies (14)–(17).*

$$V_i(0) = 0, \quad i \in I_0 \tag{14}$$

$$V_i(x) \succ 0, \quad x \in \bar{X}_i,\ x \neq 0,\ i \in I \tag{15}$$

$$\dot{V}_i(x) < 0, \quad x \in \bar{X}_i,\ x \neq 0,\ i \in I \tag{16}$$

$$V_i(x) - V_j(x) = \omega_{ij}^1 C_{ij} f_i(x) + \omega_{ij}^2 C_{ij} f_j(x), \quad x \in \bar{X}_i \bigcap \bar{X}_j,\ \forall i \in I,\ j \in N_i,\ \text{where } F_{ij} \neq 0 \tag{17}$$

where $N_i = \left\{k \in I,\ k \neq i : \bar{X}_i \bigcap \bar{X}_k \neq \phi\right\}$.

Proof. Consider $V(x)$ defined in (8) as a candidate Lyapunov function. (14) and (15) imply that $V(x)$ is positive definite. For asymptotic stability, it is sufficient to show that $V(x)$ is decreasing along all trajectories of system (7). For this purpose, consider a generic trajectory $x(t)$ of the system. At every time t, $x(t)$ can belong to a cell or the boundaries between cells. If $x(t) \in \bar{X}_i$ and *card* $\{I_x\} = 1$, then $V(x) = V_i(x)$ and (16) implies that $V(x)$ is decreasing [13]. If $x \in \bigcap_{i \in I_x} \bar{X}_i$, there are two different situations: *card* $\{I_x\} = 2$ and *card* $\{I_x\} \succ 2$. If $x \in \bigcap_{i \in I_x} \bar{X}_i$ and *card* $\{I_x\} = 2$, the boundary is a part of a line. In this case, the state trajectory can have two different behaviors: it can go to one of boundary regions or remain on the boundary. In the first case, Lemma 1 shows that when the trajectory enters one boundary region, $V(x)$ decreases across the boundary. If the trajectory remains on the boundary, based on Lemma 2, $V(x)$ is continuous on the boundary. Therefore one only needs to verify the monotonicity condition for one $i \in I_x$ as (16) implies it. If $x \in \bigcap_{i \in I_x} \bar{X}_i$ and *card* $\{I_x\} \succ 2$, the boundary is a point. In this case, the state trajectory can have two different behaviors: it can go to one of boundary regions or lie on the boundary between two regions. Lemmas 3 and 4 show that the monotonicity condition holds for these two cases, respectively. So, for system (7), by satisfying the monotonicity condition, all trajectories in X asymptotically converge to the origin. □

5. Discontinuous PWQ Lyapunov functions

In this section we analyze the stability of PWA system (4) using a discontinuous PWQ Lyapunov function. Let

$$V_i(x) = \bar{x}^T \bar{P}_i \bar{x} \quad \forall x \in \bar{X}_i \tag{18}$$

where $\bar{P}_i = \bar{P}_i^T \in R^{3\times 3}$ and

$$\bar{P}_i = \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix}$$

where $P_i \in R^{2\times 2}$, $q_i \in R^2$ and $r_i \in R$. By (18), discontinuous function $V(x)$ defined in (8), forms a discontinuous PWQ function. Conditions for the existence of a discontinuous PWQ Lyapunov function for the PWA system (4) are formulated in the next theorem.

Theorem 2. *Let $\bar{U}_i$ and $\bar{W}_i,\ i \in I$, be unknown matrices with nonnegative entries, $\bar{\omega}_{ij}^k,\ k = 1, 2$ and $i, j \in I$, be unknown vectors with suitable dimensions that have nonnegative entries and $\bar{P}_i \in R^{3\times 3},\ i \in I$, be symmetric matrices. Define*

$$\begin{aligned}
&\bar{H}_{ij} = \bar{E}_i^T \bar{\omega}_{ij}^1 \bar{C}_{ij}\bar{A}_i + \bar{E}_j^T \bar{\omega}_{ij}^2 \bar{C}_{ij}\bar{A}_j\\
&\bar{L}_i = \bar{E}_i^T \bar{U}_i \bar{E}_i\\
&\bar{M}_i = \bar{E}_i^T \bar{W}_i \bar{E}_i
\end{aligned} \tag{19}$$

If a selection of matrices $\bar{P}_i,\ \bar{U}_i,\ \bar{W}_i,\ i \in I$, and vectors $\bar{\omega}_{ij}^k,\ k = 1, 2$ for $i, j \in I$ exists, which satisfies the constraints (20)–(25), then for system (4), all trajectories started in X asymptotically converge to the origin.

