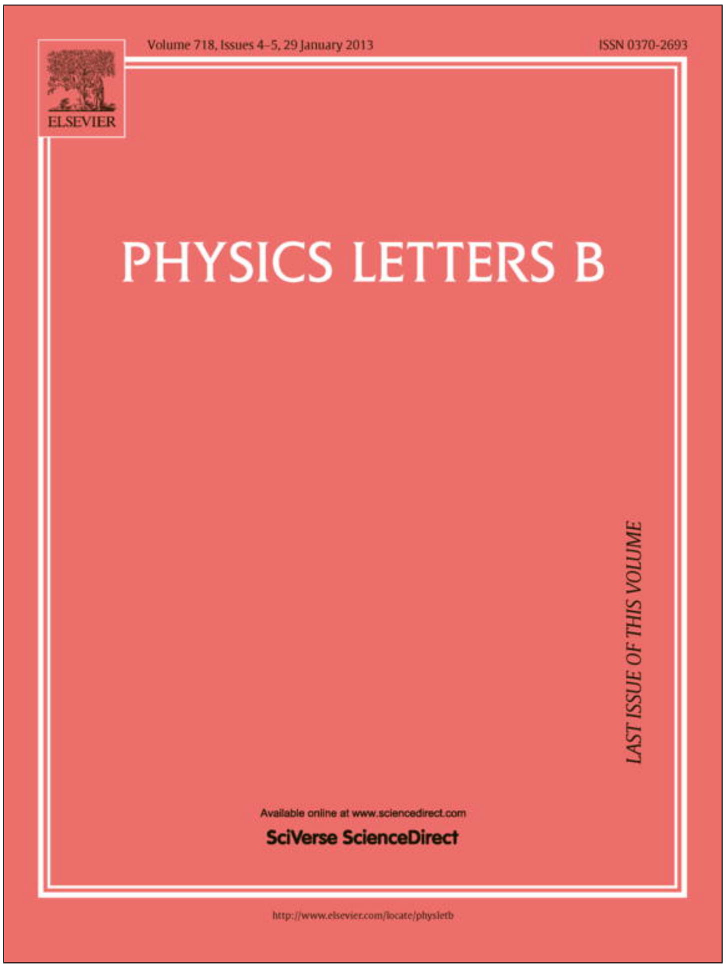
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T-duality of the Riemann curvature corrections to supergravity

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ABSTRACT

We examine the sigma model Riemann curvature corrections to the supergravity action under T-duality transformations. Using the compatibility of the effective action with on-shell linear T-duality and with the S-matrix calculations as guiding principles, we have incorporated in this action the couplings of four B-field strengths and the couplings of two Riemann curvatures and two B-field strengths at order α'^3. Using the S-matrix calculations we have also found new dilaton couplings in the string frame at this order.

1. Introduction and results

A standard method in string theory for finding the higher derivative corrections to the supergravity action [1,2] is the scattering amplitude calculation [3,4]. The α'^3 corrections to the Einstein–Hilbert action have been found in [5] by analyzing the sphere-level four-graviton scattering amplitude in type II superstring theory. The result in the eight-dimensional transverse space of the light-cone formalism, is a polynomial in the Riemann curvature tensors

$$Y \sim t^{i_1\cdots i_8} t_{j_1\cdots j_8} \mathcal{R}_{i_1 i_2}{}^{j_1 j_2} \ldots \mathcal{R}_{i_7 i_8}{}^{j_7 j_8} \tag{1}$$

where $t^{i_1\cdots i_8}$ is a tensor in eight dimensions which includes the eight-dimensional Levi-Civita tensor [5]. This $SO(8)$ invariant Lagrangian has been extended to Lorentz invariant form in [6,7]

$$\mathcal{L} = \frac{\gamma e^{-2\phi_0}}{\kappa^2}\bigg[\mathcal{R}_{hmnk}\mathcal{R}_p{}^{mn}{}_q\mathcal{R}^{hrsp}\mathcal{R}^q{}_{rs}{}^k + \frac{1}{2}\mathcal{R}_{hkmn}\mathcal{R}_{pq}{}^{mn}\mathcal{R}^{hrsp}\mathcal{R}^q{}_{rs}{}^k + \cdots\bigg] \tag{2}$$

where $\gamma = \frac{1}{8}\alpha'^3\zeta(3)$, $e^{-2\phi_0}$ is the dilaton background corresponding to the sphere-level scattering amplitude, and dots represent terms containing the Ricci and scalar curvature tensors. These terms cannot be captured by the four-graviton scattering amplitude as they are zero on-shell. They can be absorbed by the Einstein–Hilbert action in field redefinition of the metric $G \to G + \delta G$ which does not alter the scattering calculation [5]. The above R^4 couplings reproduce the sigma-model beta function [6,7].

Unlike the Einstein–Hilbert Lagrangian, there are different Lorentz invariant expressions for the Riemann curvature couplings at order α'^3 [8,9]. They are related to (2) via some identities involving the Riemann curvature tensors and some couplings involving the Ricci and scalar curvature tensors [8]. The Ricci and scalar curvature couplings can be eliminated by field redefinitions, however, the identities involving the Riemann curvature tensors hold only at four graviton levels [8]. As a result, there may be some other four Riemann curvature couplings in (2) which can be found by studying five-graviton scattering amplitude in which we are not interested in this Letter.

For the Lagrangian presented in [8,9], a proposal has been given in [9] for including the B-field and the dilaton into the action which is a prescription for generalizing the Riemann curvature tensor to include the first derivative of the B-field strength and the second derivative of the dilaton. While this prescription gives the correct B-field couplings for the Lagrangian given in [8,9], we will show that it does not work for the Lagrangian (2). In this Letter we would like to extend this Lagrangian to include the B-field and dilaton by using the compatibility of the couplings (2) with the T-duality [10–13] and by using the scattering amplitude calculations [14,5]. Similar calculations have been done in [16–25] to extend the curvature couplings on the world volume of D-brane to all other massless fields.

The outline of the Letter is as follows: We begin in Section 2 by reviewing the T-duality transformations and finding the transformation of the linearized curvature tensors under linear T-duality. In Section 3, we review the sphere-level scattering amplitude of four massless NS–NS states in type II superstring theory and reconfirm that this amplitude at order α'^3 produces the couplings (2). In Section 4, we reduce the 10-dimensional couplings (2) to 9 dimensions to find the $\mathcal{R}_y\mathcal{R}_y\mathcal{R}_y\mathcal{R}_y$ couplings where $\mathcal{R}_y$ is the Riemann

