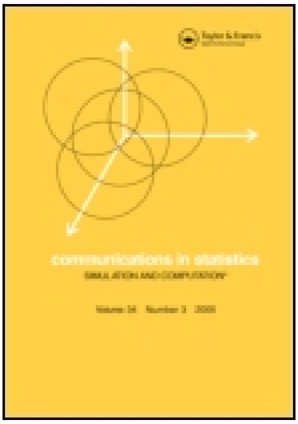

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Hadi Alizadeh noughabi [a] & Naser Reza Arghami [a]

[a] Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran

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Goodness-of-Fit Tests Based on Correcting Moments of Entropy Estimators

HADI ALIZADEH NOUGHABI AND NASER REZA ARGHAMI

Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran

In this article, we consider the entropy estimator introduced by Alizadeh Noughabi and Arghami (2010) and derive the nonparametric distribution function corresponding to our estimator as a piece-wise uniform distribution. We use the results to introduce goodness-of-fit tests for the normal and the exponential distributions. The critical values and powers for some alternatives are obtained by simulation. The powers of the proposed tests under various alternatives are compared with the competitors.

Keywords Information theory; Kullback-Leibler information; Goodness-of-fit tests; Normal; Exponential

Mathematics Subject Classification Primary 62G86; Secondary 62G10

1. Introduction

Suppose that a random variable X has a distribution function $F(x)$ with a continuous density function $f(x)$. The entropy $H(f)$ of the random variable was defined by Shannon (1948) to be

$$H(f) = -\int_{-\infty}^{\infty} f(x)\, \log f(x)\, dx.$$

The problem of estimation of $H(f)$ has been considered by many authors, including Vasicek (1976), Joe (1989), Hall and Morton (1993), van Es (1992), Correa (1995), Wieczorkowski-Grzegorewski (1999), and Alizadeh Noughabi (2010).

Among these various entropy estimators, Vasicek's sample entropy has been most widely used in developing entropy-based statistical procedures (Dudewicz and van der Meulen, 1981; Gokhale, 1983; Arizona and Ohta, 1989; Ebrahimi et al., 1992, etc.).

Vasicek (1976) proposed an estimator of entropy for one dimensional distributions. Assuming that $X_1, \ldots, X_n$ is the sample, the estimator is given by

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{2m}(X_{(i+m)} - X_{(i-m)})\right\},$$

where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$ and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are order statistics based on a random sample of size n. Vasicek (1976) established the consistency of HV_{mn} for the population

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Address correspondence to Hadi Alizadeh Noughabi, Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran; E-mail: alizadehhadi@ymail.com

entropy $H(f)$. Song (2000) obtained the asymptotic distribution of Vasicek's entropy estimator.

Ebrahimi et al. (1994) proposed a modified sample entropy as

$$HE_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{c_i m}(X_{(i+m)} - X_{(i-m)})\right\},$$

where

$$c_i = \begin{cases} 1 + \dfrac{i-1}{m}, & 1 \le i \le m, \\ 2, & m+1 \le i \le n-m, \\ 1 + \dfrac{n-i}{m}, & n-m+1 \le i \le n. \end{cases}$$

They proved that $HE_{mn} \xrightarrow{\Pr} H(f)$ as $n \to \infty$, $m \to \infty, \frac{m}{n} \to 0$.

Alizadeh Noughabi and Arghami (2010) proposed to estimate the entropy $H(f)$ of an unknown continuous probability density function f by

$$HA_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{a_i m}(X_{(i+m)} - X_{(i-m)})\right\},$$

where

$$a_i = \begin{cases} 1, & 1 \le i \le m, \\ 2, & m+1 \le i \le n-m, \\ 1, & n-m+1 \le i \le n, \end{cases}$$

and $X_{(i-m)} = X_{(1)}$ for $i \le m$ and $X_{(i+m)} = X_{(n)}$ for $i \ge n-m$. They established the consistency of HA_{mn} for the population entropy $H(f)$. They showed that their estimator has smaller mean squared error than above estimators.

The Kullback-Leibler divergence has been widely studied in statistical literature as a central index measuring quantitative similarity between two probability distributions. For given two probability distributions with density functions f and g, the KL divergence of f from g is defined (Kullback and Leibler 1951) by

$$D(f, g) = \int f(x) \log \frac{f(x)}{g(x)}\, dx.$$

Let f denote the true density function and $G = \{g(., \theta) : \theta \in \Omega\}$ be a selected statistical model for the data distribution f, where Ω is a subset of $\mathbb{R}^p$. When f actually belongs to G, the minimal value, $\min_{\theta\in\Omega} D(f, g(., \theta))$, of the KL divergence is zero. On the other hand, when f does not belong to G the minimal KL divergence is strictly positive.

In reliability studies and engineering, it is very important to test whether the underlying distribution has a particular form. Most statistical methods assume an underlying distribution in the derivation of their results. Therefore, we must check the distribution assumptions carefully.

