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Stability and bifurcation analysis in the delay-coupled nonlinear oscillators

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Abstract This paper investigates the dynamical behavior of two oscillators with nonlinearity terms, which are coupled with finite delay parameters. Each oscillator is a general class of second-order nonlinear delay-differential equations. The system of delay differential equations is analyzed by reducing the delay equations to a system of ordinary differential equations on a finite-dimensional center manifold, the corresponding to an infinite-dimensional phase space. In addition, the characteristic equation for the linear stability of the trivial equilibrium is completely analyzed and the stability region is illustrated in the parameters space. Our analysis reveals necessary coefficients of the reduced vector field on the center manifold for studying the bifurcations of the trivial equilibrium such as transcritical, pitchfork, and Hopf bifurcation. Finally, we consider the delay-coupled van der Pol equations.

Keywords Delay differential equations · Stability · Center manifold · Hopf bifurcation · Transcritical bifurcation · Pitchfork bifurcation

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1 Introduction

Nonlinear time-delay differential equations (DDEs) have been used as mathematical models for phenomena in physiology [3, 9], population dynamics [16], physics [17], climate modeling [24], and engineering [23], among others. Indeed, time delays have been introduced in order to have the output of the models more closely reflect the measured performance [7, 13, 16, 19, 21].

DDEs behave like ordinary differential equations (ODEs) on an infinite-dimensional (Banach) phase space and many results which are known for ODEs on finite-dimensional spaces have analogues in the context of DDEs. The bifurcation analysis of DDEs and ODEs are the same, although the technical details differ. Consider the neighborhood of the equilibrium solution of the nonlinear DDE; then the analysis of the linearization at the equilibrium point leads to stable, unstable and center invariant subspaces where only the stable subspace is infinite-dimensional. They are tangent to local invariant manifolds (stable, unstable, and center manifolds) and the flow near the equilibrium is exponentially attracting (stable manifold), exponentially repelling (unstable manifold), or nonhyperbolic (center manifold). Now bifurcations near equilibria are determined by the flow on the center manifold and the dimension of this manifold is determined by the number of eigenvalues of the linearization on the imaginary axis [2, 11, 12]. We focus on coupled nonlinear oscillators because they have been an interesting subject in different research fields such as physics, engineering, biology and so on [4–6, 8, 13, 15, 16, 22, 25, 28, 30, 31]. Furthermore, there have been great interests in dynamical characteristic especially the bifurcations of these oscillators with delay. For example, authors in [2, 10, 14, 18, 22, 26–29, 31] studied effect of time-delay in some coupled oscillators from the viewpoint of bifurcation.

One of the most important oscillators is van der Pol. The van der Pol equation was introduced in the 1920s as a model to describe the oscillations in the vacuum tube triode circuit, which is governed by the following second-order nonlinear oscillatory system:

$$\ddot{x}+\varepsilon\big(x^2-1\big)\dot{x}+x=f(x).$$

The dynamics of coupled van der Pol oscillators has been of interest, for example, the author in [25] used three van der Pol oscillators which are coupled to each other to model the beating of the heart, Moore-Ede [20] studied sleep dynamics by a directly coupled van der Pol oscillators and authors in [1, 14, 18, 22, 26–28] focused on the effect of time delay on the nonlinear dynamics of the system by the method of averaging together with truncation of Taylor expansions.

Wirkus [27] concerned on the following equation:

$$\begin{cases}\ddot{x}+\varepsilon(x^2-1)\dot{x}+x=\varepsilon\alpha\dot{y}(t-\tau),\\ \ddot{y}+\varepsilon(y^2-1)\dot{y}+y=\varepsilon\alpha\dot{x}(t-\tau)\end{cases} \tag{1.1}$$

and he in his thesis [28] suggested the following system for investigation:

$$\begin{cases}\ddot{x}+\varepsilon(x^2-1)\dot{x}+x=\varepsilon\alpha\dot{y}(t-\tau)+\varepsilon\beta y(t-\tau),\\ \ddot{y}+\varepsilon(y^2-1)\dot{y}+y=\varepsilon\alpha\dot{x}(t-\tau)+\varepsilon\beta x(t-\tau)\end{cases} \tag{1.2}$$

which has application to *laser dynamics*, and more generally, to the coupling of *microwave oscillators*.

Gholizade-Narm [8] studied the coupled van der Pol oscillators with different parameters—as a model of *SA-AV nodes in heart*

$$\begin{cases}\ddot{x}+\varepsilon_1(x^2-1)\dot{x}+\omega_1^2x=\alpha_1(y(t)-x(t)),\\ \ddot{y}+\varepsilon_2(y^2-1)\dot{y}+\omega_2^2y=\alpha_2(x(t)-y(t))\end{cases} \tag{1.3}$$

and

$$\begin{cases}\ddot{x}+\varepsilon_1(x^2-1)\dot{x}+\omega_1^2x=\alpha_1(y(t-\tau_1)-x(t)),\\ \ddot{y}+\varepsilon_2(y^2-1)\dot{y}+\omega_2^2y=\alpha_2(x(t-\tau_2)-y(t)).\end{cases} \tag{1.4}$$

In [8], synchronization regions for the system (1.3) and the system (1.4), where $\tau_1=\tau_2$, are obtained by using the describing function method. Furthermore, stability regions and bifurcation curves are only studied for the system (1.3) by using the perturbation method. But he did not investigate the bifurcation(s) for the system (1.4) because of complicated computations. Moreover, Zhang and Gu [29] investigated dynamics of the system (1.4) with the following conditions:

$$\omega_1=\omega_2=1,\qquad \alpha_1=\alpha_2=\alpha,\qquad \varepsilon_1=\varepsilon_2=\varepsilon.$$

They only considered Hopf bifurcation by using of the center manifold theory.

In this paper, we extend the works of Zhang [29], Wirkus ([27, 28], the case $\alpha=0$), and Gholizade-Narm [8] to a general form, i.e.,

$$\begin{cases}\ddot{x}+x=f(x,\dot{x})+\alpha_1y(t-\tau),\\ \ddot{y}+y=f_1(y,\dot{y})+\alpha_2x(t-\delta)\end{cases} \tag{1.5}$$

where the functions $f(x,\dot{x})$ and $f_1(y,\dot{y})$ are smooth. In fact, we study the effect of time-delay parameters on the nonlinear system (1.5) by the center manifold theory which reveals necessary nonlinearity terms for determining its stability regions and bifurcations (transcritical, pitchfork, Hopf).

This paper is organized as follows. Section 2 is devoted to explain center manifold theory and some information about the stability of the characteristic equation in DDEs. Section 3 concentrates on the Taylor expansion of f and f_1 in the system (1.5) and the linear stability of the trivial solution by the use of Sect. 2.2. Section 4 discusses on the bifurcations for the system (1.5) under some generic conditions on the Taylor coefficients of the functions f and f_1. We apply our results on the delay-coupled van der Pol equations.

2 Preliminary

2.1 Center manifold for DDES

In this section, we briefly state the center manifold theory for DDEs with parameters. For more details, one can refer to [2, 11, 12]. Consider the general delay-differential equation

$$\dot{Y}(t) = g\big(X(t), X(t-\tau), X(t-\delta), \mu\big) \tag{2.1}$$

where $Y = (X, \mu)$, $\mu \in R$, $X \in R^4$, and $\tau, \delta > 0$. We shall assume that g is C^r, for r large enough, and the equation admits zero as the equilibrium. Note that Eq. (2.1) should be viewed as the suspended system where the parameter μ is included as trivial dynamic ($\dot{\mu} = 0$). We separate the system (2.1) to the linear and nonlinear terms

$$\begin{cases} \dot{X}(t) = A_0(\mu)X(t) + A_1(\mu)X(t-\tau) \\ \qquad\quad + A_2(\mu)X(t-\delta) \\ \qquad\quad + G(X(t), X(t-\tau), X(t-\delta), \mu), \\ \dot{\mu} = 0 \end{cases} \tag{2.2}$$

where

$$D_{j+1}g(0,0,0,\mu_0) = \big[(D_{j+1}g)_{ik}\big]_{5\times 5}$$

and

$$A_j(\mu_0) = \big[(D_{j+1}g)_{ik}\big]_{4\times 4}$$

where $j = 0, 1, 2$ and $i, k = 1, \dots, 5$. Here, $D_j g$ means the Jacobian of g with respect to its jth component and $A_j(\mu_0)$'s are the submatrix of the matrix $D_{j+1}g(0,0,0,\mu_0)$.

Let $\mathcal{C} = C([-T_m, 0], R^{4+1})$ be the Banach space of all continuous mappings from $[-T_m, 0]$ into R^{4+1} which is equipped with the supremum norm $\|\phi\|_{T_m} = \sup_{\theta\in[-T_m,0]} |\phi(\theta)|$ for $\phi \in \mathcal{C}$ where $T_m = max\{\tau, \delta\}$.