$$\bar{P}_i = \begin{bmatrix} P_i & 0\\ 0 & 0\end{bmatrix} \quad \forall i \in I_0 \tag{20}$$

$$\bar{P}_i - \bar{L}_i \succ 0 \quad \forall i \in I,\ i \notin I_0 \tag{21}$$

$$\begin{bmatrix} I_n & 0\end{bmatrix} \left(\bar{P}_i - \bar{L}_i\right) \begin{bmatrix} I_n\\ 0\end{bmatrix} \succ 0 \quad \forall i \in I_0 \tag{22}$$

$$\bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{M}_i \prec 0 \quad \forall i \in I,\ i \notin I_0 \tag{23}$$

$$\begin{bmatrix} I_n & 0\end{bmatrix} \left(\bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{M}_i\right) \begin{bmatrix} I_n\\ 0\end{bmatrix} \prec 0 \quad \forall i \in I_0 \tag{24}$$

$$\bar{F}_{ij}^T \left(\bar{P}_i - \bar{P}_j\right) \bar{F}_{ij} = \bar{F}_{ij}^T \left(\bar{H}_{ij} + \bar{H}_{ij}^T\right) \bar{F}_{ij} \quad \forall i \in I,\ j \in N_i,\ \text{where } F_{ij} \neq 0 \tag{25}$$

where $N_i = \left\{k \in I,\ k \neq i,\ \bar{X}_i \bigcap \bar{X}_k \neq \phi\right\}$.

Proof. Consider the candidate discontinuous Lyapunov function defined by (8) and (18). (20) is equivalent to (14). Since $\bar{x}^T \bar{L}_i \bar{x} \succ 0,\ \forall x \in \bar{X}_i,\ i \in I$, (21) and (22) result in (15) [19]. In the same way, $\bar{x}^T \bar{M}_i \bar{x} \succ 0,\ \forall x \in \bar{X}_i,\ i \in I$ and by (23) and (24) we have (16). Eq. (25) implies that

$$\bar{s}^T \bar{F}_{ij}^T \left(\bar{P}_i - \bar{P}_j\right) \bar{F}_{ij} \bar{s} = \bar{s}^T \bar{F}_{ij}^T \left(\bar{H}_{ij} + \bar{H}_{ij}^T\right) \bar{F}_{ij} \bar{s} \quad \forall i \in I,\ j \in N_i,\ \text{where } F_{ij} \neq 0.$$

So $\forall x \in \bar{X}_i \bigcap \bar{X}_j$

$$\bar{x}^T \left(\bar{P}_i - \bar{P}_j\right) \bar{x} = \bar{x}^T \left(\bar{H}_{ij} + \bar{H}_{ij}^T\right) \bar{x}$$

or

$$V_i(x) - V_j(x) = \bar{x}^T \bar{E}_i^T \bar{\omega}_{ij}^1 \bar{C}_{ij} \bar{A}_i \bar{x} + \bar{x}^T \bar{E}_j^T \bar{\omega}_{ij}^2 \bar{C}_{ij} \bar{A}_j \bar{x} + \bar{x}^T \bar{A}_i^T \bar{C}_{ij}^T (\bar{\omega}_{ij}^1)^T \bar{E}_i \bar{x} + \bar{x}^T \bar{A}_j^T \bar{C}_{ij}^T (\bar{\omega}_{ij}^2)^T \bar{E}_j \bar{x}. \tag{26}$$

For $x \in \bar{X}_i \bigcap \bar{X}_j$ we have $\bar{E}_i \bar{x} \geq 0$ and $\bar{E}_j \bar{x} \geq 0$, so $(\bar{\omega}_{ij}^1)^T \bar{E}_i \bar{x}$ and $(\bar{\omega}_{ij}^2)^T \bar{E}_j \bar{x}$ are nonnegative scalars. On the other hand,

$$\bar{C}_{ij} \bar{A}_i \bar{x} = \begin{bmatrix} C_{ij} & c_{ij}\end{bmatrix} \begin{bmatrix} A_i & a_i\\ 0 & 0\end{bmatrix} \begin{bmatrix} x\\ 1\end{bmatrix} = C_{ij}\left(A_i x + a_i\right).$$

Since $A_i x + a_i$ is the vector field in the region i, it is resulted that (26) is equivalent to (17). According to Theorem 1, system (4) is locally asymptotically stable. □

Remark 1. For Theorem 2, if $\bar{\omega}_{ij}^k = 0,\ k = 1, 2$, then we will get the corresponding stability conditions based on continuous PWQ Lyapunov function.