The goodness-of-fit tests have been discussed by many authors including D'Agostino and Stephens (1986), Ascher (1990), Ahmad and Alwasel (1999), Huber-Carol et al. (2002), Li and Papadopoulos (2002), Thode (2002), Henze and Meintanis (2002a, 2002b), Henze et al. (2003), Jammalamadaka et al. (2003, 2006), Zhang and Cheng (2003), Meintanis (2004), Steele and Chaseling (2006), Meintanis (2007), Jager and Wellner (2007), Meintanis (2008), Van Rensburg and Swanepoel (2008), Raschke (2009), Meintanis (2009), Zhao et al. (2009), Grané and Fortiana (2009), Alizadeh Noughabi and Arghami (2011, 2011a, 2011b, 2011c, 2011d), and Zamanzade and Arghami (2011).

Vasicek (1976), Dudewicz and Van der Meulen (1981), Gokhale (1983), and Mack (1988) developed tests of distributional hypotheses based on the discrepancy between the sample entropy HV_{mn} and parametric entropy estimates of various distributions. Arizono and Ohta (1989) and Ebrahimi et al. (1992) proposed tests of normality and exponentiality respectively, based on an estimate of the Kullback-Liebler information which uses HV_{mn}. Next, Taufer (2002) discussed transformations to uniformity in the entropy test for exponentiality in order to increase power.

Park and Park (2003) derived the following nonparametric distribution function of Vasicek (1976)'s and Ebrahimi et al. (1994)'s estimators as:

$$g_v(x) = \begin{cases} 0 & x < \xi_1 \ or \ x > \xi_{n+1} \\ \dfrac{1}{n}\dfrac{2m}{x_{(i+m)} - x_{(i-m)}} & \xi_i < x \le \xi_{i+1} \quad i = 1, \dots, n, \end{cases}$$

where $\xi_i = (x_{(i-m)} + \cdots + x_{(i+m-1)})/2m$, and $x_{(i)} = x_{(1)}$ if $i < 1$, $x_{(i)} = x_{(n)}$ if $i > n$, and

$$g_e(x) = \begin{cases} 0 & x < \eta_1 \ or \ x > \eta_{n+1} \\ \dfrac{1}{n}\dfrac{1}{\eta_{i+1} - \eta_i} & \eta_i < x \le \eta_{i+1} \quad i = 1, \dots, n, \end{cases}$$

where

$$\eta_i = \begin{cases} \xi_{m+1} - \displaystyle\sum_{k=i}^{m} \frac{1}{m+k-1}(x_{(m+k)} - x_{(1)}) & if \ 1 \le i \le m, \\ \dfrac{\left(x_{(i-m)} + \cdots + x_{(i+m-1)}\right)}{2m} & if \ m+1 \le i \le n-m+1, \\ \xi_{n-m+1} + \displaystyle\sum_{k=n-m+2}^{i} \frac{1}{n+m-k+1}(x_{(n)} - x_{(k-m-1)}) & if \ n-m+2 \le i \le n+1, \end{cases}$$

respectively.

They introduced goodness-of-fit tests for testing normality and exponentiality based on the moments of theses nonparametric distribution functions. They also showed that the power of the test statistic based on correcting moments of Vasicek estimator is greater than the power of the test statistic based on correcting moments of Ebrahimi et al. estimator for exponentiality test, and the reverse holds for testing normality.

In Sec. 2, we derive the nonparametric distribution of Alizadeh Noughabi and Arghami estimator and then we introduce tests for normality and exponentiality based on this estimator and the nonparametric distribution corresponding to the estimator.

In Sec. 3, critical values are obtained and the powers of the proposed tests are compared with the power of the competitor tests. All simulations were carried out by using R 2.12.0 and with 10,000 replications.

2. Test Statistics Based on Correcting Moments of Alizadeh Noughabi and Arghami Entropy Estimator

In this section, we introduce goodness-of-fit test for normality and exponentiality by using the moments of the distribution corresponding to entropy estimator proposed by Alizadeh Noughabi and Arghami (2010).

The underlying nonparametric distribution function of the sample entropy has not been known yet, but we can derive the nonparametric distribution function of our estimator in the lights of Park and Park (2003) as:

$$g_a(x) = \begin{cases} 0 & x < \eta_1 \ or \ x > \eta_{n+1} \\ \dfrac{1}{n}\dfrac{1}{\eta_{i+1} - \eta_i} & \eta_i < x \leq \eta_{i+1} \quad i = 1, \ldots, n, \end{cases} \tag{1}$$

where

$$\eta_i = \begin{cases} \xi_{m+1} - \dfrac{1}{m}\displaystyle\sum_{k=i}^{m}(x_{(m+k)} - x_{(1)}) & if \ 1 \leq i \leq m, \\ \dfrac{(x_{(i-m)} + \cdots + x_{(i+m-1)})}{2m} & if \ m+1 \leq i \leq n-m+1, \\ \xi_{n-m+1} + \dfrac{1}{m}\displaystyle\sum_{k=n-m+2}^{i}(x_{(n)} - x_{(k-m-1)}) & if \ n-m+2 \leq i \leq n+1, \end{cases}$$

and $\xi_i = (x_{(i-m)} + \cdots + x_{(i+m-1)})/2m$, $x_{(i)} = x_{(1)}$ if $i < 1$, and $x_{(i)} = x_{(n)}$ if $i > n$.