We write the system (2.2) in the following DDE form

$$\frac{d}{dt}U(t) = \mathcal{L}_\mu U_t + F(U_t) \tag{2.3}$$

where $U_t(\theta) = [u(t+\theta), \mu(t+\theta)]^T \in \mathcal{C}$ for $\theta \in [-T_m, 0]$. $\mathcal{L} : \mathcal{C} \to R^{4+1}$ is the linear mapping and $F \in C^r(\mathcal{C}, R^{4+1})$, $r \geq 1$ is the nonlinear mapping. Let $u(t) = X(t)$ and $u_t(\theta) = u(t+\theta)$, then the system (2.2) is

$$\begin{cases} \frac{d}{dt}u(t) = L_\mu u_t + G(u_t, \mu), \\ \frac{d}{dt}\mu = 0. \end{cases} \tag{2.4}$$

Therefore, for every $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ and $\phi = (\varphi, \varphi_5)^T \in \mathcal{C}$, we have

$$A_0(\mu)\varphi(0) + A_1(\mu)\varphi(-\tau) + A_2(\mu)\varphi(-\delta) = L_\mu\varphi,$$
$$(L_\mu\varphi, 0)^T = \mathcal{L}_\mu\phi$$

and

$$F(\phi, \mu) = \big(G(\varphi, \mu), 0\big)^T.$$

The stability of the trivial solution of the Eq. (2.1) can be studied by the DDE of the following form:

$$\begin{cases} \dot{X}(t) = A_0(\mu)X(t) + A_1(\mu)X(t-\tau) \\ \qquad\quad + A_2(\mu)X(t-\delta), \\ \dot{\mu} = 0. \end{cases} \tag{2.5}$$

Substituting $Y(t) = Ce^{\lambda t}$ in the system (2.5), gives the following characteristic equation:

$$\lambda \cdot \det\big(\lambda I_4 - A_0(\mu) - e^{-\lambda\tau}A_1(\mu) - e^{-\lambda\delta}A_2(\mu)\big) = 0. \tag{2.6}$$

Obviously, Eq. (2.6) always has one eigenvalue on the imaginary axis. We assume that this characteristic equation has $m+1$ eigenvalues (counting multiplicity) on the imaginary axis and all other eigenvalues have negative real parts. Therefore, the space $\mathcal{C}$ can be split as $\mathcal{C} = P \oplus Q$ where $Q \subset \mathcal{C}$ is infinite-dimensional stable subspace and $P \subset \mathcal{C}$ is an $(m+1)$-dimensional center subspace tangent to the center manifold. We will denote a basis for P by the $5 \times (m+1)$ matrix Φ; the columns of Φ are the basis vectors. Also, we will consider the transpose of Eq. (2.5) with $(m+1)$-dimensional center subspace P'. We will denote a basis for P' by the $(m+1) \times 5$ matrix Ψ'. Also, we define a new basis Ψ by $\Psi = \langle\Psi', \Phi\rangle^{-1}\Psi'$ which implies $\langle\Psi, \Phi\rangle = I$. This bilinear form is defined

$$\langle\psi_i, \phi_j\rangle = \overline{\psi_i(0)}\phi_j(0) + \int_{-T_m}^{0} \overline{\psi_i(\xi+\tau)}A_1\phi_j(\xi)\,d\xi + \int_{-T_m}^{0} \overline{\psi_i(\xi+\delta)}A_2\phi_j(\xi)\,d\xi$$

where

$$\Phi = (\phi_1, \phi_2, \dots, \phi_{m+1}),$$
$$\Psi = (\psi_1, \psi_2, \dots, \psi_{m+1})^T,$$
$$\langle \Psi, \Phi \rangle = \big[\langle \psi_i, \phi_j \rangle\big]_{(m+1)\times(m+1)}$$

and $T_m = \max\{\tau, \delta\}$. This kind of basis Ψ can help us to decompose the space $\mathcal{C}$ and also reduce Eq. (2.3) on the local center manifold W^c_{loc} which is defined by

$$W^c_{\text{loc}} = \big\{\phi \in \mathcal{C} : \phi = \Phi z + h(z, F),\ z \text{ is in a neighborhood of zero in } R^{m+1}\big\}$$

where $h(z, F) \in Q$ for each z and is a C^{r-1} function with respect to z. Moreover, z satisfies the following ordinary differential equation:

$$\frac{d}{dt} z = Bz + \Psi(0) F\big(\Phi z + h(z, F)\big) \tag{2.7}$$

where the $(m+1) \times (m+1)$ matrix B satisfies the relation $\frac{d}{d\theta}\Phi = \Phi B$, [2, 11, 12]. This framework is useful to consider the transcritical and pitchfork bifurcations of the trivial solution of Eq. (1.5).

Furthermore, for analyzing the Hopf bifurcation for Eq. (2.1), we only consider the first equation of the system (2.4)

$$\frac{d}{dt} u(t) = L_\mu u_t + G(u_t, \mu). \tag{2.8}$$

Next, we define

$$\big(A(\mu)\varphi\big)(\theta) = \begin{cases} \frac{d\varphi}{d\theta}, & \theta \in [-T_m, 0), \\ L_\mu \varphi, & \theta = 0 \end{cases}$$

and

$$(R_\mu \varphi)(\theta) = \begin{cases} 0, & \theta \in [-T_m, 0), \\ G(\varphi, \mu), & \theta = 0 \end{cases}$$

where $\varphi \in C^1([-T_m, 0], R^4)$. Since $\frac{du_t}{d\theta} = \frac{du_t}{dt}$, we have

$$\dot{u}_t = A(\mu) u_t + R_\mu u_t. \tag{2.9}$$

It is necessary to compute eigenvectors q, q^* associated with simple purely imaginary eigenvalues, $\pm i\beta$, of the first equation of the system (2.5) and its transpose. We then normalize q and q^* so that $\langle q^*, q\rangle = 1$.

Let u_t be the solution of Eq. (2.9) at the critical value $\mu = \mu_0$. Define

$$\begin{cases} z(t) = \langle q^*, u_t \rangle, \\ W(t, \theta) = u_t(\theta) - 2\,\text{Re}\{z(t) q(\theta)\}. \end{cases}$$

This definition can help us to reduced Eq. (2.8) on the center manifold which is

$$\dot{z}(t) = i\beta z(t) + \bar{q}^*(0) G(u_t, \mu_0) \tag{2.10}$$

and determines properties of the bifurcating periodic orbits in the center manifold at the critical value $\mu = \mu_0$; see [2, 11].

2.2 Stability of characteristic equation

Studying the system (1.5) at the trivial solution needs to investigate the following general transcendental polynomial equation:

$$\lambda^2 + p\lambda + r + q\, e^{-\lambda T} = 0, \quad q \neq 0. \tag{2.11}$$

Therefore, in this subsection we concentrate on this equation. We substitute $\lambda = iw$ in Eq. (2.11) in order to find the roots of Eq. (2.11) with zero real parts. Notice that

- if $w = 0$, then $r + q = 0$,
- if $w \neq 0$, then

  $$\begin{cases} \cos(wT) = \frac{w^2 - r}{q}, \\ \sin(wT) = \frac{pw}{q}. \end{cases}$$

  Thus, $wT = \arcsin\{\frac{pw}{q}\}$ and

  (i) if $\frac{w^2-r}{q} \geq 0$ and $wT_j := wT + 2j\pi$, then $\sin(wT_j) = \sin(wT) \Rightarrow wT_j = \arcsin\{\frac{pw}{q}\} + 2j\pi$,
  (ii) if $\frac{w^2-r}{q} < 0$ and $wT_j := -wT + (2j+1)\pi$, then $\sin(wT_j) = \sin(wT) \Rightarrow wT_j = -\arcsin\{\frac{pw}{q}\} + (2j+1)\pi$.

Therefore, one can write

$$w_\pm = \frac{\sqrt{2}}{2}\Big\{\big(2r - p^2\big) \pm \big[\big(2r - p^2\big)^2 - 4\big(r^2 - q^2\big)\big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}},$$

$$T_j^{\pm} = \begin{cases} \frac{1}{w_\pm}\{\arcsin\{\frac{p}{q}w_\pm\} + 2j\pi\}; & \frac{w_\pm^2 - r}{q} \geq 0, \\ \frac{1}{w_\pm}\{-\arcsin\{\frac{p}{q}w_\pm\} + (2j+1)\pi\}; & \\ \frac{w_\pm^2 - r}{q} < 0 & \end{cases}$$

with $j \in \{0, 1, 2, \ldots\}$ such that $T_j^{\pm} > 0$.