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tensor with one Killing index. The consistency of these couplings with the linear T-duality is used to find the following couplings:

$$\mathcal{L}\supset\frac{\gamma e^{-2\phi_0}}{16\kappa^2}\big[-\mathcal{H}_{hpr;k}\mathcal{H}_n{}^{pr}{}_{;q}\mathcal{H}^{kms;q}\mathcal{H}_m{}^n{}_s{}^{;h}-\mathcal{H}_{hmn;k}\mathcal{H}^{hnp;q}\mathcal{H}^k{}_{ps;r}\mathcal{H}^m{}_q{}^{s;r}+\mathcal{H}_{kmn;h}\mathcal{H}^{hpq;n}\mathcal{H}^k{}_{ps;r}\mathcal{H}^m{}_q{}^{s;r}\big] \tag{3}$$

where $\mathcal{H}$ is the B-field strength, $\mathcal{H}_{abc}=B_{ab,c}+B_{ca,b}+B_{bc,a}$. As usual, the commas and the semicolons represent partial and covariant derivatives, respectively. We have also explicitly confirmed the above couplings with the S-matrix element of four B-fields in type II superstring theory. In Section 5, we consider the consistency of the couplings $\mathcal{RRR}_y\mathcal{R}_y$ and $\mathcal{HHH}_y\mathcal{H}_y$ with the linear T-duality to find $\mathcal{RRHH}$ couplings. The couplings $\mathcal{HHR}_y\mathcal{R}_y$ and $\mathcal{RRH}_y\mathcal{H}_y$ must be the T-duality transformations of $\mathcal{HHH}_y\mathcal{H}_y$ and $\mathcal{RRR}_y\mathcal{R}_y$, respectively. Moreover, the couplings $\mathcal{H}_y\mathcal{H}_y\mathcal{R}_y\mathcal{R}_y$ and $\mathcal{HH}_y\mathcal{RR}_y$ each must be invariant under the T-duality transformations. Imposing these conditions and using on-shell relations, we have found the following couplings:

$$\begin{aligned}\mathcal{L}\supset\frac{\gamma e^{-2\phi_0}}{2\kappa^2}\big[&\mathcal{R}_{hmkn}\mathcal{R}^{mpnq}\mathcal{H}^{krs}{}_{;q}\mathcal{H}_{prs}{}^{;h}\\&-2\mathcal{R}_{hrps}\mathcal{R}^{qrks}\mathcal{H}^h{}_{kn;m}\mathcal{H}^{np}{}_q{}^{;m}\\&+2\mathcal{R}_{mpnq}\mathcal{R}^{qrks}\mathcal{H}^h{}_k{}^{n;m}\mathcal{H}_h{}^p{}_{s;r}+\mathcal{R}_{mnpq}\mathcal{R}^{qrks}\mathcal{H}^{hmn}{}_{;k}\mathcal{H}_h{}^p{}_{s;r}\\&+2\mathcal{R}_{mpnq}\mathcal{R}^{qrks}\mathcal{H}_k{}^{mn;h}\mathcal{H}^p{}_{rs;h}+\mathcal{R}_{hmkn}\mathcal{R}^{mpnq}\mathcal{H}^h{}_{ps;r}\mathcal{H}^k{}_q{}^{s;r}\\&-2\mathcal{R}_{mpnq}\mathcal{R}^{qrks}\mathcal{H}_k{}^{mn;h}\mathcal{H}_h{}^p{}_{s;r}+2\mathcal{R}_{hmkn}\mathcal{R}^{mpnq}\mathcal{H}^k{}_{qs;r}\mathcal{H}_p{}^{rs;h}\\&-6\mathcal{R}_{hrps}\mathcal{R}^{qrks}\mathcal{H}^h{}_{kn;m}\mathcal{H}^{mn}{}_q{}^{;p}\big]\end{aligned} \tag{4}$$

In Section 6, we discuss the dilaton couplings. We argue that many terms of the dilaton amplitudes are reproduced by transforming the string frame couplings (2) and (4) to the Einstein frame. However, there are some terms in the scattering amplitudes that cannot be reproduced in this way. The scattering amplitude of two dilatons and two gravitons produces the following couplings as well as the couplings in (2):

$$\mathcal{L}\supset-\frac{\gamma e^{-2\phi_0}}{16\kappa^2}\big[\mathcal{R}^{hk}{}_{mn}\mathcal{R}^{mnpq}\Phi_{;hp}\Phi_{;kq}+2\mathcal{R}^h{}_m{}^k{}_n\mathcal{R}^{mpnq}\Phi_{;hp}\Phi_{;kq}+2\mathcal{R}^h{}_m{}^k{}_n\mathcal{R}^{qmpn}\Phi_{;hp}\Phi_{;kq}\big] \tag{5}$$

The scattering amplitude of four dilatons produces the following couplings:

$$\mathcal{L}\supset-\frac{\gamma e^{-2\phi_0}}{64\kappa^2}\big[\Phi_{;hn}\Phi^{;hs}\Phi^{;nq}\Phi_{;qs}-\Phi_{;mn}\Phi^{;mn}\Phi_{;rs}\Phi^{;rs}\big] \tag{6}$$

And the scattering amplitude of two dilatons and two B-fields produces the couplings:

$$\begin{aligned}\mathcal{L}\supset\frac{\gamma e^{-2\phi_0}}{16\kappa^2}\big[&6\Phi_{;hp}\Phi_{;kq}\mathcal{H}^{hkn;m}\mathcal{H}_{mn}{}^{q;p}\\&+2\Phi_{;hp}\Phi_{;kq}\mathcal{H}^{hkn;m}\mathcal{H}_n{}^{pq}{}_{;m}-\Phi_{;hk}\Phi_{;pq}\mathcal{H}^{hpn;m}\mathcal{H}^{kq}{}_{n;m}\\&-2\Phi_{;hk}\Phi_{;pq}\mathcal{H}^{kqn;m}\mathcal{H}^p{}_{mn}{}^{;h}-\Phi_{;hk}\Phi_{;pq}\mathcal{H}^{kmn;q}\mathcal{H}^p{}_{mn}{}^{;h}\big]\end{aligned}$$

In Section 7, we briefly discuss our results.

2. T-duality

The full set of nonlinear T-duality transformations have been found in [11]. When the T-duality transformation acts along the Killing coordinate y, the massless NS–NS fields transform as:

$$\begin{aligned}&e^{2\tilde\Phi}=\frac{e^{2\Phi}}{G_{yy}};\qquad \tilde G_{yy}=\frac{1}{G_{yy}}\\&\tilde G_{\mu y}=\frac{B_{\mu y}}{G_{yy}};\qquad \tilde G_{\mu\nu}=G_{\mu\nu}-\frac{G_{\mu y}G_{\nu y}-B_{\mu y}B_{\nu y}}{G_{yy}}\\&\tilde B_{\mu y}=\frac{G_{\mu y}}{G_{yy}};\qquad \tilde B_{\mu\nu}=B_{\mu\nu}-\frac{B_{\mu y}G_{\nu y}-G_{\mu y}B_{\nu y}}{G_{yy}}\end{aligned} \tag{7}$$

where μ,ν denote any coordinate directions other than y. In above transformation the metric is given in the string frame. If y is identified on a circle of radius ρ, *i.e.*, $y\sim y+2\pi\rho$, then after T-duality the radius becomes $\tilde\rho=\alpha'/\rho$. The string coupling $g=e^{\phi_0}$ is also shifted as $\tilde g=g\sqrt{\alpha'}/\rho$.