Derivation of g_a is quite intuitive, but it can be shown, directly from the definition of entropy, that Alizadeh Noughabi and Arghami estimator is in fact the entropy of a continuous distribution with probability density function (pdf) $g_a(x)$.

Remark 2.1 The first two moments of the above distribution can be derived to be

$$\left(\frac{1}{n}\right)\left(\frac{\eta_1}{2} + \sum_{i=2}^{n}\eta_i + \frac{\eta_{n+1}}{2}\right) \text{ and } \left(\frac{1}{3n}\right)\sum_{i=1}^{n}\left(\eta_{i+1}^2 + \eta_i^2 + \eta_i\eta_{i+1}\right), \text{ respectively.}$$

2.1. *Testing Normality*

Given a random sample $X_1, \ldots, X_n$ from a continuous probability distribution F with a density $f(x)$, over the real line and with mean μ and variance $\sigma^2 < \infty$, the hypothesis of interest is

$$H_0 : f(x) = f_0(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad for\ some\ (\mu, \sigma) \in \Theta, \ x \in \mathbb{R},$$

where μ and σ are unspecified and $\Theta = \mathbb{R} \times \mathbb{R}^+$. The alternative to H_0 is

H_1 : The data do not come from a normal distribution.

The asymmetric Kullback-Leibler distance of f from f_0 is:

$$D(f, f_0) = \int f(x;\mu,\sigma)\log\frac{f(x;\mu,\sigma)}{f_0(x;\mu,\sigma)}dx$$

$$= -H(f) + \log\sqrt{2\pi\sigma^2} + \frac{1}{2\sigma^2}E_f(X-\mu)^2,$$

$D(f, f_0)$ is minimum (zero) if and only if $f = f_0$, where

$$D(f, f_0) = \log\sqrt{2\pi E_f(X-\mu)^2} + 0.5 - H(f),$$

which can be estimated by

$$TA_{mn} = \log\sqrt{2\pi\hat{\sigma}_a^2} + 0.5 - HA_{mn},$$

where $\hat{\sigma}_a^2 = Var_{g_a}(X)$. We reject H_0 for large values of TA_{mn}.

The test statistic is invariant with respect to location and scale transformations. To see this, note that if x_i's are multiplied by a constant $c > 0$ and are added by a constant d then η_i's in Eq. (1) are multiplied by c and are added by d and thus $\hat{\sigma}_a$ is multiplied by c. On the other hand, it is obvious that if the observations are multiplied by $c > 0$ and are added by a constant d, HA_{mn} is increased by $\ln c$. Thus, TA_{mn} remains invariant.

2.2. *Testing Exponentiality*

Given a random sample $X_1, \ldots, X_n$ from a continuous probability distribution F with a density $f(x)$ over a non negative support and with mean $\theta < \infty$, the hypothesis of interest is

$$H_0 : f(x) = f_0(x;\theta) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right), \quad \textit{for some } \theta \in \Theta, \quad x \geq 0,$$

where θ is unspecified and $\Theta = \mathbb{R}^+$. The alternative to H_0 is

H_1 : The data do not come from an exponential distribution.

The asymmetric Kullback-Leibler distance of f from f_0 is:

$$D(f, f_0) = \int f(x;\theta)\log\frac{f(x;\theta)}{f_0(x;\theta)}dx$$

$$= -H(f) + \log\theta + \frac{1}{\theta}E_f(X),$$

$D(f, f_0)$ is minimum (zero) if and only if H_0 holds, where

$$D(f, f_0) = \log E_f(X) + 1 - H(f),$$

which can be estimated by

$$TA_{mn} = \log\hat{\theta}_a + 1 - HA_{mn},$$

where $\hat{\theta}_a = E_{g_a}(X)$. We reject H_0 for large values of TA_{mn}.

Similar to the argument of Sec. 2.1, we can see the test statistic is invariant with respect to scale transformations.

3. Simulation Study

3.1. *Critical Values*

In order to compute the proposed test statistic for a given data set, one needs to specify the order of spacings m. Since n is known, it is obvious that m may be taken as a function of n.

In practice, a general guide for the choice of m for a fixed n would be valuable to the users. However, simulations show that the optimal m(in terms of power) also depends on the alternative that one may have in mind and the optimal m's for different alternatives is different. We observe that there is no m that is optimal for all alternatives. Therefore if one wants to guard against all alternatives a compromise should be made.