Notice that $w_\pm$ can be finite for each given r, p, q. Therefore, they are fixed. But $T_j^{\pm}$'s depend on the sign of $\frac{w_\pm^2 - r}{q}$ and the values of r, p, q, and j.

If $T = 0$, then Eq. (2.11) becomes the second-order polynomial equation

$$\lambda^2 + p\lambda + r + q = 0 \tag{2.12}$$

that it is easy to find its roots. From the relation between roots of Eqs. (2.11) and (2.12), we have the following lemma.

Lemma 1 (i) *If either* $(r^2 - q^2) > 0$ *and* $(2r - p^2) < 0$, *or* $(2r - p^2)^2 - 4(r^2 - q^2) < 0$ *holds, then the number of roots of Eq.* (2.11) *with positive real parts is the same as that of Eq.* (2.12) *for all* $T \geq 0$.

(ii) *If either* $r^2 - q^2 < 0$, *or* $2r - p^2 > 0$ *and* $(2r - p^2)^2 - 4(r^2 - q^2) = 0$ *holds, then the number of roots of Eq.* (2.11) *with positive real parts is the same as that of Eq.* (2.12) *for* $T \in [0, \widetilde{T}_0^+)$. *Also, t Eq.* (2.11) *has a pair of simple purely imaginary roots* $\pm i w_+$ *at* $T = \widetilde{T}_j^+$.

(iii) *If* $r^2 - q^2 > 0$, $(2r - p^2) > 0$ *and* $(2r - p^2)^2 - 4(r^2 - q^2) > 0$ *hold, then the number of roots of t Eq.* (2.11) *with positive real parts is the same as that of Eq.* (2.12) *for* $T \in [0, \widetilde{T}_0)$ *and Eq.* (2.11) *has a pair of simple purely imaginary roots* $\pm i w_+$ $(\pm i w_-)$ *at* $T = \widetilde{T}_0^+$ $(T = \widetilde{T}_0^-)$, *where* $T_0 = \min\{\widetilde{T}_0^+, \widetilde{T}_0^-\}$.

Proof See [26]. □

Moreover, Baptisini and T'aboas stated a necessary and sufficient stability condition for the equation

$$(\lambda^2 + p\lambda + r)e^{T\lambda} + q = 0, \quad T > 0 \tag{2.13}$$

such that all the roots of the equation have negative real parts.

Theorem 2 *Consider the vector* $v(b) = (pb, r - b^2)$, $b \geq 0$. *A necessary and sufficient condition for Eq.* (2.13) *to be stable for any* $T > 0$ *is that* $|v(b)| > q$, *for any* $b > 0$, *if* $q > 0$ *or* $|v(b)| > -q$, *for any* $b \geq 0$, *if* $q < 0$.

Proof See [3]. □

3 Linear stability of system (1.5)

Let us consider the following DDE:

$$\begin{cases} \ddot{x} + x = f(x, \dot{x}) + \alpha_1 y(t - \tau), \\ \ddot{y} + y = f_1(y, \dot{y}) + \alpha_2 x(t - \delta) \end{cases} \tag{3.1}$$

where $\tau, \delta > 0$ are the delay parameters and f, f_1 are arbitrary smooth functions have origin as the equilibrium solution. It is convenient to write (3.1) as the four-dimensional first-order system

$$\begin{cases} \dot{x}_1 = x_2(t), \\ \dot{x}_2 = -x_1(t) + \alpha_1 y_1(t - \tau) + f(x_1, x_2), \\ \dot{y}_1 = y_2(t), \\ \dot{y}_2 = -y_1(t) + \alpha_2 x_1(t - \delta) + f_1(y_1, y_2). \end{cases} \tag{3.2}$$

Let

$$\begin{aligned} f(x_1, x_2) &= \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1^2 + \gamma_4 x_1 x_2 + \gamma_5 x_2^2 \\ &\quad + \gamma_6 x_1^3 + \gamma_7 x_1^2 x_2 + \gamma_8 x_1 x_2^2 + \gamma_9 x_2^3 \\ &\quad + O(4), \\ f_1(y_1, y_2) &= \eta_1 y_1 + \eta_2 y_2 + \eta_3 y_1^2 + \eta_4 y_1 y_2 + \eta_5 y_2^2 \\ &\quad + \eta_6 y_1^3 + \eta_7 y_1^2 y_2 + \eta_8 y_1 y_2^2 + \eta_9 y_2^3 \\ &\quad + O(4) \end{aligned}$$

and γ_i, η_i's are the Taylor coefficients of functions f, f_1, respectively. For simplifying the computations, we assume that the Taylor coefficients $D_i f(0, 0)$ and $D_i f_1(0, 0)$, $i = 1, 2$ are equal. Then the characteristic equation of the linearization of system (3.2) at the trivial solution is

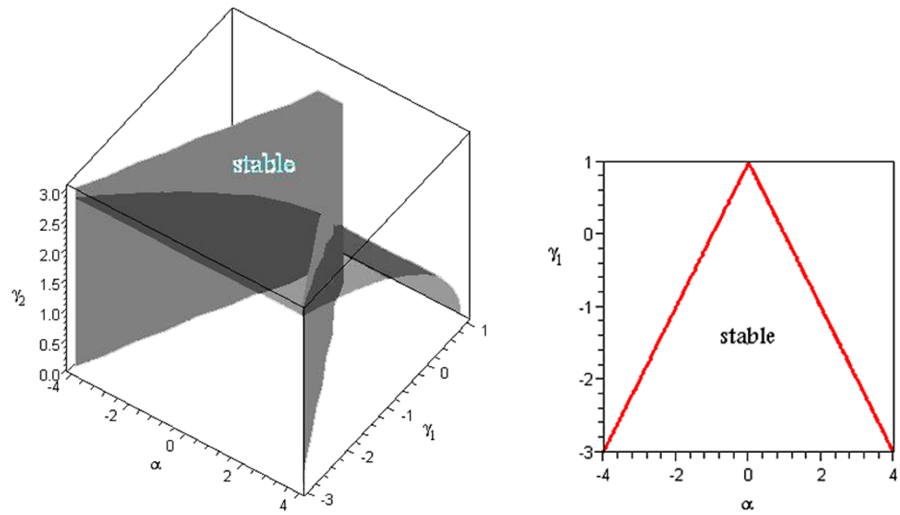
$$\det\bigl(\Delta(\lambda, \tau, \delta)\bigr) = \bigl(\lambda^2 - \lambda\gamma_2 + (1 - \gamma_1)\bigr)^2 - \alpha_1\alpha_2 e^{-\lambda(\tau+\delta)}.$$

Let α_1 and α_2 have the same sign. If we denote $\tau + \delta = 2T$ and $\alpha_1\alpha_2 = \alpha^2$, then the characteristic equation is

$$\det\bigl(\Delta(\lambda, \tau, \delta)\bigr) = \Delta_1.\Delta_2 \tag{3.3}$$

where

$$\Delta_1 = \bigl[\lambda^2 - \lambda\gamma_2 + (1 - \gamma_1) - \alpha e^{-\lambda T}\bigr],$$
$$\Delta_2 = \bigl[\lambda^2 - \lambda\gamma_2 + (1 - \gamma_1) + \alpha e^{-\lambda T}\bigr].$$



(a) (b)

Fig. 1 Maple simulation of stable region in $(\gamma_1, \alpha, \gamma_2)$-space and (γ_1, α)-space. (**a**) The trivial solution of system (3.1) is stable in the region $\{(\gamma_1, \alpha, \gamma_2) | 2(1-\gamma_1) - \gamma_2^2 < 0, (1-\gamma_1)^2 - \alpha^2 > 0, -3 \le \gamma_1 \le 1, -4 \le \gamma_2 \le 3, -4 \le \alpha \le 4\}$. (**b**) In the region $\{(\gamma_1, \alpha) | (1-\gamma_1)^2 - \alpha^2 > 0, -3 \le \gamma_1 \le 1, 4 \le \alpha \le 4\}$ the trivial solution of system (3.1) is stable

These determine the local stability of the trivial solution of the system (3.2). We see that Eq. (3.3) is stable if and only if equations Δ_1 and Δ_2 satisfy Theorem 2, where

$$p := -\gamma_2, \qquad q := \pm\alpha, \qquad r := 1 - \gamma_1 \tag{3.4}$$

and

$$\begin{aligned} r^2 - q^2 &= (1-\gamma_1)^2 - \alpha^2, \\ 2r - p^2 &= 2(1-\gamma_1) - \gamma_2^2, \\ \big(2r - p^2\big)^2 - 4\big(r^2 - q^2\big) &= \gamma_2^4 + 4\alpha^2 - 4(1-\gamma_1)\gamma_2^2. \end{aligned} \tag{3.5}$$

Now we state the following lemma.