We would like to study the T-dual Ward identity [27,24] of the scattering amplitude of four gravitons, so we need the above transformations at the linear order. Assuming that the NS–NS fields are small perturbations around the background, *i.e.*,

$$\begin{aligned}&G_{\mu\nu}=\eta_{\mu\nu}+2\kappa h_{\mu\nu};\qquad G_{yy}=\frac{\rho^2}{\alpha'}(1+2\kappa h_{yy})\\&G_{\mu y}=2\kappa h_{\mu y}\\&B_{\mu\nu}=2\kappa b_{\mu\nu};\qquad B_{\mu y}=2\kappa b_{\mu y}\\&\Phi=\phi_0+\sqrt{2}\kappa\phi\end{aligned} \tag{8}$$

the transformations (7) take the following linear form for the perturbations:

$$\begin{aligned}&\sqrt2\tilde\phi=\sqrt2\phi-h_{yy};\qquad \tilde h_{yy}=-h_{yy};\qquad \tilde h_{\mu y}=b_{\mu y}\\&\tilde b_{\mu y}=h_{\mu y};\qquad \tilde h_{\mu\nu}=h_{\mu\nu};\qquad \tilde b_{\mu\nu}=b_{\mu\nu}\end{aligned} \tag{9}$$

To study the linear T-duality of the couplings (2), it is convenient to find the T-duality transformation of the linearized curvature tensors. The Riemann tensor at the linear order in graviton is given by

$$R_{abcd}=\kappa(h_{ad,bc}+h_{bc,ad}-h_{ac,bd}-h_{bd,ac}) \tag{10}$$

In the case that one of its indices is the y-index, *i.e.*, R_{abcy} where y is the Killing direction along which the T-duality is to be performed, it becomes

$$R_{abcy}=\kappa(h_{ay,bc}+h_{bc,ay}-h_{ac,by}-h_{by,ac})=\kappa(h_{ay,bc}-h_{by,ac}) \tag{11}$$

where the second equality assumes that all fields are independent of the T-dual coordinate y. This becomes after T-duality,

$$\kappa(b_{ay,bc}-b_{by,ac}) \tag{12}$$

to which one may trivially add $\kappa b_{ba,yc}$ since the fields are assumed independent of y, and hence complete the exterior derivative. Therefore, the linearized transformation of the Riemann tensor with one y-index is

$$R_{abcy}\leftrightarrow-\kappa H_{aby,c} \tag{13}$$

where $H_{\mu\nu\rho}=b_{\mu\nu,\rho}+b_{\rho\mu,\nu}+b_{\nu\rho,\mu}$. Here the arrow goes in both directions since the derivation can clearly be run in reverse and hence these two expressions are exchanged under T-duality.

In the case that two indices of the Riemann tensor are the y indices, *i.e.*, R_{ayby}, it becomes

$$R_{ayby}=-\frac{\kappa\rho^2}{\alpha'}h_{yy,ab}\leftrightarrow\frac{\kappa\rho^2}{\alpha'}h_{yy,ab}=-R_{ayby} \tag{14}$$

where again derivatives of h with respect to y have been dropped and added in order to complete the curvatures. Note that due to the complete antisymmetric of the B-field strength no analogous terms with a double y index will be relevant. That is $H_{ayy}=0$ by the antisymmetry of the indices. Similarly, $R_{abyy}=0$ by antisymmetry.

The transformation of the Ricci curvature tensor involves the dilaton as well as the B-field strength. To see this consider the case that none of the indices of the Ricci tensor carries the y index. It transforms as

$$\begin{aligned} R_{ab} &= \eta^{cd}R_{cadb}+\frac{\alpha'}{\rho^2}R_{yayb} \\ &= \eta^{cd}R_{cadb}-\kappa h_{yy,ab}\to \eta^{cd}R_{cadb}+\kappa h_{yy,ab} \\ &= R_{ab}+2\kappa h_{yy,ab} \end{aligned} \tag{15}$$

where in the first identity we have separated the contracted indices along and orthogonal to y. The last term is not tensor, so there must be another term whose T-duality cancels that term. Using the linear transformation of the dilaton (9), one finds the following combination is invariant:

$$R_{ab}+2\sqrt{2}\kappa\phi_{,ab}\leftrightarrow R_{ab}+2\sqrt{2}\kappa\phi_{,ab} \tag{16}$$

Similarly the transformation of the Ricci curvature when it carries one or two y indices, and the transformation of the scalar curvature are

$$\begin{aligned} &R_{ay}\leftrightarrow \kappa H_{aby}{}^{,b} \\ &R_{yy}\leftrightarrow -R_{yy} \\ &R+4\sqrt{2}\kappa\phi_{,a}{}^{a}\leftrightarrow R+4\sqrt{2}\kappa\phi_{,a}{}^{a} \end{aligned} \tag{17}$$

The last transformation in particular indicates that the supergravity Lagrangian must include a Laplacian of the dilaton to be invariant under the T-duality (see Eq. (1.10) in [28] for the presence of such term in the T-dual Lagrangian at leading order of α').

To extend a coupling to a set of couplings which are invariant under linear T-duality, we first use the dimensional reduction to reduce the 10-dimensional couplings to 9-dimensional couplings, *i.e.*, separate the indices along and orthogonal to y, and then apply the above T-duality transformations. If the original coupling is not invariant under the T-duality, one must add new terms to make them invariant.

3. Four-point amplitude

The scattering amplitude of four massless NS–NS states with polarization tensors ε^{ab} in covariant formalism is given by [14]

$$\begin{aligned} \mathcal{A} = &-\frac{\alpha'^3\kappa^2e^{-2\phi_0}}{16}\frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)} \\ &\times \varepsilon_1^{a_1b_1}\varepsilon_2^{a_2b_2}\varepsilon_3^{a_3b_3}\varepsilon_4^{a_4b_4}K_{a_1a_2a_3a_4}K_{b_1b_2b_3b_4} \end{aligned} \tag{18}$$

There is also a factor of delta function $\delta^{10}(k_1+k_2+k_3+k_4)$ imposing conservation of momentum. The Mandelstam variables $s=-4\alpha' k_1\cdot k_2$, $t=-4\alpha' k_1\cdot k_3$, $u=-4\alpha' k_2\cdot k_3$ satisfy $s+t+u=0$, and

$$\begin{aligned} K_{a_1a_2a_3a_4} = 4\big[&-k_2.k_1k_3.k_1\eta_{a1a4}\eta_{a2a3}-k_2.k_1k_2.k_3\eta_{a1a3}\eta_{a2a4} \\ &-k_2.k_3k_3.k_1\eta_{a1a2}\eta_{a3a4}+k_2.k_1\eta_{a1a4}(k_1)_{a2}(k_1)_{a3} \\ &+k_3.k_1\eta_{a1a4}(k_1)_{a2}(k_1)_{a3}-k_2.k_1\eta_{a2a4}(k_1)_{a3}(k_2)_{a1} \\ &-k_3.k_1\eta_{a2a4}(k_1)_{a3}(k_2)_{a1}+k_3.k_1\eta_{a1a4}(k_1)_{a2}(k_2)_{a3} \\ &-k_3.k_1\eta_{a2a4}(k_2)_{a1}(k_2)_{a3}+k_2.k_1\eta_{a1a2}(k_1)_{a3}(k_2)_{a4} \\ &+k_3.k_1\eta_{a1a2}(k_1)_{a3}(k_2)_{a4}+k_3.k_1\eta_{a1a2}(k_2)_{a3}(k_2)_{a4} \\ &-k_2.k_1\eta_{a3a4}(k_1)_{a2}(k_3)_{a1}-k_3.k_1\eta_{a3a4}(k_1)_{a2}(k_3)_{a1} \\ &+k_2.k_1\eta_{a2a4}(k_2)_{a3}(k_3)_{a1}-k_2.k_1\eta_{a2a3}(k_2)_{a4}(k_3)_{a1} \\ &+k_2.k_1\eta_{a1a4}(k_1)_{a3}(k_3)_{a2}+k_3.k_1\eta_{a3a4}(k_2)_{a1}(k_3)_{a2} \\ &+k_2.k_1\eta_{a1a3}(k_2)_{a4}(k_3)_{a2}-k_2.k_1\eta_{a3a4}(k_3)_{a1}(k_3)_{a2} \\ &+k_2.k_1\eta_{a1a3}(k_1)_{a2}(k_3)_{a4}+k_3.k_1\eta_{a1a3}(k_1)_{a2}(k_3)_{a4} \\ &-k_3.k_1\eta_{a2a3}(k_2)_{a1}(k_3)_{a4}+k_3.k_1\eta_{a1a2}(k_2)_{a3}(k_3)_{a4} \\ &+k_2.k_1\eta_{a1a3}(k_3)_{a2}(k_3)_{a4}\big] \end{aligned} \tag{19}$$