Based on the simulations presented in this article, we propose the following heuristic formula for choosing m, subject to n:

$$m = \left[\sqrt{n} - 1\right],$$

where $[x]$ means the integer part of x. The tests attain good (not best) powers for all alternative distributions for these values of m.

We observe that with increasing n, an optimal choice of m also increases, while the ratio m/n tends to zero.

We reject H_0 at the significance level α if $TA_{mn} \geq C(\alpha)$, where the critical point $C(\alpha)$ is determined by the $\alpha-$quantile of the distribution of the TA_{mn} statistic under the hypothesis H_0.

For the proposed values of m, we used Monte Carlo methods with 10,000 replicates from standard normal (exponential) distribution to obtain the critical values of the proposed procedure for α equal to 0.01, 0.05, and 0.10. Table 1 gives the critical values $C(\alpha)$ for various sample sizes and for testing normality and exponentiality. Note that the critical

Table 1
Critical values of the TA_{mn} statistic for testing normality and exponentiality (for optimal m)

	For testing normality			For testing exponentiality		
n	$C(0.01)$	$C(0.05)$	$C(0.10)$	$C(0.01)$	$C(0.05)$	$C(0.10)$
5	1.3717	0.9404	0.7624	1.6621	1.2008	0.9749
10	0.5943	0.4422	0.3722	0.6663	0.4785	0.3899
15	0.4937	0.3738	0.3185	0.5137	0.3844	0.3229
20	0.3643	0.2805	0.2475	0.3628	0.2636	0.2143
25	0.3130	0.2434	0.2147	0.2881	0.2020	0.1656
30	0.2817	0.2255	0.1988	0.2553	0.1886	0.1563
40	0.2403	0.1943	0.1743	0.2116	0.1526	0.1282
50	0.2196	0.1805	0.1615	0.1855	0.1303	0.1079

values do not depend on the unknown parameters, because we show that TA_{mn} is invariant under location-scale (scale) transformation of the observations.

For sample sizes $4 \leq n \leq 100$, the approximate critical values of normality test may be computed by

$$C(\alpha) \approx a(\alpha) + b(\alpha)\exp(1/n)\,,$$

where

$$\begin{array}{lll} a(0.01) = -5.2743 & a(0.05) = -3.60359 & a(0.10) = -2.86228 \\ b(0.01) = 5.3853 & b(0.05) = 3.70662 & b(0.10) = 2.96175. \end{array}$$

The above formula is based on regression of $C(\alpha)$ on n for $n = 4, 5, \ldots, 100$. ($R^2 \approx 0.96$).

Similarly, the approximate critical values of exponentiality test may be computed by

$$C(\alpha) \approx a(\alpha) + b(\alpha)\exp(1/n)\,,$$

where $R^2 \approx 0.97$ and

$$\begin{array}{lll} a(0.01) = -6.9789 & a(0.05) = -5.00396 & a(0.10) = -4.12279 \\ b(0.01) = 7.0236 & b(0.05) = 5.03629 & b(0.10) = 4.14987. \end{array}$$

For large n, $C(\alpha) \approx 0$. Thus, unless the data perfectly fit the normal (exponential) curve and $TA_{mn} = 0$, the null hypothesis will be rejected.

Figures 1 and 2 show the critical values of TA_{mn}-statistic for various sample sizes and at the significance levels $\alpha = 0.01,\ 0.05$ and 0.10, for normality and exponentiality, respectively. Those are useful for when n is not tabulated.

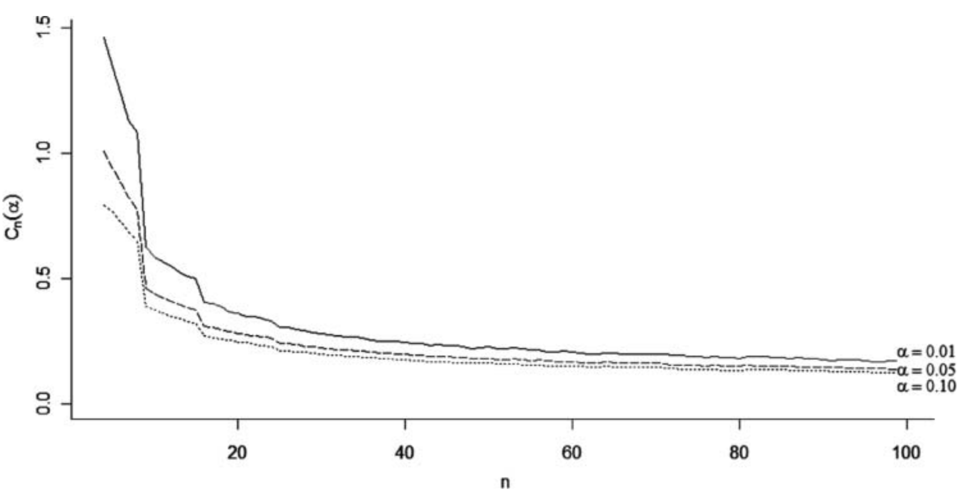

Figure 1. Critical values of TA_{mn}-statistic against sample size and at the significance levels $\alpha = 0.01,\ 0.05$, and 0.10, for normality test.