Lemma 3 *All the roots of Eq.* (3.3) *have negative real parts, if and only if*

$$|\nu(b)| > \alpha \quad \textit{for any } b > 0$$

where $\nu(b) = (-\gamma_2 b, (1-\gamma_1) - b^2)$. *Moreover,*

(i) *if* $2(1-\gamma_1) - \gamma_2^2 < 0$ *and* $(1-\gamma_1)^2 - \alpha^2 > 0$, *then this inequality holds for any* $b > 0$, [*see Fig.* 1(a)],
(ii) *if this inequality holds for any* $b > 0$, *then* $(1-\gamma_1)^2 - \alpha^2 > 0$, [*see Fig.* 1(b)].

Proof Applying Theorem (2) on Δ_1 and Δ_2 imply that they have roots with negative real parts iff

$$|\nu(b)| > |q| \quad \text{for any } b > 0$$

where $|\nu(b)| = (-\gamma_2 b, (1-\gamma_1) - b^2)$ and $q = \pm\alpha$. This inequality yields to

$$b^4 + \gamma_2^2 b^2 - 2(1-\gamma_1)b^2 + (1-\gamma_1)^2 > \alpha^2.$$

It is obvious that if $2(1-\gamma_1) - \gamma_2^2 < 0$ and $(1-\gamma_1)^2 - \alpha^2 > 0$, then $|\nu(b)| > |q|$ for any $b > 0$. On the other hand, if the inequality $|\nu(b)| > |q|$ holds for any $b > 0$, then we have $(1-\gamma_1)^2 - \alpha^2 > 0$, $(b \to 0)$. □

4 Bifurcations for system (1.5)

Investigating the local bifurcations of the trivial solution of the system (1.5) needs to obtain the eigenvalues of Eq. (3.3), that is to say $\Delta_1.\Delta_2$. By using Sect. 2.2, Eq. (3.3) has a pair of simple purely imaginary roots when

$$w_\pm = \frac{\sqrt{2}}{2}\Big\{\big(2(1-\gamma_1) - \gamma_2^2\big) \pm \big[\gamma_2^4 + 4\alpha^2 - 4(1-\gamma_1)\gamma_2^2\big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}} \tag{4.1}$$

at $T = T_j^\pm$, such that

$$T_j^+ = \begin{cases} \frac{1}{w_+}\big\{2\pi - \arcsin\big\{\frac{\gamma_2}{|\alpha|} w_+\big\} + j\pi\big\}; \\ \quad w_+^2 - r \ge 0, \\ \frac{1}{w_+}\big\{\arcsin\big\{\frac{\gamma_2}{|\alpha|} w_+\big\} + j\pi\big\}; \quad w_+^2 - r < 0 \end{cases} \tag{4.2}$$

and

$$T_j^- = \frac{1}{w_-}\left\{\arcsin\left\{\frac{\gamma_2}{|\alpha|}w_-\right\} + j\pi\right\} \tag{4.3}$$

where $j \in \{0, 1, 2, \dots\}$ and $T_j^\pm > 0$. Lemma 1 implies the following lemma for the characteristic equation (3.3).

Lemma 4 *Suppose that $w_\pm$ and $T_j^\pm$ are defined by* (4.1), (4.2), *and* (4.3).

(i) *If $(1-\gamma_1)^2 - \alpha^2 = 0$, then $\lambda = 0$ is a root of Eq.* (3.3) *which implies the occurrence of the local bifurcation, such as transcritical, ...*
(ii) *If either $(1-\gamma_1)^2 - \alpha^2 < 0$, or $2(1-\gamma_1) - \gamma_2^2 > 0$ and $\gamma_2^4 + 4\alpha^2 - 4(1-\gamma_1)\gamma_2^2 = 0$, then the system* (3.2) *undergoes a Hopf bifurcation at the origin when $T = T_j^+$, $j = 0, 1, \dots$.*
(iii) *If $(1-\gamma_1)^2 - \alpha^2 > 0$, $2(1-\gamma_1) - \gamma_2^2 > 0$ and $\gamma_2^4 + 4\alpha^2 - 4(1-\gamma_1)\gamma_2^2 > 0$, then the system* (3.2) *undergoes a Hopf bifurcation at the origin when $T = T_j^\pm$, $j = 0, 1, \dots$.*

Proof (i) Obviously, $\lambda = 0$ is a root of Eq. (3.3) if

$$(1-\gamma_1) = \alpha \quad \text{or} \quad (1-\gamma_1) = -\alpha.$$

For (ii) and (iii), it is enough to use Lemma 1. □

Note that if we take T as bifurcation parameter, then differentiation $\lambda(T)$ with respect to T leads to

$$\frac{d\lambda}{dT} = \frac{-\lambda(\lambda^2 - r - p\lambda)}{(p + T\lambda^2 + Tr) + \lambda(2 + Tp)}. \tag{4.4}$$

$\mathrm{Re}(\frac{d\lambda}{dT})$ and $\mathrm{Im}(\frac{d\lambda}{dT})$ at critical value of the time delay are

$$\mathrm{Re}\left(\frac{d\lambda}{dT}\right)\Big|_{T=T_j^\pm} = \frac{w_\pm^2(\gamma_2^4 + 4\alpha^2 - 4(1-\gamma_1)\gamma_2^2)^{1/2}}{[-\gamma_2 + T_j^\pm(1-\gamma_1 - w_\pm^2)]^2 + [2 - \gamma_2 T_j^\pm]^2 w_\pm^2}, \tag{4.5}$$

$$\begin{aligned}\mathrm{Im}\left(\frac{d\lambda}{dT}\right)\Big|_{T=T_j^\pm} &= \big\{w_\pm\big(w_\pm^2 - 1 + \gamma_1\big)\big(-\gamma_2 - T_j^\pm\big(w_\pm^2 - 1 + \gamma_1\big)\big) \\ &\quad - \gamma_2 w_\pm^3\big(2 - T_j^\pm\gamma_2\big)\big\} \\ &\quad \times \big\{\big[-\gamma_2 + T_j^\pm\big(1 - \gamma_1 - w_\pm^2\big)\big]^2 \\ &\quad + \big[2 - \gamma_2 T_j^\pm\big]^2 w_\pm^2\big\}^{-1}.\end{aligned} \tag{4.6}$$

These relations use in Sect. 4.2 in order to study the Hopf bifurcation for system (3.2).

4.1 Transcritical and Pitchfork bifurcations for system (1.5)

By Lemma 4, zero solution for system (3.2) can undergo a local bifurcation when $\alpha = 1-\gamma_1$ or $-\alpha = 1-\gamma_1$. Let $\mu = \alpha - (1-\gamma_1)$ be the bifurcation parameter. We rewrite system (3.2) as

$$\begin{cases}\dot{x}_1 = x_2(t), \\ \dot{x}_2 = (\mu - \alpha)x_1(t) + \alpha_1 y_1(t-\tau) + \gamma_2 x_2 \\ \qquad + \gamma_3 x_1^2 + \gamma_4 x_1 x_2 + \gamma_5 x_2^2 + \gamma_6 x_1^3 + \gamma_7 x_1^2 x_2 \\ \qquad + \gamma_8 x_1 x_2^2 + \gamma_9 x_2^3 + O(4), \\ \dot{y}_1 = y_2(t), \\ \dot{y}_2 = (\mu - \alpha)y_1(t) + \alpha_2 x_1(t-\delta) + \gamma_2 y_2 + \eta_3 y_1^2 \\ \qquad + \eta_4 y_1 y_2 + \eta_5 y_2^2 + \eta_6 y_1^3 + \eta_7 y_1^2 y_2 + \eta_8 y_1 y_2^2 \\ \qquad + \eta_9 y_2^3 + O(4), \\ \dot{\mu} = 0.\end{cases} \tag{4.7}$$

The linearization of system (4.7) at origin is

$$\begin{cases}\dot{X}(t) = A_0 X(t) + A_1 X(t-\tau) + A_2 X(t-\delta), \\ \dot{\mu} = 0\end{cases} \tag{4.8}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\alpha & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\alpha & \gamma_2 \end{pmatrix}$$

and

$$A_1 = [a_{ij}] = \begin{cases}\alpha_1, & i = 2,\ j = 3, \\ 0, & \text{otherwise},\end{cases}$$

$$A_2 = [a_{ij}] = \begin{cases}\alpha_2, & i = 4,\ j = 1, \\ 0, & \text{otherwise}.\end{cases}$$

