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Probabilistic G-Metric space and some fixed point results

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Abstract

In this note we introduce the notions of generalized probabilistic metric spaces and generalized Menger probabilistic metric spaces. After making our elementary observations and proving some basic properties of these spaces, we are going to prove some fixed point result in these spaces.

Keywords: Probabilistic G-metric; Menger probabilistic G-metric; G-Contraction map.

1 Introduction and preliminaries

Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization, and approximation theory. Different generalizations of the notion of a metric space have been proposed by Gähler [5, 6] and by Dhage [1, 2]. However, Ha et al [7]. have pointed out that the results obtained by Gähler for his 2-metrics are independent, rather than generalizations, of the corresponding results in metric spaces, while in [11] the current authors have pointed out that Dhages notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid.

In 2006 the concept of generalized metric space was introduced [12]. For more results in these spaces one can see [9] and [10].

On the other hand, in 1942, Menger [13] introduced the notion of probabilistic metric space (briefly PM-space) as a generalization of metric space. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. The development of fixed point theory in PM-spaces was due to Schweizer and Sklar [15, 16]. Fixed point theory has been always an active area of research since 1922 with the celebrated Banach contraction fixed point theorem. One can see [3], [4], [8], [14], [17],[19] for some fixed point results in probabilistic metric spaces.

Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following axioms:

1 $G(x,y,z) = 0$ if $x = y = z$,

2 $G(x,x,y) > 0$, for all $x,y \in X$, with $x \neq y$,

3 $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$, with $z \neq y$,

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4 $G(x,y,z) = G(x,z,y) = G(y,z,x) = G(z,y,x) = ...$(symmetry in all three variables),

5 $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a *generalized metric*, or, more specifically a *G-metric* on X, and the pair (X,G) is called a *G-metric space* (see [12]). A sequence (x_n) in a G-metric space (X,G) is said to be *G-convergent* to x if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$; which means that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$. Also a sequence (x_n) is called *G-Cauchy* if for a given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq N$; that is if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

We may construct G-metrics using an ordinary metric. Indeed if (X,D) is a metric space, then define

$$G_s(x,y,z) = d(x,y) + d(z,y) + d(x,z).$$
$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\},$$

for all $x,y,z \in X$. One can verify that G_s and G_m are G-metric.

A *distribution function* is a function $F : [-\infty,\infty] = \bar{\mathbb{R}} \to [0,1]$ that is nondecreasing and left continuous on $\mathbb{R}$; moreover, $F(-\infty) = 0$ and $F(\infty) = 1$.
The set of all the distribution functions is denoted by Δ and the set of those distribution functions such that F(0) = 0 is denoted by Δ^+.
A natural ordering in Δ is defined by $F \leq G$ whenever $F(x) \leq G(x)$, for every $x \in \mathbb{R}$. The maximal element in this order for Δ^+ is ε_0, where for $-\infty \leq a \leq \infty$ the distribution function ε_a is defined by

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if} \quad -\infty \leq x \leq a \\ 1 & \text{if} \quad a < x \leq \infty. \end{cases}$$

A binary operation on Δ^+ which is commutative, associative, nondecreasing in each place, and has ε_0 as identity, is said to be *triangle function*.

Also a *probabilistic metric space* (abbreviated, PM-space) is an ordered triple (S,F,τ) where S is a nonempty set, τ is a triangle function and $F : S \times S \to \Delta^+$ ($F(p,q)$ is denoted by $F_{p,q}$) satisfies the following conditions:

1) $F_{p,p} = \varepsilon_0$,

2) If $p \neq q$, then $F_{p,q} \neq \varepsilon_0$,

3) $F_{p,q} = F_{q,p}$,

4) $F_{p,r} \geq \tau(F_{p,q}, F_{q,r})$,
for every $p,q,r \in S$.

If 1), 3), 4) and are satisfied, then (S,F,τ) is called a *probabilistic pseudo-metric space*.