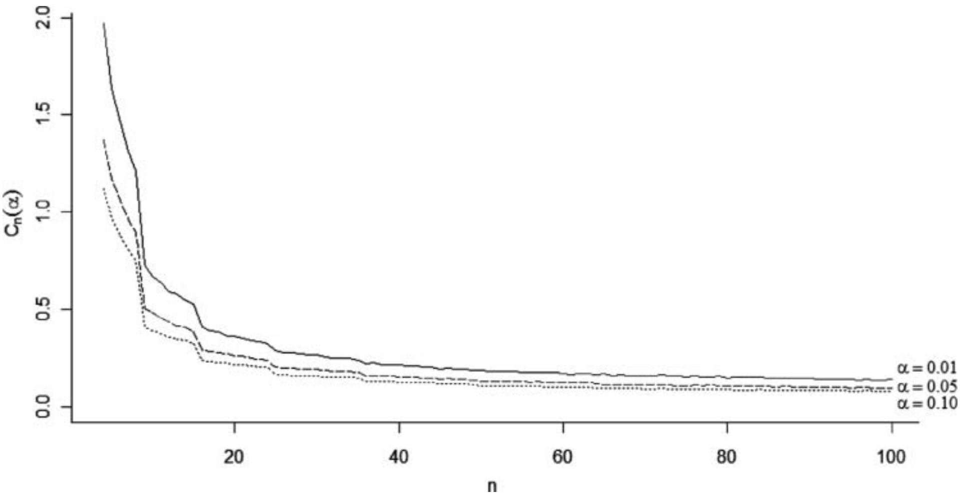

Figure 2. Critical values of TA_{mn}-statistic against sample size and at the significance levels $\alpha = 0.01,\ 0.05$, and 0.10, for exponentiality test.

3.2. *Competitor Tests*

We chose the competitor tests from the tests of normality and exponentiality introduced in Park and Park (2003). The test statistics of competitor tests are as follows.

1. For testing normality:

$$TV_{mn} = \log\sqrt{2\pi\hat{\sigma}_v^2} + 0.5\ - HV_{mn},$$
$$TE_{mn} = \log\sqrt{2\pi\hat{\sigma}_e^2} + 0.5\ - HE_{mn},$$

   where $\hat{\sigma}_v^2 = Var_{g_v}(X)$ and $\hat{\sigma}_e^2 = Var_{g_e}(X)$.
2. For testing exponentiality:

$$TV_{mn} = \log\hat{\theta}_v + 1\ - HV_{mn}\theta$$
$$TE_{mn} = \log\hat{\theta}_e + 1\ - HE_{mn},$$

where $\hat{\theta}_v = E_{g_v}(X)$ and $\hat{\theta}_e = E_{g_e}(X)$.

Since the tests based on the empirical distribution function for normality are important tests which are commonly used in practice, we also compare the power of the entropy-based tests of normality with the said tests. These tests are Cram-er von Mises (*CH*), Kolmogorov-Smirnov (*D*), Anderson-Darling (A^2), and Kuiper (V).

Moreover, we consider the famous Shapiro-Wilk (W) which is a regression test. The W test is somewhat complicated and there are coefficients to be determined for calculating W. The coefficients are constants tabulated in Shapiro and Wilk (1965) for n up to 50. For $n > 50$, the values of coefficients are approximated by the methods given in Shapiro and Wilk (1965) and Royston (1982). For further study about this test see Shapiro and Wilk (1965, 1968).

Note that entropy tests and Shapiro-Wilk test are exact, but the other four tests are approximate in the sense that the actual size of the test is only approximately equal to the nominal size.

Also, for testing exponentiality we chose some competitor tests from the class of tests of exponentiality discussed in Henze and Meintanis (2005). These tests are Kolmogorov: Smirnov (D), Cramer–von Mises (CH), test statistic proposed by D'Agostino and Stephens, (S) (1986), Cox and Oakes (CO) (1984), Epps and Pulley (EP) (1986), Henze (HE) (1993), test statistic ($T_{n,a}^{(1)}$) proposed by Henze and Meintanis (2005).

3.3. *Power Comparison*

3.3.1. *Testing Normality.* We compute the powers of the tests based on TV_{mn}, TE_{mn}, and TA_{mn} statistics by means of Monte Carlo simulations under 20 alternatives. These alternatives were used by Esteban et al. (2001) in their study of power comparisons of several tests for normality. The alternatives can be divided into four groups, depending on the support and shape of their densities. From the point of view of applied statistics, natural alternatives to normal distribution are in groups I and II. For the sake of completeness, we also consider groups III and IV. This fact gives additional insight to understand the behaviour of the new test statistic TA_{mn}.