Theorem 5 *System* (4.7) *has the transcritical bifurcation if*

$$-2\gamma_2 + 2\alpha T_m \neq 0, \qquad \gamma_3 + \eta_3 \frac{\alpha}{\alpha_1} \neq 0,$$

$$\gamma_6 + \eta_6 \left(\frac{\alpha}{\alpha_1}\right)^2 = 0$$

and the pitchfork bifurcation if

$$-2\gamma_2 + 2\alpha T_m \neq 0, \qquad \gamma_3 + \eta_3 \frac{\alpha}{\alpha_1} = 0,$$

$$\gamma_6 + \eta_6 \left(\frac{\alpha}{\alpha_1}\right)^2 \neq 0.$$

Proof Obviously, the linear system (4.8) has eigenvalues $\lambda_1 = \lambda_2 = 0$, which one of them is corresponding to the equation $\dot{\mu} = 0$ in this system. We compute eigenvectors associated with these eigenvalues for the linear system (4.8). Therefore, bases for the center subspace of system (4.8) and its transpose are

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \frac{\alpha}{\alpha_1} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Psi' = \begin{pmatrix} -\gamma_2 & 1 & \gamma_2(\frac{\alpha}{\alpha_2}) & \frac{\alpha}{\alpha_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By using the following bilinear form,

$$\langle \psi, \phi \rangle = \overline{\psi(0)}\phi(0) + \int_{-T_m}^{0} \overline{\psi(\xi + \tau)} A_1 \phi(\xi)\, d\xi + \int_{-T_m}^{0} \overline{\psi(\xi + \delta)} A_2 \phi(\xi)\, d\xi$$

we have

$$\Psi = \begin{pmatrix} -\gamma_2 l & l & \gamma_2 l(\frac{\alpha_1}{\alpha}) & l\frac{\alpha_1}{\alpha} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$l = (-2\gamma_2 + 2\alpha T_m)^{-1} \tag{4.9}$$

and $T_m = \max\{\tau, \delta\}$. Now, we consider the local coordinates $z = (u, \mu)^T$ on the center manifold. By using Sect. 2.1, the nonlinear terms in system (4.7) and the matrix B can be written as follows:

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$F\left(\left(u, 0, \frac{\alpha}{\alpha_1}u, 0, \mu\right)^T\right) = \left(0, \mu u + \gamma_3 u^2 + \gamma_6 u^3, 0, \frac{\alpha}{\alpha_1}\mu u + \left(\frac{\alpha}{\alpha_1}\right)^2 \eta_3 u^2 + \left(\frac{\alpha}{\alpha_1}\right)^3 \eta_6 u^3, 0\right)^T.$$

Thus, we get the following system on the center manifold:

$$\begin{cases} \dot{u} = l(2\mu u + (\gamma_3 + \eta_3 \frac{\alpha}{\alpha_1})u^2 + (\gamma_6 + \eta_6 (\frac{\alpha}{\alpha_1})^2)u^3), \\ \dot{\mu} = 0. \end{cases} \tag{4.10}$$

By definition (4.9), system (4.10) is

$$\dot{u} = \frac{1}{-2\gamma_2 + 2\alpha T_m}\left[2\mu u + \left(\gamma_3 + \eta_3 \frac{\alpha}{\alpha_1}\right)u^2 + \left(\gamma_6 + \eta_6\left(\frac{\alpha}{\alpha_1}\right)^2\right)u^3\right]$$

which implies the transcritical bifurcation if

$$-2\gamma_2 + 2\alpha T_m \neq 0, \qquad \gamma_3 + \eta_3 \frac{\alpha}{\alpha_1} \neq 0,$$

$$\gamma_6 + \eta_6 \left(\frac{\alpha}{\alpha_1}\right)^2 = 0$$

and the pitchfork bifurcation if

$$-2\gamma_2 + 2\alpha T_m \neq 0, \qquad \gamma_3 + \eta_3 \frac{\alpha}{\alpha_1} = 0,$$

$$\gamma_6 + \eta_6 \left(\frac{\alpha}{\alpha_1}\right)^2 \neq 0.$$

□

4.2 Hopf bifurcation for system (1.5)

We now suppose that conditions (ii) or (iii) in Lemma 4 hold, then system (3.2) has a pair of purely imaginary roots $\pm i w$ at the critical value of time delay $T = T_*$.

Introduce the new parameter $\mu = T - T_*$ and let $t \to \frac{t}{T}$. We rewrite system (3.2) as

$$\begin{cases} \dot{x}_1 = (\mu + T_*)x_2(t), \\ \dot{x}_2 = (\mu + T_*)(-x_1(t) + \alpha_1 y_1(t-\tau_1) + f(x_1, x_2)), \\ \dot{y}_1 = (\mu + T_*)y_2(t), \\ \dot{y}_2 = (\mu + T_*)(-y_1(t) + \alpha_2 x_1(t-\delta_1) + f_1(y_1, y_2)) \end{cases} \tag{4.11}$$

where $\tau_1 = \frac{\tau}{T}$ and $\delta_1 = \frac{\delta}{T}$. By using Sect. 2.1, we want to obtain the center manifold for system (4.11). Let $\mathcal{C} = C([-T_{11}, 0], R^4)$ be the phase space with the supremum norm $\|\phi\| = \sup_{\theta\in[-T_{11},0]} |\phi(\theta)|$ where $T_{11} = \max\{\tau_1, \delta_1\}$. Assume that $u(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$, then as usual, $u_t \in \mathcal{C}$ is defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-T_{11}, 0]$. By Sect. 2.1, this system can be written as

$$\dot{u}(t) = L_\mu u_t + G(u_t, \mu) \tag{4.12}$$

and for every $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathcal{C}$, we have

$$L_\mu \phi = (\mu + T_*)\big(A_0\phi(0) + A_1\phi(-\tau_1) + A_2\phi(-\delta_1)\big) \tag{4.13}$$

also

$$\begin{aligned} G(\phi, \mu) = (\mu + T_*)\big(&0, \gamma_3\phi_1^2 + \gamma_4\phi_1\phi_2 + \gamma_5\phi_2^2 \\ &+ \gamma_6\phi_1^3 + \gamma_7\phi_1^2\phi_2 + \gamma_8\phi_1\phi_2^2 \\ &+ \gamma_9\phi_2^3, 0, \eta_3\phi_3^2 + \eta_4\phi_3\phi_4 + \eta_5\phi_4^2 \\ &+ \eta_6\phi_3^3 + \eta_7\phi_3^2\phi_4 + \eta_8\phi_3\phi_4^2 + \eta_9\phi_4^3\big)^T. \end{aligned}$$

On the other hand, by using Sect. 2.1 for $\phi \in C^1([-T_{11}, 0], R^4)$, we have

$$(A(\mu)\phi)(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-T_{11}, 0), \\ L_\mu\phi, & \theta = 0 \end{cases} \tag{4.14}$$

and

$$(R_\mu\phi)(\theta) = \begin{cases} 0, & \theta \in [-T_{11}, 0), \\ G(\phi, \mu), & \theta = 0. \end{cases}$$

Then system (4.11) is equivalent to

$$\dot{u}_t = A(\mu)u_t + R_\mu u_t \tag{4.15}$$

Furthermore, define

$$(A^*(0)\psi)(\xi) = \begin{cases} -\frac{d\psi}{d\xi}, & \xi \in (0, T_{11}], \\ T_*(A_0^T\psi(0) + A_1^T\psi(\tau_1) \\ \quad + A_2^T\psi(\delta_1)), & \xi = 0 \end{cases}$$

for $\psi \in C^1([0, T_{11}], (R^4)^*)$ at $\mu = 0$.

The system (4.11) has a pair of purely imaginary eigenvalues $\mp iwT_*$ at $\mu = 0$ and its corresponding eigenvectors $A(0)$ and $A^*(0)$ are

$$q(\theta) = \big(q_1(0), q_2(0), q_3(0), q_4(0)\big)^T e^{iwT_*\theta},$$
$$q^*(\xi) = D\big(q_1^*(0), q_2^*(0), q_3^*(0), q_4^*(0)\big)e^{iwT_*\xi}$$

such that

$$\begin{aligned} q(0) &= \begin{pmatrix} \frac{i\alpha_1 e^{-iw\tau_1 T_*}}{w(K+iL)} \\ \frac{-\alpha_1 e^{-iw\tau_1 T_*}}{(K+iL)} \\ \frac{-i}{w} \\ 1 \end{pmatrix}, \\ q^*(0) &= D\begin{pmatrix} \frac{(iw+\gamma_2)(K-iL)e^{-iw\tau_1 T_*}}{\alpha_1} \\ \frac{-(K-iL)e^{-iw\tau_1 T_*}}{\alpha_1} \\ -(iw+\gamma_2) \\ 1 \end{pmatrix}^T \end{aligned} \tag{4.16}$$

where

$$K = w^2 - 1 + \gamma_1, \qquad L = w\gamma_2.$$

Moreover, D obtains from the condition $\langle q^*, q\rangle = 1$, where this bilinear form is [2, 12]

$$\begin{aligned} \langle q^*, q\rangle = \overline{q^*}(0)q(0) &+ \int_{-T_{11}}^{0} \overline{q^*}(s+\tau_1)A_1 q(s)\,ds \\ &+ \int_{-T_{11}}^{0} \overline{q^*}(s+\delta_1)A_2 q(s)\,ds \end{aligned}$$

thus

$$D = \frac{w(K-iL)}{(-4Kw) + i(2K\gamma_2 + K^2 + L^2 + \alpha^2 e^{2iwT_*})}. \tag{4.17}$$

Now we introduce the coordinates for the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq. (4.15).