The on-shell conditions are $k_i^2=k_i\cdot\varepsilon_i=\varepsilon_i\cdot k_i=0$. The polarization tensor is symmetric and traceless for graviton, antisymmetric for B-field and for dilaton it is

$$\varepsilon^{ab}=\frac{\phi}{\sqrt{8}}\big(\eta^{ab}-k^a\ell^b-k^b\ell^a\big) \tag{20}$$

where ℓ^a is an auxiliary vector which satisfies $k\cdot\ell=1$ and ϕ is the dilaton polarization which is one. In equation (19) we have used the conservation of momentum to write the amplitude in terms of momentum k_1,k_2,k_3. We have also used the on-shell conditions to rewrite $k_1\cdot\varepsilon_4=-k_2\cdot\varepsilon_4-k_3\cdot\varepsilon_4$, similarly for $\varepsilon_4\cdot k_1$. We have normalized the amplitude (18) to be consistent with the normalization factor in the couplings (2).

The coupling (1) has been found in [5] from the amplitude (18) by expanding the gamma functions at low energy

$$\frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)}=-\frac{2^9}{stu}-2\zeta(3)+\cdots \tag{21}$$

The first term corresponds to the massless poles in the four-point function which are reproduced by the Einstein–Hilbert action [15], and the second term,

$$\begin{aligned} \Delta\mathcal{A} = {}&\gamma\kappa^2e^{-2\phi_0}\varepsilon_1^{a_1b_1}\varepsilon_2^{a_2b_2}\varepsilon_3^{a_3b_3}\varepsilon_4^{a_4b_4}K_{a_1a_2a_3a_4}K_{b_1b_2b_3b_4}\delta^{10} \\ &\times(k_1+k_2+k_3+k_4) \end{aligned} \tag{22}$$

corresponds to the coupling (1) [5]. The explicit form of the above amplitude has too many terms to write them all. It has almost all structures of the contractions of the four polarization tensors and the eight momenta. Let us mention which structures the amplitude (22) does not have. The structure of (19) dictates that $\Delta\mathcal{A}$ does not have $(k\cdot\varepsilon\cdot k)^4$, $k\cdot k(k\cdot\varepsilon\cdot k)^2k\cdot\varepsilon\cdot\varepsilon\cdot k$ and $(k\cdot k)^3k\cdot\varepsilon\cdot k\mathrm{Tr}[\varepsilon\cdot\varepsilon\cdot\varepsilon]$ structures. Obviously, it does not have structures which contain $\mathrm{Tr}[\varepsilon]$ either.

The couplings (2) have been found in [6,7] by writing the eight-dimensional tensor $t^{i_1\cdots i_8}t_{j_1\cdots j_8}$ in terms of 10-dimensional tensors. These couplings can also be verified by explicit comparison with the amplitude (22). To this end, one has to calculate the four-graviton amplitude from (2) which is

$$\begin{aligned} A(1,2,3,4) = \gamma\kappa^2e^{-2\phi_0}\Big[&(R_1)_{hmnk}(R_2)_p{}^{mn}{}_q(R_3)^{hrsp}(R_4)^q{}_{rs}{}^k \\ &+\frac{1}{2}(R_1)_{hkmn}(R_2)_{pq}{}^{mn}(R_3)^{hrsp}(R_4)^q{}_{rs}{}^k+\cdots\Big] \end{aligned} \tag{23}$$

where dots represent the 23 other permutation terms. In above amplitude the subscripts 1, 2, 3, 4 are the particle labels, and

$$(R_1)^{hmnk}=-\big(\varepsilon_1^{hk}k_1^mk_1^n+\varepsilon_1^{mn}k_1^hk_1^k-\varepsilon_1^{hn}k_1^mk_1^k-\varepsilon_1^{mk}k_1^hk_1^n\big) \tag{24}$$

Using the on-shell conditions to write the field theory amplitude in terms of k_1,k_2,k_3 and write $k_1\cdot\varepsilon_4$ in terms of $k_2\cdot\varepsilon_4$ and $k_3\cdot\varepsilon_4$, as in string theory amplitude (18), we have found exact agreement with the string theory amplitude when the polarization tensors are symmetric.

4. $(\partial H)^4$ couplings

The scattering amplitude of four B-fields can be read from (18) by using antisymmetric polarizations ε^{ab}. A proposal for the B-field couplings in field theory is given in [9] which is the replacement $R_{abcd} \to R_{abcd} + \kappa e^{-\phi_0/2} H_{ab[c,d]}$. This proposal gives the $(\partial H)^4$ couplings at order α'^3 by using the following replacement for the Riemann curvature:

$$R_{abcd} \to \kappa e^{-\phi_0/2} H_{ab[c,d]} = \kappa e^{-\phi_0/2}(b_{ad,bc} + b_{bc,ad} - b_{ac,bd} - b_{bd,ac}) \qquad (25)$$

We have explicitly check that while the above replacement in the Lagrangian given in [9] produces correctly the string amplitude (22), this replacement in the Lagrangian (2) does not produce correctly the B-field couplings. In particular, the above replacement in the Lagrangian (2) produces terms with structure $k \cdot k(k \cdot \varepsilon \cdot k)^2 k \cdot \varepsilon \cdot \varepsilon \cdot k$ whereas the string theory amplitude (22) does not produce such structure. The reason for this apparently inconsistency is that the identity that relates the Lagrangian given in [9] to (2) is not an identity any more when one uses the above replacement for the Riemann curvatures in that identity. For example, the Bianchi identity for the curvature is not an identity when one uses the replacement (25). In this Letter, we would like to find the B-field couplings corresponding to the couplings (2) by using the compatibility of this Lagrangian with the linear T-duality transformations.

The S-matrix elements in string theory must satisfy the Ward identity corresponding to the T-duality [27,24]. This means the scattering amplitude (18) must be invariant under linear T-duality transformations (9) on the quantum fluctuations and must be invariant under nonlinear T-duality transformation (7) on the background fields. One can easily verify that the background factor $e^{-2\phi_0}\delta^{10}(k_1+k_2+k_3+k_4) = (2\pi\rho)e^{-2\phi_0}\delta^9(k_1+k_2+k_3+k_4)$ where ρ is the radius of the circle along which the T-duality is implemented, is invariant under the T-duality. The amplitude (22) which is the string amplitude at order α'^3, has no massless pole so the T-dual Ward identity dictates that couplings in the spacetime must be invariant under the linear T-duality. In spacetime, the invariance of the background under the nonlinear T-duality appears as the invariant of the factor $e^{-2\phi_0}\sqrt{-G}$ in the action.