Group I. Support $(-\infty, \infty)$, symmetric.

- Student t with 1 degree of freedom (i.e., the standard Cauchy);
- Student t with 3 degrees of freedom;
- Standard logistic;
- Standard double exponential.

Group II. Support $(-\infty, \infty)$, asymmetric.

- Gumbel with parameters $\alpha = 0$(location) and $\beta = 1$ (scale);
- Gumbel with parameters $\alpha = 0$ (location) and $\beta = 2$ (scale);
- Gumbel with parameters $\alpha = 0$ (location) and $\beta = 1/2$ (scale).

Group III. Support $(0, \infty)$.

- Exponential with mean 1;
- Gamma with parameters $\beta = 1$ (scale) and $\alpha = 2$ (shape);
- Gamma with parameters $\beta = 1$ (scale) and $\alpha = 1/2$ (shape);
- Lognormal with parameters $\mu = 0$ (scale) and $\sigma = 1$ (shape);
- Lognormal with parameters $\mu = 0$ (scale) and $\sigma = 2$ (shape);
- Lognormal with parameters $\mu = 0$ (scale) and $\sigma = 1/2$ (shape);
- Weibull with parameters $\beta = 1$ (scale) and $\alpha = 1/2$ (shape);
- Weibull with parameters $\beta = 1$ (scale) and $\alpha = 2$ (shape).

Group IV. Support (0,1).

- Uniform;
- Beta (2,2);
- Beta (0.5,0.5);
- Beta (3,1.5);
- Beta (2,1).

Table 2
Powers of the tests of size 0.05 for sample sizes $n = 10$ and $n = 20$ under alternatives from Group I

n	Alternatives	TV_{mn}	TE_{mn}	TA_{mn}	CH	D	V	W	A^2
10	$t_{(1)}$	0.375	0.460	0.507	**0.618**	0.580	0.589	0.594	**0.618**
20	$t_{(1)}$	0.684	0.786	0.858	0.880	0.847	0.865	0.869	**0.882**
10	$t_{(3)}$	0.082	0.112	0.134	0.182	0.164	0.163	0.187	**0.190**
20	$t_{(3)}$	0.121	0.205	0.301	0.309	0.260	0.277	**0.340**	0.327
10	Logistic	0.048	0.058	0.065	0.080	0.073	0.071	0.082	**0.083**
20	Logistic	0.046	0.064	0.095	0.106	0.087	0.090	**0.123**	0.113
10	Double exponential	0.053	0.077	0.094	0.158	0.142	0.142	0.150	**0.159**
20	Double exponential	0.062	0.129	0.229	0.270	0.224	0.242	0.264	**0.274**

Under each alternative we generated 10,000 samples of size 10 and 20. We evaluated for each sample and for several values of the parameter m the statistics (TV_{mn}, TE_{mn}, and TA_{mn}) and the power of the corresponding test was estimated by the frequency of the event the statistic is in the critical region". The power estimates are given in Tables 2–5.

For each sample size and alternative, the bold type in these tables indicates the maximal power.

Tables 2, 3, and 4 indicate a clear and uniform superiority of our procedure to Vasicek's (1976) and Ebrahimi et al.'s (1994). In Groups I, II, and III it is seen that the proposed test TA_{mn} has the most power among the entropy-based tests. In Group IV the test TV_{mn} has the most power.

Among all tests, we can see the tests A^2, W, TA_{mn}, and TV_{mn} have the most power in groups I, II, III, and IV, respectively.

3.3.2. *Testing Exponentiality.* To facilitate the comparisons of the powers of the present tests with those of the existing tests, we selected the same alternatives listed in Henze and Meintanis (2005) and their choices of parameters:

- the Weibull distribution with density $\theta x^{\theta-1}\exp(-x^{\theta})$, denoted by *Weibull*($\theta$);
- the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $Gamma(\theta)$;

Table 3
Powers of the tests of size 0.05 for sample sizes $n = 10$ and $n = 20$ under alternatives from Group II

n	Alternatives	TV_{mn}	TE_{mn}	TA_{mn}	CH	D	V	W	A^2
10	Gumbel (0,1)	0.092	0.111	0.124	0.137	0.121	0.117	**0.153**	0.147
20	Gumbel (0,1)	0.176	0.237	0.279	0.249	0.203	0.194	**0.313**	0.273
10	Gumbel (0,2)	0.095	0.109	0.121	0.136	0.121	0.117	**0.150**	0.144
20	Gumbel (0,2)	0.178	0.240	0.280	0.252	0.203	0.195	**0.315**	0.276
10	Gumbel (0,1/2)	0.091	0.110	0.123	0.139	0.118	0.117	**0.154**	0.147
20	Gumbel (0,1/2)	0.177	0.236	0.277	0.249	0.203	0.194	**0.314**	0.275