By Sect. 2.1, one can define

$$\begin{cases} z(t) = \langle q^*, u_t \rangle, \\ W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \end{cases} \tag{4.18}$$

On the center manifold C_0, $W(t,\theta)$ can be written as follows:

$$\begin{aligned} W(t,\theta) &= W\big(z(t), \bar{z}(t), \theta\big) \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\overline{z^2}}{2} + \cdots \end{aligned} \tag{4.19}$$

where z and $\bar{z}$ are local coordinates for the center manifold C_0 in the direction of q^* and $\bar{q}^*$. For the solution $u_t \in C_0$ of Eq. (4.15), we have

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{u}_t \rangle = \langle q^*, A(0)u_t + R_0 u_t \rangle \\ &= i\beta z(t) + \bar{q}^*(0)G(u_t, 0) \\ &= i\beta z(t) + \bar{q}^*(0)G\big(W(t,\theta) + 2\,\mathrm{Re}\{z(t)q(\theta)\}, 0\big) \\ &= i\beta z(t) + \bar{q}^*(0)h(z,\bar{z}) \\ &= i\beta z(t) + g(z,\bar{z}) \end{aligned} \tag{4.20}$$

where $\beta = wT_*$, $\mu = 0$, $G(u_t, 0) = G(u_t, T_*) = h(z,\bar{z})$ and

$$g(z,\bar{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\overline{z^2}}{2} + \cdots.$$

Equations (4.15), (4.18), and (4.20) imply

$$\begin{aligned} \dot{W} &= \dot{u_t} - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= A(0)u_t + R_0 u_t - A(0)zq - A(0)\bar{z}\bar{q} - g(z,\bar{z})q \\ &\quad - \bar{g}(z,\bar{z})\bar{q} \\ &= A(0)W + R_0 u_t - 2\,\mathrm{Re}\{\bar{q}^*(0)h(z,\bar{z})q\} \\ &= \begin{cases} A(0)W - 2\,\mathrm{Re}\{\bar{q}^*(0)h(z,\bar{z})q\}, & \theta \in [T_{11}, 0), \\ A(0)W - 2\,\mathrm{Re}\{\bar{q}^*(0)h(z,\bar{z})q\} + h(z,\bar{z}), \\ \quad \theta = 0 \end{cases} \end{aligned}$$

then it is rewritten as

$$\dot{W} = A(0)W + H(z,\bar{z},\theta) \tag{4.21}$$

where

$$\begin{aligned} H(z,\bar{z},\theta) &= H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\overline{z^2}}{2} \\ &\quad + H_{30}\frac{z^3}{6} + \cdots. \end{aligned} \tag{4.22}$$

On the other hand, the following equation comes from (4.19):

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{4.23}$$

From comparing Eqs. (4.23) and (4.21), we see that there exist the following equalities between coefficients of $W(z,\bar{z},\theta)$ and $H(z,\bar{z},\theta)$:

$$\begin{cases} (A(0) - 2i\beta)W_{20}(\theta) = -H_{20}(\theta), \\ A(0)W_{11}(\theta) = -H_{11}(\theta), \\ (A(0) + 2i\beta)W_{02}(\theta) = -H_{02}(\theta). \end{cases} \tag{4.24}$$

In addition, for $\theta \in [-T_{11}, 0)$, we have

$$\begin{aligned} &H(z,\bar{z},\theta) \\ &\quad = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \\ &\quad = -\left(g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\overline{z^2}}{2} + \cdots\right) \\ &\qquad \times q(\theta) \\ &\qquad - \left(\bar{g}_{20}(\theta)\frac{\overline{z^2}}{2} + \bar{g}_{11}(\theta)z\bar{z} + \bar{g}_{02}(\theta)\frac{z^2}{2} + \cdots\right) \\ &\qquad \times \bar{q}(\theta) \\ &\quad = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\overline{z^2}}{2} \\ &\qquad + H_{30}\frac{z^3}{6} + \cdots \end{aligned}$$

therefore,

$$\begin{cases} \bar{H}_{02}(\theta) = H_{20}(\theta), \\ H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{cases} \tag{4.25}$$

Now, we can compute $W_{11}(\theta)$ and $W_{20}(\theta)$ for $\theta \in [-T_{11}, 0)$. By (4.14) at $\mu = 0$, we have

$$\big(A(0)W_{ij}\big)(\theta) = \frac{dW_{ij}(\theta)}{d\theta} \tag{4.26}$$

for $W_{11}(\theta)$ and $W_{20}(\theta)$. Thus, Eqs. (4.24), (4.25), and (4.26), imply the following first-order differential equation:

$$\dot{W}_{20}(\theta) = 2i\beta W_{20}(\theta) + g_{20}q(0)e^{i\beta\theta} + \bar{g}_{02}\bar{q}(0)e^{-i\beta\theta} \tag{4.27}$$

which has the following solution:

$$W_{20}=\frac{3C\beta e^{2i\beta\theta}+3ig_{20}q(0)e^{i\beta\theta}+i\bar{g}_{02}\bar{q}(0)e^{-i\beta\theta}}{3\beta} \tag{4.28}$$

similarly, we have

$$\dot{W}_{11}(\theta)=g_{11}q(0)e^{i\beta\theta}+\bar{g}_{11}\bar{q}(0)e^{-i\beta\theta} \tag{4.29}$$

with the solution

$$W_{11}=\frac{E\beta-ig_{11}q(0)e^{i\beta\theta}+i\bar{g}_{11}\bar{q}(0)e^{-i\beta\theta}}{\beta} \tag{4.30}$$

where C, E are both four-dimensional vectors and can be determined by setting $\theta=0$ in $H(z,\bar{z},\theta)$. By the continuity of W_{20} and W_{11} on $[-T_{11},0)$, we get

$$W_{20}(0)=\frac{3C\beta+3ig_{20}q(0)+i\bar{g}_{02}\bar{q}(0)}{3\beta}, \tag{4.31}$$

$$W_{11}(0)=\frac{E\beta-ig_{11}q(0)+i\bar{g}_{11}\bar{q}(0)}{\beta}. \tag{4.32}$$

Let

$$u_t(0)=\big(u_1(0),u_2(0),u_3(0),u_4(0)\big)^T$$

where

$$\begin{aligned}u_i(0)&=zq_i(0)+\bar{z}\bar{q}_i(0)+W_{20}^i(0)\frac{z^2}{2}\\&\quad+W_{11}^i(0)z\bar{z}+W_{02}^i(0)\frac{\overline{z^2}}{2}+\cdots,\qquad i=1,2,3,4.\end{aligned} \tag{4.33}$$

In addition, we have

$$\begin{aligned}H(z,\bar{z},0)&=-2\operatorname{Re}\{\bar{q}^*(0)h(z,\bar{z})q(0)\}+h(z,\bar{z})\\&=-g(z,\bar{z})q(0)-\bar{g}(z,\bar{z})\bar{q}(0)+h(z,\bar{z})\end{aligned} \tag{4.34}$$

Substituting (4.33) into (4.20) and comparing (4.34) with (4.22) yield

$$H_{20}(0)=-g_{20}q(0)-\bar{g}_{02}\bar{q}(0)+h_{20},$$
$$H_{11}(0)=-g_{11}q(0)-\bar{g}_{11}\bar{q}(0)+h_{11}$$

where

$$h_{20}=T_*\begin{pmatrix}0\\ \gamma_3{q_1}^2+\gamma_5{q_2}^2+\gamma_4q_1q_2\\ 0\\ \eta_3{q_3}^2+\eta_5{q_4}^2+\eta_4q_3q_4\end{pmatrix}$$

and

$$h_{11}=T_*\begin{pmatrix}0\\ 2\gamma_3\bar{q}_1q_1+2\gamma_5\bar{q}_2q_2+\gamma_4(\bar{q}_1q_2+q_1\bar{q}_2)\\ 0\\ 2\eta_3\bar{q}_3q_3+2\eta_5\bar{q}_4q_4+\eta_4(\bar{q}_3q_4+q_3\bar{q}_4)\end{pmatrix}.$$

By (4.14) at $\mu=0$, we have

$$\big(A(0)W_{ij}\big)(0)=L_0W_{ij}(0)$$

for $W_{11}(\theta)$ and $W_{20}(\theta)$ at $\theta=0$. Hence, (4.24) implies

$$\begin{aligned}&T_*\big(A_0W_{20}(0)+A_1W_{20}(-\tau_1)+A_2W_{20}(-\delta_1)\big)\\&\quad=2i\beta W_{20}(0)-H_{20}(0)\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}&T_*\big(A_0W_{11}(0)+A_1W_{11}(-\tau_1)+A_2W_{11}(-\delta_1)\big)\\&\quad=-H_{11}(0).\end{aligned} \tag{4.36}$$