In section 2, we introduce the notion generalized probabilistic metric space. Then Some examples and elementary properties of these spaces are discussed. In section 3, generalized Menger probabilistic G-metric space is studied. Finally in section 4, some fixed point theorem in generalized Menger probabilistic metric spaces are investigated.

2 Probabilistic G-metric space

Definition 2.1. *Suppose X is a nonempty set, τ is a triangle function and $G : X \times X \times X \to \Delta^+$, is a mapping satisfying*

G_1 $G(p,p,p) = \varepsilon_0$,

G_2 *if $p \neq q$, then $G(p,p,q) \neq \varepsilon_0$,*

G_3 *if $q \neq r$, then $G(p,p,q) \geq G(p,q,r)$,*

G_4 $G(p,q,r) = G(p,r,q) = G(q,r,p) = ...$,

G_5 $G(p,q,r) \geq \tau(G(p,s,s),G(s,q,r))$,

for all $p,q,r,s \in X$. Then (X,G,τ) is called a generalized probabilistic metric space *(or briefly, probabilistic G-metric space). (X,G,τ) is called a* probabilistic pseudo G-metric space *if G_1, G_3, G_4 and G_5 are satisfied.*

A probabilistic G-metric space (X,G,τ) is said to be symmetric if for every $x,y \in X$,

$$G(x,y,y) = G(y,x,x).$$

A probabilistic G-metric space (X,G,τ) is called proper *if $\tau(\varepsilon_a,\varepsilon_b) \geq \varepsilon_{a+b}$, for all $a,b \in [0,\infty)$*

In the following two examples, we construct two probabilistic G-metric space using a PM-space and a G-metric space, respectively.

Example 2.1. *With $\tau(F,G) = \min\{F,G\}$, let (X,F,τ) be a probabilistic metric space. If $G_M : X^3 \to \Delta^+$ is defined by $G_m(p,q,r) = \min\{F_{p,q},F_{p,r},F_{q,r}\}$, then (X,G_M,τ) is a probabilistic G-metric space.*
Indeed if $p = q = r$ then

$$G_m(p,q,r) = \min\{F_{p,q},F_{p,r},F_{q,r}\} = \min\{\varepsilon_0,\varepsilon_0,\varepsilon_0\} = \varepsilon_0.$$

Also for $p \neq q$ by definition of probabilistic metric, $F_{p,q} \neq \varepsilon_0$, so

$$G_m(p,p,q) = \min\{F_{p,q},F_{p,p}\} = \min\{F_{p,q},\varepsilon_0\} = F_{p,q} \geq \varepsilon_0.$$

Now if $q \neq r$ then

$$G_m(p,p,q) = \min\{F_{p,q},\varepsilon_0\} = F_{p,q} \geq \min\{F_{p,q},F_{p,r},F_{q,r}\} = G_m(p,q,r).$$

Commutativity of G_m is trivial by commutativity of F. For proving G_5, let $p,q,r,s \in X$. We have

$$\min\{G_m(p,s,s),G_m(s,q,r)\} = \min\{F_{p,s},F_{s,s},F_{s,q},F_{s,r},F_{q,r}\}.$$

Thus

$$\begin{aligned} G_m(p,q,r) &= \min\{F_{p,q},F_{q,r},F_{p,r}\} \\ &\geq \min\{\min\{F_{p,s},F_{s,q}\},F_{q,r},\min\{F_{p,s},F_{s,r}\}\} \\ &= \min\{\min\{F_{p,s},F_{s,s}\},\min\{F_{s,q},F_{q,r},F_{s,r}\}\} \\ &= \tau(G_m(p,s,s),G_m(s,q,r)). \end{aligned}$$

Example 2.2. *Let (X,F) be a G-metric space. For every $p,q,r \in X$, define*

$$G_{p,q,r} = \varepsilon_{F_{p,q,r}}. \tag{2.1}$$

Also let $\tau$is a triangle function for which

$$\tau(\varepsilon_a,\varepsilon_b) \leq \varepsilon_{a+b},$$

for all $a,b \in \mathbb{R}^+$. Then it is straightforward to show that (X,G,τ) is a probabilistic G-metric space.