Table 4
Powers of the tests of size 0.05 for sample sizes $n = 10$ and $n = 20$ under alternatives from Group III

n	Alternatives	TV_{mn}	TE_{mn}	TA_{mn}	CH	D	V	W	A^2
10	Exponential	0.397	0.454	**0.477**	0.390	0.301	0.360	0.442	0.416
20	Exponential	0.830	0.865	**0.870**	0.724	0.586	0.696	0.836	0.773
10	Gamma (2)	0.151	0.185	0.213	0.210	0.175	0.180	**0.239**	0.225
20	Gamma (2)	0.429	0.508	**0.533**	0.425	0.326	0.353	0.532	0.467
10	Gamma (1/2)	0.762	0.794	**0.810**	0.672	0.540	0.662	0.735	0.703
20	Gamma (1/2)	0.992	**0.993**	**0.993**	0.952	0.879	0.957	0.984	0.970
10	Lognormal (0,1)	0.519	0.581	**0.616**	0.554	0.463	0.524	0.603	0.578
20	Lognormal (0,1)	0.910	0.934	**0.937**	0.881	0.778	0.857	0.932	0.904
10	Lognormal (0,2)	0.933	0.945	**0.951**	0.896	0.826	0.892	0.920	0.909
20	Lognormal (0,2)	**1.000**	**1.000**	**1.000**	0.9978	0.991	0.9976	0.9996	0.9987
10	Lognormal (0,1/2)	0.144	0.181	0.208	0.220	0.182	0.187	**0.245**	0.233
20	Lognormal (0,1/2)	0.364	0.445	0.485	0.427	0.337	0.346	**0.517**	0.463
10	Weibull (1/2)	0.923	0.935	**0.940**	0.855	0.758	0.854	0.894	0.875
20	Weibull (1/2)	**1.000**	**1.000**	**1.000**	0.9957	0.9818	0.9962	0.9992	0.9979
10	Weibull (2)	0.073	0.074	0.080	0.079	0.074	0.068	**0.084**	0.083
20	Weibull (2)	0.126	0.143	0.145	0.120	0.103	0.095	**0.156**	0.132

- the lognormal low $Lognormal(0, \theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2} \exp(-(\log x)^2/(2\theta^2))$;
- the half-normal HN distribution with density $\Gamma(2/\pi)^{1/2} \exp(-x^2/2)$;
- the uniform distribution with density 1, $0 \le x \le 1$;
- the modified extreme value $EV(\theta)$, with distribution function $1 - \exp(\theta^{-1}(1 - e^x))$;
- the linear increasing failure rate law $LF(\theta)$ with density $(1 + \theta x)\exp(-x - \theta x^2/2)$;
- Dhillon's (1981) law $DL(\theta)$ with distribution function $1 - \exp(-(\log(x + 1))^{\theta+1})$;
- Chen's (2000) distribution $CH(\theta)$, with distribution function $1 - \exp(2(1 - e^{x^\theta}))$.

Table 5
Powers of the tests of size 0.05 for sample sizes $n = 10$ and $n = 20$ under alternatives from Group IV

n	Alternatives	TV_{mn}	TE_{mn}	TA_{mn}	CH	D	V	W	A^2
10	Uniform	**0.181**	0.158	0.129	0.074	0.066	0.081	0.082	0.080
20	Uniform	**0.443**	0.391	0.258	0.144	0.100	0.150	0.200	0.171
10	Beta(2,2)	**0.084**	0.071	0.064	0.044	0.046	0.048	0.042	0.046
20	Beta(2,2)	**0.136**	0.112	0.064	0.058	0.053	0.064	0.053	0.058
10	Beta(1/2,1/2)	**0.514**	0.481	0.451	0.229	0.162	0.237	0.299	0.268
20	Beta(1/2,1/2)	**0.910**	0.891	0.824	0.509	0.318	0.490	0.727	0.618
10	Beta(3,1/2)	0.656	0.686	**0.704**	0.542	0.418	0.530	0.609	0.576
20	Beta(3,1/2)	0.980	**0.984**	0.983	0.875	0.746	0.879	0.948	0.913
10	Beta(2,1)	**0.170**	0.164	0.162	0.115	0.100	0.109	0.130	0.126
20	Beta(2,1)	**0.428**	0.423	0.358	0.232	0.174	0.202	0.306	0.261