Substituting (4.25), (4.28), and (4.31) into (4.35) imply

$$\begin{aligned}h_{20}&=\frac{ig_{20}}{\beta}\triangle(0,i\beta)q(0)+\frac{i\bar{g}_{02}}{3\beta}\triangle(0,-i\beta)\bar{q}(0)\\&\quad+\triangle(0,2i\beta)C\end{aligned}$$

since $\triangle(0,i\beta)q(0)=0$, then

$$C=\triangle^{-1}(0,i2\beta)h_{20}$$

and by using (4.25), (4.30) , (4.32), and (4.36), we have

$$E=\triangle^{-1}(0,0)h_{11}.$$

By using the above analysis, W_{20} and W_{11} can be computed. After substituting them in Eq. (4.20), g_{20}, g_{11}, g_{02}, and g_{21} can be determined, then we can compute the following quantities [2, 11, 12]:

$$\begin{aligned}C_1(0)&=\frac{i}{2\beta}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)\\&\quad+\frac{g_{21}}{2},\end{aligned} \tag{4.37}$$

$$K_2 = -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(T_*)\}}, \tag{4.38}$$

$$\beta_2 = 2\,\mathrm{Re}\{C_1(0)\}, \tag{4.39}$$

$$T_2 = -\frac{\mathrm{Im}\{C_1(0)\} + K_2\,\mathrm{Im}\{\lambda'(T_*)\}}{\beta} \tag{4.40}$$

where $\beta = wT_*$. Indeed:

- K_2 determines the directions of the Hopf bifurcation.
- β_2 determines the stability of the bifurcating periodic solutions.
- T_2 determines the increasing or decreasing of the period of the bifurcating solutions.

Since the computations are very complicated, we assume additional condition on the system (1.5). Let $f,\ f_1$ be odd functions, i.e.,

$$f(-x_1,-x_2) = -f(x_1,x_2)$$

and

$$f_1(-y_1,-y_2) = -f_1(y_1,y_2).$$

Therefore,

$$g_{20} = g_{11} = g_{02} = 0$$

and

$$\begin{aligned} g_{21} = T_*\big\{&\bar{q}_2^*(0)\big[3\eta_6\bar{q}_3q_3^2 + 3\eta_9\bar{q}_4q_4^2 \\ &+ \eta_7\big(2\bar{q}_3q_3q_4 + q_3^2\bar{q}_4\big) \\ &+ \eta_8\big(2\bar{q}_4q_4q_3 + q_4^2\bar{q}_3\big)\big] \\ &+ \bar{q}_4^*(0)\big[3\gamma_6\bar{q}_3q_3^2 + 3\gamma_9\bar{q}_4q_4^2 \\ &+ \gamma_7\big(2\bar{q}_3q_3q_4 + q_3^2\bar{q}_4\big) \\ &+ \gamma_8\big(2\bar{q}_4q_4q_3 + q_4^2\bar{q}_3\big)\big]\big\}. \end{aligned} \tag{4.41}$$

Substituting the values of q and q^* from (4.16) into (4.41) lead to

$$\begin{aligned} \mathrm{Re}\big\{C_1(0)\big\} = \frac{T_*}{2}\bigg\{&U\bigg[\bigg(\frac{K\alpha_1}{K^2+L^2}\bigg)^2\bigg(3\gamma_9 + \frac{\gamma_7}{w^2}\bigg) \\ &+ \bigg(3\eta_9 + \frac{\eta_7}{w^2}\bigg)\bigg] \\ &+ V\bigg[\bigg(\frac{K\alpha_1}{K^2+L^2}\bigg)^2\bigg(\frac{3\gamma_6}{w^3} + \frac{\gamma_8}{w}\bigg) \\ &+ \bigg(\frac{\alpha_1}{K^2+L^2}\bigg)^2(2KL)\bigg(3\gamma_9 + \frac{\gamma_7}{w^2}\bigg)\bigg] \\ &+ V\bigg[\frac{3\eta_6}{w^3} + \frac{\eta_8}{w}\bigg]\bigg\} \end{aligned}$$

where

$$\begin{aligned} U &= |D|^{-1}\big\{Kw\big(-4Kw - \sin(2T_*w)\alpha^2\big) \\ &\quad - wL\big(2K\gamma_2 + K^2 + L^2 + \cos(2T_*w)\alpha^2\big)\big\}, \\ V &= |D|^{-1}\big\{wL\big(-4Kw - \sin(2T_*w)\alpha^2\big) \\ &\quad + wK\big(2K\gamma_2 + K^2 + L^2 + \cos(2T_*w)\alpha^2\big)\big\}, \\ |D| &= \big[\big(4Kw + \sin(2T_*w)\alpha^2\big)^2 \\ &\quad + \big(2K\gamma_2 + K^2 + L^2 + \cos(2T_*w)\alpha^2\big)^2\big]^{1/2}. \end{aligned}$$

These computations and results state the proof of the following theorem.

Theorem 6 *Suppose that the system* (3.2) *satisfied Lemma* 4((ii) *or* (iii)). *Then the Hopf bifurcation occurs for system* (3.2) *and according to the sign of* K_2, T_2, β_2, *we have*:

- *if* $K_2 > 0$ ($K_2 < 0$), *then the Hopf bifurcation is supercritical* (*subcritical*) *and the bifurcating periodic solutions exist at* $T = T_*$,
- *if* $\beta_2 < 0$ ($\beta_2 > 0$), *then the bifurcating periodic solutions are stable* (*unstable*),
- *if* $T_2 > 0$ ($T_2 < 0$), *then the period of the bifurcating solutions increases* (*decreases*).

5 An application example

The dynamics of coupled nonlinear oscillators has received much attention over the last years; we study two van der Pol oscillators with linear coupling

$$\begin{cases} \ddot{x} + \varepsilon(x^2-1)\dot{x} + \omega_1^2 x = \alpha_1(y(t-\tau) - x(t)), \\ \ddot{y} + \varepsilon(y^2-1)\dot{y} + \omega_2^2 y = \alpha_2(x(t-\delta) - y(t)). \end{cases} \tag{5.1}$$

This model is suggested in [28] *as the microwave model where* $\omega_i^2 + \alpha_i = 1,\ i = 1,2$. *Also, this model is considered as the SA-AV nodes in heart which its synchronization is investigated in* [8] *by the describing function method. The advantage of consideration* *of this example is existence (nonexistence) of the bifurcations*. Now, we rewrite this model as follows:

$$\begin{cases} \dot{x}_1 = x_2(t), \\ \dot{x}_2 = (-\omega_1^2 - \alpha_1)x_1(t) + \alpha_1 y_1(t-\tau) \\ \qquad + \varepsilon x_2 - \varepsilon x_1^2 x_2, \\ \dot{y}_1 = y_2(t), \\ \dot{y}_2 = (-\omega_2^2 - \alpha_2)y_1(t) + \alpha_2 x_1(t-\delta) \\ \qquad + \varepsilon y_2 - \varepsilon y_1^2 y_2. \end{cases} \tag{5.2}$$

From comparing system (5.2) with system (3.2), it is easy to see that functions f, f_1 are odd and

$$\begin{cases} -1+\gamma_1 = -\omega_1^2 - \alpha_1 = -\omega_2^2 - \alpha_2, \\ \gamma_2 = \gamma_7 = \eta_7 = \varepsilon, \\ \gamma_i = \eta_i = 0, \quad i = 3,4,5,6,8,9. \end{cases} \tag{5.3}$$

By the results of Sect. 4.1, the transcritical and pitchfork bifurcations do not occur in this model.

If $\varepsilon = 0$, $(1-\gamma_1) > 0$, and $(1-\gamma_1)^2 - \alpha^2 > 0$, then by using Lemma 4(iii), system (5.2) can undergo a Hopf bifurcation. But g_{20}, g_{11}, g_{02}, and g_{21} are equal to zero. Therefore, $C_1(0)$, K_2, β_2, and T_2 are also equal to zero, or equivalently the Hopf bifurcation has zero radius.