6. Examples

Example 1. Consider the following saturated system with the unit saturation:

$$\begin{cases} \dot{x}_1 = -2x_1 + x_2\\ \dot{x}_2 = -3x_1 + x_2 - sat(x_1 + x_2)\end{cases} \tag{27}$$

where the function $sat(x)$ is defined as

$$sat(x) = \begin{cases} 1 & x \geq 1\\ x & -1 < x < 1\\ -1 & x \leq -1.\end{cases}$$

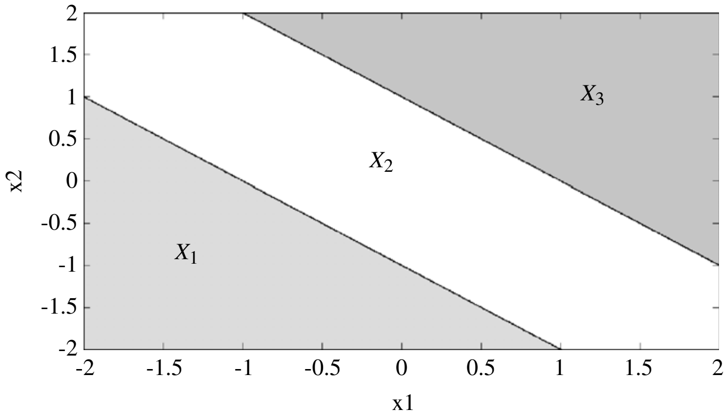

Fig. 4. Polyhedral partition of the state space.

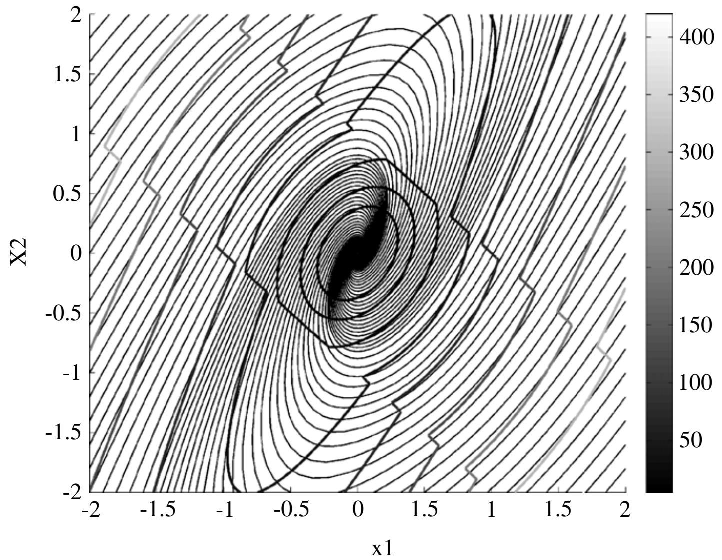

Fig. 5. Phase portraits and level curves of the Lyapunov function associated to system (27).

As specified in Fig. 4, the function domain is partitioned to three sub-domains. The stability of the system is verified by discontinuous PWQ Lyapunov functions. A feasible solution was calculated using CVX, a package for specifying and solving convex programs [20]. Level curves of $V(x)$ and phase portraits of the system are depicted in Fig. 5. The Color-bar, on the right side of Fig. 5, specifies the value of $V(x)$ on the level curves. Discontinuity of level curves on boundaries of sub-regions verifies that the Lyapunov function is discontinuous on boundaries. Due to asymptotic stability, each state trajectory of the system converges to the origin. By tracking each state trajectory converging to the origin, the monotonicity of the Lyapunov function can be observed via level curves. Interestingly, the boundaries are not traversed by trajectories in one direction and on a part of the boundary $\bar{X}_i \bigcap \bar{X}_j$, the direction of the vector field is from X_i to X_j and on the other part, the direction of the vector field is from X_j to X_i. It is resulted that the functions $V_k(x),\ k = 1, 2, 3$ are obtained automatically such that on a part of the boundary $V_i \succ V_j$ and on the other part $V_j \succ V_i$.

Example 2. Consider the saturated system (28) with unit saturation

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 + e\,\mathrm{sat}(x_1 + x_2). \end{cases} \tag{28}$$

We generated 1000 PWA systems randomly like the mentioned system such that for each system the following subsystem (the subsystem which includes the origin) is stable

$$\dot{x} = \begin{bmatrix} a & b \\ c+e & d+e \end{bmatrix} x, \quad -1 \le x_1 + x_2 \le 1.$$

It is clear that there is not any guarantee for the stability of these PWA systems, but each analysis method that could detect more stable systems is more flexible and less conservative. Table 1 shows the results for continuous and discontinuous PWQ Lyapunov functions. This example verifies the less conservativeness of discontinuous Lyapunov functions, well.

Table 1
Stable systems identified by different approaches.

Type of Lyapunov function	The number of identified stable systems
Continuous PWQ	521
Discontinuous PWQ	788

7. Conclusions

In this paper, the problem of stability analysis for continuous PWA systems via discontinuous Lyapunov functions has been studied. New stability conditions are derived via linear matrix inequalities. Since the continuous PWQ Lyapunov function is a special case of discontinuous PWQ Lyapunov functions, it is clear that the proposed analysis method in this paper is more powerful and less conservative than the analysis method based on continuous PWQ Lyapunov functions; this fact was verified through an example. The proposed procedure can be applied to all planar continuous PWA systems. It is also applicable to the systems in which the switching surfaces traversed by the trajectories in more than one direction.

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