Table 6
Powers of the tests of size 0.05 for sample size $n = 20$

Alternatives	TV_{mn}	TE_{mn}	TA_{mn}	D	CH	S	CO	EP	$T_{n,a}^{(1)}$
$Weibull(0.8)$	0.061	0.086	0.132	0.17	0.20	0.24	**0.28**	0.24	0.04
$Weibull(1.4)$	0.312	0.287	0.236	0.28	0.34	0.35	0.37	0.36	**0.45**
$Gamma(0.4)$	0.506	0.563	0.630	0.71	0.76	0.76	**0.91**	0.76	0.33
$Gamma(1.0)$	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
$Gamma(2.0)$	0.463	0.439	0.383	0.40	0.47	0.46	0.54	0.48	**0.55**
$Lognormal(0, 0.8)$	0.429	0.446	**0.453**	0.30	0.33	0.24	0.33	0.25	0.27
$Lognormal(0, 1.5)$	0.415	0.497	0.587	0.58	0.62	**0.67**	0.60	**0.67**	0.18
HN	0.192	0.171	0.126	0.18	0.21	0.21	0.19	0.21	**0.33**
$Uniform$	0.827	0.785	0.687	0.52	0.66	0.70	0.50	0.66	**0.86**
$CH(0.5)$	0.329	0.389	0.464	0.56	0.61	0.63	**0.80**	0.63	0.23
$CH(1.0)$	0.150	0.128	0.090	0.13	0.14	0.15	0.13	0.15	**0.25**
$CH(1.5)$	0.762	0.725	0.635	0.67	0.79	0.84	0.81	0.84	**0.91**
$LF(2.0)$	0.247	0.222	0.168	0.24	0.28	0.29	0.25	0.28	**0.42**
$LF(4.0)$	0.347	0.314	0.245	0.34	0.41	0.42	0.37	0.42	**0.56**
$EV(0.5)$	0.147	0.128	0.095	0.13	0.14	0.15	0.13	0.15	**0.25**
$EV(1.5)$	0.396	0.354	0.279	0.35	0.43	0.46	0.37	0.45	**0.63**
$DL(1.0)$	**0.252**	**0.252**	0.240	0.20	0.23	0.19	0.25	0.20	0.25
$DL(1.5)$	0.618	0.601	0.550	0.56	0.65	0.62	**0.72**	0.64	0.67

These alternatives were used by Henze and Meintanis (2005) and Grané and Fortiana (2010) in their study of power comparisons of several tests for exponentiality. According to Henze and Meintanis, these distributions comprise of widely used alternatives to the exponential model and include densities f with decreasing hazard rates (DHR) $f(x)/[1 - F(x)]$, increasing hazard rates (IHR) as well as models with non-monotone hazard functions.

We estimated the powers of the tests based on 10,000 samples of size n equal to 10 and 20. Table 6 shows the estimated powers at significance level $\alpha = 0.05$.

For each alternative, the bold type in Table 6 indicates the statistic achieving the maximal power.

We observe that the proposed test based on Alizadeh Noughabi and Arghami entropy estimator, TA_{mn}, perform very well compared with the other entropy tests for Weibull (0.8), Gamma (0.4), lognormal, and $CH(0.5)$, alternatives. However, for almost all other alternatives the test proposed by Park and Park (2003) based on Vasicek's entropy estimator, TV_{mn}, has the greatest power. No single test can be said to perform best for testing exponentiality against all alternatives. In general, among all tests the test statistic $T_{n,a}^{(1)}$ proposed by Henze and Meintanis (2005) has the most power.

4. Conclusions

In this article, we first derived the nonparametric distribution corresponding to our entropy estimator. We next used this nonparametric distribution to obtain the test statistics for normality and exponentiality. We introduced a new test for normality and compared the power of the proposed test with competitor tests, using Monte Carlo computations for sample sizes $n = 10$ and $n = 20$. We observed that each of the tests TV_{mn}, A^2, W, and

TA_{mn} can be most powerful, depending on the type of the alternative. The test TV_{mn} is most powerful against alternatives with the support (0, 1)(Group IV). The test A^2 is most powerful against symmetric alternatives with the support $(-\infty, \infty)$ (Group I). The test TA_{mn} (the proposed test) is most powerful against alternatives in Group III with the support $(0, \infty)$. The test W is most powerful against asymmetric alternatives in Group II with the support $(-\infty, \infty)$.

Based on these observations, we can formulate the following recommendations for the application of the studied tests in practice.

1. Use the statistic TV_{mn}, based on Vasicek entropy estimator, if the presumed alternatives are supported by the bounded interval (0, 1).
2. Use the statistic TA_{mn}, based on Alizadeh Noughabi and Arghami entropy estimator, if the presumed alternatives are supported by $(0, \infty)$.
3. Use the statistic A^2, if the presumed alternatives are symmetric and supported by $(-\infty, \infty)$.
4. Use the statistic W, if the presumed alternatives are asymmetric and supported by $(-\infty, \infty)$.

We also introduced a goodness-of-fit test for exponentiality based on the nonparametric distribution corresponding to Alizadeh Noughabi and Arghami (2010) estimator. We compared the power of the proposed test with the competitor tests using Monte Carlo computations. We observed that the test based on the test statistic $T^{(1)}_{n,a}$ proposed by Henze and Meintanis (2005) is, in general, the most powerful.

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