Now let us apply the linear T-duality on the quantum fluctuations in (2) to find the couplings of four B-fields. We first use the dimensional reduction to reduce the action to 9 dimensions, and then apply the T-duality transformations on them. The terms in which the Riemann tensors carry one Killing index y are the following[1]:

$$\frac{\gamma e^{-2\phi_0}}{2\kappa^2}[-8R_{knhy}R_{nqpy}R_{ksqy}R_{pshy} + 4R_{hkny}R_{pqny}R_{ksqy}R_{pshy} - 2R_{mnky}R_{mnpy}R_{ksry}R_{psry} - 4R_{knmy}R_{mpny}R_{ksry}R_{psry}] \qquad (26)$$

We have to find new couplings of four H such that their dimensional reduction transform to the above couplings under the linear T-duality (13). Consider the following couplings:

$$\frac{\gamma\kappa^2 e^{-2\phi_0}}{2}[-2H_{hpr,k}H_{npr,q}H_{kms,q}H_{mns,h} - 2H_{hmn,k}H_{hnp,q}H_{kps,r}H_{mqs,r} + 2H_{kmn,h}H_{hpq,n}H_{kps,r}H_{mqs,r}] \qquad (27)$$

The dimensional reduction of these couplings produces the following terms:

$$\frac{\gamma\kappa^2 e^{-2\phi_0}}{2}[-8H_{kmy,q}H_{mny,h}H_{hpy,k}H_{npy,q} + 4H_{hmy,k}H_{hpy,q}H_{kpy,r}H_{mqy,r} - 4H_{mny,h}H_{hpy,n}H_{msy,r}H_{psy,r} - 2H_{hny,k}H_{hny,q}H_{ksy,r}H_{qsy,r}] \qquad (28)$$

which are the transformation of (26) under the T-duality transformation (13). Therefore, the couplings (27) are the prediction of T-duality for the couplings of four ∂H at order α'^3. We have also calculated its scattering amplitude and find exact agreement with the string theory amplitude (22) when the polarization tensors are antisymmetric. Since both the above couplings and the $(\partial H)^4$ couplings in [9] are reproduced by the string theory amplitude (22), they must be identical up to some identities. Extending the linearized couplings (27) to nonlinear, one finds the couplings in (3).

5. $R^2(\partial H)^2$ couplings

There has been one consistency condition for the couplings (27), *i.e.*, under the dimensional reduction its $H_yH_yH_yH_y$ terms must be transformed to (26) under T-duality (13). So it was relatively easy to find these terms. The dimensional reduction of the couplings $RRHH$ however must satisfy four consistency conditions: 1-Their RRH_yH_y terms must transform under T-duality (13) to the RRR_yR_y terms of the couplings (2). 2-Their HHR_yR_y terms must transform under T-duality (13) to the HHH_yH_y terms of the couplings (27). 3-Their HH_yRR_y terms must be invariant under (13). 4-Their $H_yH_yR_yR_y$ terms must be invariant. So it is nontrivial to find such couplings.

Let us consider the RRR_yR_y terms of the dimensional reduction of the couplings (2) which are given by

$$\frac{\gamma e^{-2\phi_0}}{2\kappa^2}[-4R_{knhy}R_{nqpy}R_{hrps}R_{qrks} + 2R_{hkny}R_{pqny}R_{hrps}R_{qrks} + 2R_{hkmn}R_{mnpq}R_{ksqy}R_{pshy} + 4R_{hmkn}R_{mpnq}R_{ksqy}R_{pshy} + 4R_{mnpq}R_{mnky}R_{qrks}R_{psry} + 8R_{knmy}R_{mpnq}R_{qrks}R_{psry}] \qquad (29)$$

One may use the T-duality transformation (13) to find RRH_yH_y terms and then extend the y-index in them to a complete index. In this way one can find the $RRHH$ couplings which are consistent with the above couplings. However, it turns out that their HH_yRR_y terms would not be invariant under T-duality. They would not be consistent with the S-matrix element (22) either. That means there must be some other terms as well as those found by using the transformation (13).

The only possibility for extending the transformation (13) is to add the trivial term $H_{abc,y}$, *i.e.*,

$$R_{abcy} \to -\kappa H_{aby,c} + \alpha H_{abc,y} \qquad (30)$$

where the coefficient α is an arbitrary constant. The above extra term in the transformation of the Riemann tensor is zero because of the implicit assumption in the T-duality transformations that fields are independent of the Killing coordinate. However, in extending the y-index to a complete index that term makes nontrivial contribution. So we use the above transformation for the Riemann tensor in the couplings (29) and then extend the y-index to a complete index. In doing this one has to use different constants for the coefficients α's in each replacement because a priori we do not know which replacement has such extra term.[2] We have

[1] From now on we use only subscripts indices and the repeated indices are contracted with the flat metric.

[2] The replacement (25) corresponds to $\alpha = 1$. However, as we mentioned in the previous section such replacement in the Lagrangian (2) does not produce correctly the B-field couplings.

tried to find such coefficients by imposing T-duality transformations. We could not find a consistent set of coefficients in this way unless we use the on-shell relations. Alternatively, one may find these constants by comparing the result with the on-shell S-matrix element (22). We find the following result:

$$\begin{aligned}
&\frac{\gamma e^{-2\phi_0}}{2}\big[4R_{hmkn}R_{mpnq}H_{krs,q}H_{prs,h} - 8R_{hrps}R_{qrks}H_{hkn,m}H_{npq,m}\\
&\quad + 8R_{mpnq}R_{qrks}H_{hkn,m}H_{hps,r} + 4R_{mnpq}R_{qrks}H_{hmn,k}H_{hps,r}\\
&\quad + 8R_{mpnq}R_{qrks}H_{kmn,h}H_{prs,h} + 4R_{hmkn}R_{mpnq}H_{hps,r}H_{kqs,r}\\
&\quad - 8R_{mpnq}R_{qrks}H_{kmn,h}H_{hps,r} + 8R_{hmkn}R_{mpnq}H_{kqs,r}H_{prs,h}\\
&\quad - 24R_{hrps}R_{qrks}H_{hkn,m}H_{mnq,p}\big]
\end{aligned}\tag{31}$$

Plus some other terms that their coefficients cannot be fixed by the four-point function calculations. They, however, cancels each other when we write them in terms of h and b instead of their field strengths. That means there are identities at four NS–NS level that cancel the terms that are not fixed by the S-matrix calculations. Those identities may not hold at five NS–NS level. As a result, there may be some other couplings that their coefficients can be fixed by analyzing the five-point functions. We have checked that the couplings (31) satisfy the above four T-duality constraints. In checking these constraints, one has to use the on-shell conditions. Since both the above couplings and the $R^2(\partial H)^2$ couplings in [9] are produced by the string theory amplitude (22), they must be identical up to some identities. The nonlinear extension of the couplings (31) appears in (4).

6. Dilaton couplings

The string frame couplings (2) and (4) can produce various dilaton couplings when transforming them to the Einstein frame. One may then expect that the dilaton S-matrix elements at order α'^3 are reproduced by these couplings in the Einstein frame. In this section we are going to show that the dilaton couplings in the Einstein frame do not fully reproduce the string theory amplitudes. Hence, the string frame field theory should contain some new dilaton couplings.