We assume that $\varepsilon \neq 0$. Let

$$\begin{aligned} &\omega_1 = \omega_2 = 3, \qquad \alpha_1 = \alpha_2 = -5, \\ &\tau = 1.05, \qquad \delta = 0.802, \qquad \varepsilon = 4. \end{aligned} \tag{5.4}$$

By combining (3.4), (5.3), and (5.4), we have

$$p = -4, \qquad q = \pm 5, \qquad r = 4 \tag{5.5}$$

and according to Lemma (4)(ii), system (5.2) undergoes the Hopf bifurcation at origin. In fact, its characteristic equation has eigenvalues $\pm i(w=1)$ at $T_* = T_0 = 0.926$ (by (4.1) and (4.2)). We define $\mu = T - T_*$ and rewrite system (5.2) as the form (4.11)

$$\begin{cases} \dot{x}_1 = (\mu + T_*)x_2(t), \\ \dot{x}_2 = (\mu + T_*)((-\omega_1^2 - \alpha_1)x_1(t) + \alpha_1 y_1(t-\tau_1) \\ \qquad + \varepsilon x_2 - \varepsilon x_1^2 x_2), \\ \dot{y}_1 = (\mu + T_*)y_2(t), \\ \dot{y}_2 = (\mu + T_*)((-\omega_2^2 - \alpha_2)y_1(t) + \alpha_2 x_1(t-\delta_1) \\ \qquad + \varepsilon y_2 - \varepsilon y_1^2 y_2) \end{cases} \tag{5.6}$$

where $\tau_1 = \frac{\tau}{T}$ and $\delta_1 = \frac{\delta}{T}$. This system has the eigenvalues $\pm i T_*(\pm i w T_*)$ at $\mu = 0$. Its eigenvectors are computed by using (4.16) and (4.17)

$$\begin{cases} q(0) = (-0.0449 + 0.9989i, -0.9989 \\ \qquad + 0.0449i, -i, 1)^T, \\ q^*(0) = D(4.0409 + 0.8190i, -0.9989 \\ \qquad + 0.0449i, -4-i, 1), \\ D = -0.1769 + 0.0841i. \end{cases} \tag{5.7}$$

It is sufficient to get g_{21} since $g_{20} = g_{11} = g_{02} = 0$. By the formula (4.41), (5.3) , (5.4), and (5.7), we have

$$g_{21} = -0.01468 + 0.02916i. \tag{5.8}$$

The following values are computed by using (4.4), (4.37), (4.38), (4.39), and (4.40):

$$\lambda'(\mu = 0) = \lambda'(T_*) = 0.0449 - 4.0548i, \tag{5.9}$$

$$C_1(0) = -0.00734 + 0.01458i, \tag{5.10}$$

$$K_2 = 0.1634, \tag{5.11}$$

$$\beta_2 = -0.01468, \tag{5.12}$$

$$T_2 = 0.7. \tag{5.13}$$

Hence, the system on the center manifold states as follows:

$$\begin{aligned} \dot{z} &= \big(0.0449\mu + i(0.926 - 4.0548\mu)\big)z \\ &\quad + (-0.00734 + 0.01458i)z^2\bar{z} \end{aligned} \tag{5.14}$$

which is satisfied (4.20) at $\mu = 0$. According to Theorem 6, (5.11), (5.12), and (5.13), the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable and the period of the bifurcating solutions increases (see Fig. 2).

Remark Although our example is for special values of ω_1, ω_2, α_1, α_2, ε, τ, and δ, we can change them and obtain the similar results. For example, If we have

$$\begin{aligned} &\omega_1^2 = \frac{1}{2}, \qquad \omega_2^2 = \frac{2}{3}, \\ &\alpha_1 = \frac{3}{2}, \qquad \alpha_2 = \frac{4}{3}, \qquad \varepsilon = 1 \end{aligned} \tag{5.15}$$

then system (5.2) undergoes the Hopf bifurcation at $T = 0.785$ ($\tau = 0.9$, $\delta = 0.67$) and at $T = 6.656$ ($\tau = 3.15$, $\delta = 3.5$) when its characteristic equation has eigenvalues $\pm i$ and $\pm 1.44i$, respectively.

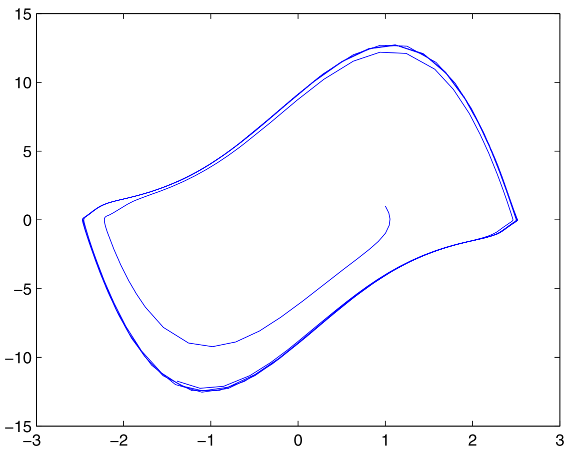


Fig. 2 Stable limit cycle with $\omega_1 = \omega_2 = 3$, $\alpha_1 = \alpha_2 = -5$, $\tau = 1.05$, $\delta = 0.802$, $\varepsilon = 4$

- At $T = 0.785$ and eigenvalues $\pm i$, the reduced system on the center manifold is

$$\dot{z} = \big(2.661\mu + i(0.785 - 1.57\mu)\big)z + (-0.2249 + 0.0809i)z^2\bar{z} \quad (5.16)$$

and

$$K_2 = 0.0845, \qquad \beta_2 = -0.4999, \qquad T_2 = 0.862. \quad (5.17)$$

Therefore, the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, and the period of the bifurcating solutions increases (see Fig. 3).

- At $T = 6.656$ and eigenvalues $\pm 1.44i$, the reduced system on the center manifold is

$$\dot{z} = \big(1.247\mu + i(1.44 - 2.73\mu)\big)z + (0.2025 + 0.562i)z^2\bar{z} \quad (5.18)$$

and

$$K_2 = -0.162, \qquad \beta_2 = 0.4051, \qquad T_2 = -0.1047. \quad (5.19)$$

Therefore, the Hopf bifurcation is subcritical, the bifurcating periodic solutions are unstable and the period of the bifurcating solutions decreases (see Fig. 4).

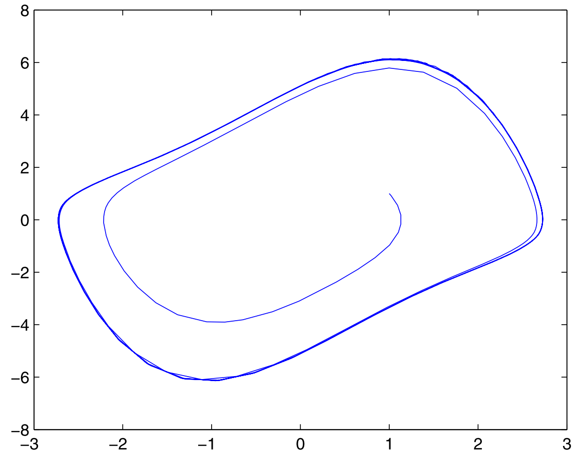


Fig. 3 Stable limit cycle with $\omega_1 = \sqrt{\frac{1}{2}}$, $\omega_2 = \sqrt{\frac{2}{3}}$, $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{4}{3}$, $\tau = 0.9$, $\delta = 0.67$, $\varepsilon = 1$

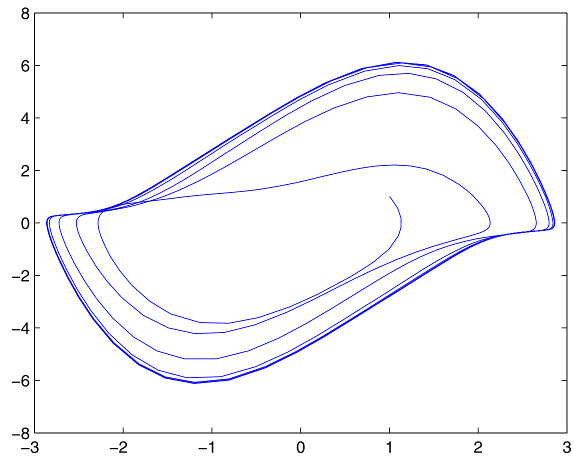


Fig. 4 Unstable limit cycle with $\omega_1 = \sqrt{\frac{1}{2}}$, $\omega_2 = \sqrt{\frac{2}{3}}$, $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{4}{3}$, $\tau = 3.15$, $\delta = 3.5$, $\varepsilon = 1$

6 Conclusion

In this paper, system (1.5) of coupled nonlinear oscillators is considered. The effect of the time delay on the linear stability of the system is investigated. By analyzing the associated characteristic equation, we derived the necessary and sufficient conditions on the trivial equilibrium which guarantee stability. Based on the center manifold theory, we reduced system (1.5) on the finite dimensional system of ordinary differential equations when the characteristic equation had eigenvalues on the imaginary axis. Moreover, the bifurcations of the trivial equilibrium (transcritical, pitchfork, Hopf) are studied. We applied our results on the delay-coupled van der Pol equations.

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