The S-matrix element of one dilaton and three gravitons in string theory side is given by (22) in which one of the polarizations is (20) and the other three are symmetric and traceless. On the other hand, the scattering amplitude of four symmetric polarization tensors must satisfy the Ward identity, that is, if one replaces each polarization tensor by $\varepsilon^{ab} \to k^a\zeta^b + k^b\zeta^a$ where ζ^a is an arbitrary vector, the amplitude must be zero. This indicates that the term $-k^a\ell^b - k^b\ell^a$ in the dilaton polarization (20) must disappear in the string amplitude of one dilaton and three symmetric polarization tensors, *i.e.*, the dilaton polarization is effectively $\varepsilon_1^{ab} = \phi_1\eta^{ab}/\sqrt{8}$. This replacement cancels many terms in (22). The surviving terms are the following:

$$\begin{aligned}
\Delta\mathcal{A} = \frac{\gamma\kappa^2 e^{-2\phi_0}}{2\sqrt{8}}&\big[16(k_2.k_3)^2(k_3.k_1)^2\,\mathrm{Tr}[\varepsilon_2]\,\mathrm{Tr}[\varepsilon_3.\varepsilon_4]\\
&+ 16(k_2.k_1)^2k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 16(k_3.k_1)^2k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32(k_3.k_1)^2k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 16(k_3.k_1)^2k_2.\varepsilon_3.k_2k_2.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32(k_3.k_1)^2k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32(k_3.k_1)^2k_2.\varepsilon_3.k_2k_2.\varepsilon_4.k_3\,\mathrm{Tr}[\varepsilon_2]\\
&+ 16(k_3.k_1)^2k_2.\varepsilon_3.k_2k_3.\varepsilon_4.k_3\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32(k_2.k_1)^2k_3.k_1k_1.\varepsilon_3.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 64k_2.k_1(k_3.k_1)^2k_1.\varepsilon_3.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32(k_3.k_1)^3k_1.\varepsilon_3.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_2\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_3\,\mathrm{Tr}[\varepsilon_2]\\
&+ 32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_3\,\mathrm{Tr}[\varepsilon_2]\big]\phi_1\\
&+ (2\leftrightarrow 3) + (2\leftrightarrow 4)
\end{aligned}\tag{32}$$

which are zero when the polarizations are traceless. Note that the terms in (22) which contain the trace of four polarization tensors, *e.g.*, $\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_4]$, are canceled when one of the polarization is replace by $\phi\eta_{ab}/\sqrt{8}$.

The string scattering amplitude produces couplings in the Einstein frame, so in the field theory side we consider the transformation of the string couplings (2) to the Einstein frame, *i.e.*, $G_{ab} = e^{\Phi/2}G^E_{ab}$. At the linear order it gives $h_{ab} = h^E_{ab} + \phi\eta_{ab}\sqrt{8}$, and in terms of the linearized Riemann curvature it becomes $R_{ab}{}^{cd} = R^E_{ab}{}^{cd} - \kappa\eta_a^{[c}\phi_{,b]}{}^{d]}/\sqrt{8}$. In the field theory couplings (2), one must then replace one of the polarizations by $\phi\eta_{ab}/\sqrt{8}$, hence, one again finds zero result for the scattering amplitude of one dilaton and three gravitons. So it confirms that there is no coupling of one dilaton and three gravitons in the string frame [26,8] or in the Einstein frame. This is not the case, however, for the couplings of two dilatons and two gravitons as we shall see below.

The scattering amplitude of two dilatons and two symmetric tensors in string theory side can be read from the amplitude (32) by replacing the polarization ε_2 with (20). Apart from the terms containing the trace of ε_2 which is $\mathrm{Tr}[\varepsilon_2] = \phi_2\sqrt{8}$, the auxiliary term $-k_2^a\ell_2^b - k_2^b\ell_2^a$ in the dilaton polarization (20) cancels in the terms in the last line of (32), hence, effectively for these terms the dilaton polarization is $\phi_2\eta_{ab}/\sqrt{8}$. The amplitude becomes

$$\begin{aligned}
\Delta\mathcal{A} = \frac{\gamma\kappa^2 e^{-2\phi_0}}{2}&\big[16(k_2.k_3)^2(k_3.k_1)^2\,\mathrm{Tr}[\varepsilon_3.\varepsilon_4]\\
&+ 16(k_2.k_1)^2k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2\\
&+ 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2\\
&+ 16(k_3.k_1)^2k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2\\
&+ 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_2\\
&+ 32(k_3.k_1)^2k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_2\\
&+ 16(k_3.k_1)^2k_2.\varepsilon_3.k_2k_2.\varepsilon_4.k_2\\
&+ 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3\\
&+ 32(k_3.k_1)^2k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3\\
&+ 32(k_3.k_1)^2k_2.\varepsilon_3.k_2k_2.\varepsilon_4.k_3\\
&+ 16(k_3.k_1)^2k_2.\varepsilon_3.k_2k_3.\varepsilon_4.k_3\\
&+ 32(k_2.k_1)^2k_3.k_1k_1.\varepsilon_3.\varepsilon_4.k_2\\
&+ 64k_2.k_1(k_3.k_1)^2k_1.\varepsilon_3.\varepsilon_4.k_2 + 32(k_3.k_1)^3k_1.\varepsilon_3.\varepsilon_4.k_2\\
&+ 32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_2 + 32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_2\\
&+ 32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_3 + 32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_3
\end{aligned}$$

$$+2(k_2.k_1)^2\big((k_2.k_1)^2+2k_2.k_1k_3.k_1$$
$$+2(k_3.k_1)^2\big)\mathrm{Tr}[\varepsilon_3]\mathrm{Tr}[\varepsilon_4]\big]\phi_1\phi_2 \tag{33}$$

where the terms in the last line of (32) appear in the last line of the above amplitude. Apart from these terms which are zero for graviton, all other terms are in fact the terms of the scattering amplitude (22) which include[3] $\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2]=\phi_1\phi_2$.

In transforming the couplings (2) to the Einstein frame, one transforms $h_{ab}h^{ba}=h^E_{ab}(h^E)^{ba}+\frac{10}{8}\phi^2$, or in terms of polarization it becomes $\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2]=\mathrm{Tr}[\varepsilon_1^E\cdot\varepsilon_2^E]+\frac{10}{8}\phi_1\phi_2$. So the above amplitude is not fully reproduced by transforming the string frame couplings (2) to the Einstein frame, *i.e.*, 5/4 of the above amplitude is reproduced by (2) and −1/4 of it is a new dilaton coupling in the string frame.

To find the field theory couplings corresponding to the above amplitude, we have to find the couplings in (2) which have $h_{ab}h^{ba}$ and use the replacement $h_{ab}h^{ba}\to\phi^2$ in them. On the other hand, in the dimensional reduction the term $h_{yy}h^{yy}$ is a component of $h_{ab}h^{ba}$. Hence, to find the couplings corresponding to the above amplitude, we have to find the $h_{yy}h^{yy}$-terms in the dimensional reduction of (2) and use the replacement $h_{yy}h^{yy}\to\phi^2$ in them. The dimensional reduction produces the following terms:

$$\frac{\gamma e^{-2\phi_0}}{2\kappa^2}\big[R_{hkmn}R_{mnpq}R_{hypy}R_{kyqy}+2R_{hmkn}R_{mpnq}R_{hypy}R_{kyqy}$$
$$+2R_{hrps}R_{qrks}R_{hyky}R_{pyqy}\big] \tag{34}$$

Using the fact that $R_{hypy}=-\frac{\rho^2\kappa}{\alpha'}h_{yy,hp}$, one finds the couplings corresponding to the amplitude (33) to be

$$\frac{\gamma e^{-2\phi_0}}{2}\big[R_{hkmn}R_{mnpq}\phi_{,hp}\phi_{,kq}+2R_{hmkn}R_{mpnq}\phi_{,hp}\phi_{,kq}$$
$$+2R_{hrps}R_{qrks}\phi_{,hk}\phi_{,pq}\big] \tag{35}$$

We have also checked it explicitly that the above couplings produce the amplitude (33). These couplings are also invariant under linear T-duality because the T-dual extension of the second derivative of the dilaton (16) contains the Ricci tensor which is zero on-shell. The nonlinear extension of the above couplings with the factor of −1/4 appears in (5).

The scattering amplitude of three dilatons and one symmetric tensor is given by the scattering amplitude of two dilatons and two symmetric tensors (33) in which one of the symmetric tensor is (20). For the term in the last line one must replace $\mathrm{Tr}[\varepsilon_3]=\phi_3\sqrt{8}$ and for all other terms one must replace $(\varepsilon_3)_{ab}=\phi_3\eta_{ab}/\sqrt{8}$. The result is

$$\Delta\mathcal{A}=2\sqrt{2}\gamma\kappa^2e^{-2\phi_0}\big[\big((k_2.k_1)^2$$
$$+k_2.k_1k_3.k_1+(k_3.k_1)^2\big)^2\mathrm{Tr}[\varepsilon_4]\big]\phi_1\phi_2\phi_3 \tag{36}$$

which is zero when the polarization tensor is traceless. Hence, there is no coupling of three dilatons and one graviton.

The scattering amplitude of four dilatons is given by the above amplitude in which the trace of ε_4 is replace by $\mathrm{Tr}[\varepsilon_4]=\phi_4\sqrt{8}$, *i.e.*,

$$\Delta\mathcal{A}=8\gamma\kappa^2e^{-2\phi_0}\big[\big((k_2.k_1)^2+k_2.k_1k_3.k_1+(k_3.k_1)^2\big)^2\big]\phi_1\phi_2\phi_3\phi_4 \tag{37}$$

The above terms are in fact the terms of the scattering amplitude of four symmetric tensors (22) which include the trace of two polarization tensors, *e.g.*, $\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2]\mathrm{Tr}[\varepsilon_3\cdot\varepsilon_4]=\phi_1\phi_2\phi_3\phi_4$. On the other hand, the $R_{yy}R_{yy}R_{yy}R_{yy}$ terms of the dimensional reduction of (2) produce the trace of two polarization tensors. In fact the traces of four polarization tensors *e.g.*, $\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_4]$ in the amplitude (22) are canceled when the polarizations commute inside the trace which is the case for the component ε_{yy} which appears in the couplings $R_{yy}R_{yy}R_{yy}R_{yy}$. So the couplings corresponding to the above amplitude can be read from $R_{yy}R_{yy}R_{yy}R_{yy}$ which are

$$\frac{\gamma e^{-2\phi_0}}{2\kappa^2}\big[-2R_{hyny}R_{hysy}R_{nyqy}R_{qysy}+2R_{myny}R_{myny}R_{rysy}R_{rysy}\big] \tag{38}$$

Inspired by these couplings, one finds the couplings corresponding to (37) to be

$$\frac{\gamma\kappa^2e^{-2\phi_0}}{2}\big[-2\phi_{,hn}\phi_{,hs}\phi_{,nq}\phi_{,qs}+2\phi_{,mn}\phi_{,mn}\phi_{,rs}\phi_{,rs}\big] \tag{39}$$

We have also checked the above couplings by direct comparison with the amplitude (37). In transforming the couplings (2) and (5) to the Einstein frame one transforms $\mathrm{Tr}[\varepsilon_i\cdot\varepsilon_j]\mathrm{Tr}[\varepsilon_k\cdot\varepsilon_l]$ in (2) to $(\frac{10}{8})^2\phi_i\phi_j\phi_k\phi_l$, and $-\frac{1}{4}\phi_1\phi_2\mathrm{Tr}[\varepsilon_3\cdot\varepsilon_4]$ in (5) to $-\frac{5}{8}\phi_1\phi_2\phi_3\phi_4$. So 25/16 − 10/16 of the above amplitude is reproduced by transforming the couplings (2) and (5) to the Einstein frame and 1/16 of it is a new dilaton coupling in the string frame. The nonlinear extension of (39) with the factor of 1/16 appears in (6).

We finally consider the couplings involving the dilaton and the B-field. The scattering amplitude of two symmetric tensors and two B-fields has no trace of one symmetric tensor, consequently, the scattering amplitude of one dilaton, one symmetric tensor and two B-fields is given by the former amplitude in which one of the symmetric tensor is replaced by $\phi\eta_{ab}/\sqrt{8}$. The result is

$$\Delta\mathcal{A}=\frac{\gamma\kappa^2e^{-2\phi_0}}{2\sqrt{8}}\big[32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3$$
$$+32(k_3.k_1)^2k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3$$
$$-32(k_2.k_1)^2k_3.k_1k_1.\varepsilon_3.\varepsilon_4.k_2$$
$$-64k_2.k_1(k_3.k_1)^2k_1.\varepsilon_3.\varepsilon_4.k_2$$
$$-32(k_3.k_1)^3k_1.\varepsilon_3.\varepsilon_4.k_2-32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_2$$
$$-32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_2-32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_3$$
$$-32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_3-16(k_2.k_1)^2(k_3.k_1)^2\mathrm{Tr}[\varepsilon_3.\varepsilon_4]$$
$$-32k_2.k_1(k_3.k_1)^3\mathrm{Tr}[\varepsilon_3.\varepsilon_4]$$
$$-16(k_3.k_1)^4\mathrm{Tr}[\varepsilon_3.\varepsilon_4]\big]\mathrm{Tr}[\varepsilon_2]\phi_1+\cdots \tag{40}$$

where dots refer to the terms which are not proportional to $\mathrm{Tr}[\varepsilon_2]$. They are reproduced by transforming the couplings (31) to the Einstein frame. So there is no coupling of one dilaton, one graviton and two B-fields in the string frame.

The scattering amplitude of two dilatons and two B-fields is given by the amplitude (40) in which the symmetric polarization is (20). The terms in which the polarization appears as $\mathrm{Tr}[\varepsilon_2]$, are invariant under the Ward identity associated with the symmetric tensor. For these terms one should replace $\mathrm{Tr}[\varepsilon_2]=\phi_2\sqrt{8}$. The result is the following:

$$\Delta\mathcal{A}_1=\frac{\gamma\kappa^2e^{-2\phi_0}}{2}\big[16k_3.k_1k_3.k_2\big(-2k_1.\epsilon_3.k_2k_2.\epsilon_4.k_3$$
$$+k_2.k_1\big(2k_1.\epsilon_3.\epsilon_4.k_2+k_3.k_1\mathrm{Tr}[\epsilon_3.\epsilon_4]\big)$$

[3] Note that the proposal given in [9] that extends the Riemann curvature to include the dilaton, *i.e.*, $R_{ab}{}^{cd}\to R_{ab}{}^{cd}-\kappa\eta_a^{[c}\phi_{,b]}{}^{d]}/\sqrt{8}$, is equivalent to extension $h_{ab}\to h_{ab}+\phi\eta_{ab}/\sqrt{8}$. This gives $\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2]\to\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2]+\phi_1\phi_2$ in the eight-dimensional transverse space of the light-cone formalism.

$$+ k_3.k_1\big(2(k_1.\epsilon_3.\epsilon_4.k_2 + k_2.\epsilon_3.\epsilon_4.k_2 + k_2.\epsilon_3.\epsilon_4.k_3) + k_3.k_1\mathrm{Tr}[\epsilon_3.\epsilon_4]\big)\big)\big]\phi_1\phi_2 \tag{41}$$

The other terms separately satisfy the Ward identity associated with the symmetric tensor. So $-k_2^a\ell_2^b - k_2^b\ell_2^a$ in the dilaton polarization (20) cancels in these terms, hence, effectively the dilaton polarization is $\phi_2\eta_{ab}/\sqrt{8}$. The result in this case is

$$\Delta\mathcal{A}_2 = \frac{\gamma\kappa^2 e^{-2\phi_0}}{2}\big[-8(k_2.k_1)^2\big(-2k_1.\epsilon_3.k_2k_2.\epsilon_4.k_3 + k_2.k_1\big(2k_1.\epsilon_3.\epsilon_4.k_2 + k_3.k_1\mathrm{Tr}[\epsilon_3.\epsilon_4]\big) + k_3.k_1\big(2(k_1.\epsilon_3.\epsilon_4.k_2 + k_2.\epsilon_3.\epsilon_4.k_2 + k_2.\epsilon_3.\epsilon_4.k_3) + k_3.k_1\mathrm{Tr}[\epsilon_3.\epsilon_4]\big)\big)\big]\phi_1\phi_2 \tag{42}$$

The above amplitude is reproduced by transforming the couplings (31) to the Einstein frame.

To find the couplings corresponding to the amplitude (41), we note that these terms are the terms of the scattering amplitude of two symmetric tensors and two B-fields which are proportional to $\mathrm{Tr}[\varepsilon_1\cdot\varepsilon_2] = \phi_1\phi_2$. So the couplings corresponding to (41) may be read from the $HHR_{yy}R_{yy}$ terms of the dimensional reduction of the couplings (31) which are

$$\frac{\gamma e^{-2\phi_0}}{2}\big[-24R_{hypy}R_{kyqy}H_{hkn,m}H_{mnq,p} - 8R_{hypy}R_{kyqy}H_{hkn,m}H_{npq,m} + 4R_{hyky}R_{pyqy}H_{hps,r}H_{kqs,r} + 8R_{hyky}R_{pyqy}H_{kqs,r}H_{prs,h} + 4R_{hyky}R_{pyqy}H_{krs,q}H_{prs,h}\big]$$

Inspired by this, one finds the following couplings of two dilatons and two B-fields:

$$\frac{\gamma\kappa^2 e^{-2\phi_0}}{2}\big[-24\phi_{,hp}\phi_{,kq}H_{hkn,m}H_{mnq,p} - 8\phi_{,hp}\phi_{,kq}H_{hkn,m}H_{npq,m} + 4\phi_{,hk}\phi_{,pq}H_{hps,r}H_{kqs,r} + 8\phi_{,hk}\phi_{,pq}H_{kqs,r}H_{prs,h} + 4\phi_{,hk}\phi_{,pq}H_{krs,q}H_{prs,h}\big]$$

We have checked explicitly that the above couplings produce the amplitude (40). Here again 5/4 of the above couplings are reproduced by transforming the couplings (31) to the Einstein frame, and $-1/4$ of them are new couplings. The nonlinear extension of these couplings appear in (7).

7. Discussion

In this Letter we have extended the sigma model Riemann curvature couplings (2) to include the B-field and the dilaton couplings. We have found these new couplings by imposing the consistency of the couplings (2) with the linear T-duality and by the S-matrix calculations. The T-duality in these couplings is satisfied on-shell. Even in the absence of the B-field, the couplings (2) satisfy the standard T-duality only on-shell. The reason is that the dimensional reduction of the couplings (2) contains the following term:

$$R_{kyny}R^{n}{}_{yqy}R_{rysy}R^{qrks}\eta^{yy}\eta^{yy}\eta^{yy} \tag{43}$$

which is not invariant under the T-duality (14). However, using the same calculation as we have done in (23) one finds the on-shell amplitude corresponding to this coupling is zero. This may be the reason that the $RRHH$ couplings (31) are also invariant under on-shell T-duality.

In general one expects the effective actions to be invariant under off-shell T-duality. So the effective action which includes the supergravity at order α'^0 and the Riemann curvature corrections (2) at order α'^3 should be invariant under an off-shell T-duality which receives quantum corrections. In fact there are different sets of Riemann curvature corrections which are related to each others via some couplings involving the Ricci and scalar curvatures [8]. These terms can be eliminated by field redefinitions involving higher derivative terms. The field redefinitions at the same time changes the standard form of the T-duality (7) to a non-standard form which receives the higher derivative corrections. So one expects one set of Riemann curvature corrections to be invariant under the standard T-duality transformations, and all other sets to be invariant under the non-standard T-duality transformations.

We have found four NS–NS couplings which are related to the four-graviton couplings (2) by on-shell linear T-duality transformations. However, there are ambiguities in the couplings (2) which can be fixed by studying the five-graviton amplitudes. For example, the four Riemann curvature couplings $\epsilon_{10}\cdot\epsilon_{10}RRRR$ can be added to (2) because this term has its first non-zero contribution at five gravitons [29]. The sigma-model approach implies that this term appears in the effective action [30]. It would be interesting to find the couplings which are related to the couplings $\epsilon_{10}\cdot\epsilon_{10}RRRR$ under T-dual Ward identity.

We have found the couplings (3) and (4) by using the fact that the S-matrix elements should satisfy the T-dual Ward identity [27,24]. On the other hand the S-matrix elements should satisfy the S-dual Ward identity [27,31–33]. Using this identity, one may extend the couplings we have found in this Letter to include the R–R couplings as well. The couplings involving the R–R two-form can easily be included in (3) and (4) by replacing $e^{-\phi_0}H_{abc}H_{def}$ with the following S-duality invariant expression:

$$e^{-\phi_0}H_{abc,d}H_{efg,h} \to e^{-\phi_0}H_{abc,d}H_{efg,h} + e^{\phi_0}F_{abc,d}F_{efg,h}$$

where F is the field strength of the R–R two-form. Similar extension for the D-brane couplings at order α'^2 has been verified by explicit calculations in [34]. A representation for the $R^2(\partial F)^2$ couplings have been found in [35]. It has been shown in [35] that this representation is the same as the $R^2(\partial F)^2$ couplings that one finds by using the above extension in the $R^2(\partial H)^2$ couplings in [9]. The $R^2(\partial F)^2$ terms that we have found are then the same as the couplings found in [35] up to some identities. One may use the consistency of the above R–R two-form couplings with the linear T-duality to find all other R–R couplings at order α'^3. One may also extend the four-point couplings at order α'^3 to arbitrary order of α' using the prescription given in [36].

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