Also if a proper probabilistic G-metric is of the form (2.1), then $F_{p,q,r}$ is a G-metric. Suppose (X,G,τ) is a proper probabilistic G-metric space and there exists a function $F : X \times X \times X \to \mathbb{R}^+$, such that

$$G_{p,q,r} = \varepsilon_{F_{p,q,r}}$$

then (X,F) is a G-metric space.
Indeed in this case

$$\varepsilon_0 = G_{p,p,p} = \varepsilon_{F_{p,p,p}} = \varepsilon_0,$$

so $F_{p,p,p} = 0$. If $p \neq q$ then

$$\varepsilon_0 \neq G_{p,p,q} = \varepsilon_{F_{p,p,q}},$$

which implies that $F_{p,p,q} \neq 0$. Also if $q \neq r$ then the fact that $G_{p,p,q} \geq G_{p,q,r}$ implies that $F_{p,p,q} \leq F_{p,q,r}$. Commutativity of F follows from commutativity of G. For proving

$$F_{p,q,r} \leq F_{p,s,s} + F_{s,q,r},$$

we note that G is proper, so,

$$\varepsilon_{F_{p,q,r}} = G_{p,q,r} \geq \tau(\varepsilon_{F_{p,s,s}}, \varepsilon_{F_{s,q,r}}) \geq \varepsilon_{F_{p,s,s}+F_{s,q,r}}$$

which implies that (X,G) is a G-metric space.

In the following proposition, it is proved that we may construct a probabilistic G-metric space using a pseudo probabilistic G-metric space. To do this, we introduce the following relation:
Let (X,G,τ) be a probabilistic pseudo G-metric space. For $p,q \in X$, we say $p \sim q$ if and only if

$$G(p,p,q) = \varepsilon_0 \quad \text{and} \quad G(p,q,q) = \varepsilon_0.$$

This relation is an equivalence relation. Indeed if $p \sim q$ and $q \sim r$, then

$$G(p,p,q) = \varepsilon_0 \quad , G(p,q,q) = \varepsilon_0 \text{ and } G(q,q,r) = \varepsilon_0, \quad G(r,r,q) = \varepsilon_0$$

But G is a probabilistic pseudo G-metric, so

$$G(p,p,r) = G(r,p,p) \geq \tau(G(r,q,q), G(q,p,p)) = \tau(\varepsilon_0, \varepsilon_0) = \varepsilon_0,$$

which implies that $G(p,p,r) \geq \varepsilon_0$. Now maximality of ε_0 implies that $G(p,p,r) = \varepsilon_0$. Similarly $G(p,r,r) = \varepsilon_0$. This prove that $\sim$ is transitive. The other properties of $\sim$ to be an equivalence relation is trivial.

Proposition 2.1. *Let (X,G,τ) be a probabilistic pseudo G-metric space, for every $p \in S$, let $p^\star$ denote the equivalence class of p and let $X^\star$ denotes the set of these equivalence classes. Then the expression*

$$G^\star(p^\star, q^\star, r^\star) = G(p,q,r), \qquad p \in p^\star, q \in q^\star, r \in r^\star$$

define a function $G^\star$ from $X^\star \times X^\star \times X^\star$ into Δ^+ and the triple $(X^\star, G^\star, \tau)$ is a probabilistic G-metric space, the quotient space of (X,G,τ).

Proof. First we prove that $G^\star$ is well defined, i.e. if $r, r' \in p^\star$, $q, q' \in q^\star$ and $p, p' \in p^\star$, then

$$G(p,q,r) = G(p',q',r').$$

Since $q \sim q'$, $p \sim p'$ and $r \sim r'$ and τ is a triangular function, we have

$$\begin{aligned} G(p,q,r) &\geq \tau(G(p,p',p'), G(p',q,r)) = G(p',q,r) \\ &\geq \tau(G(q,q',q'), G(q',p',r)) = G(q',p',r) \\ &\geq \tau(G(r,r',r'), G(r',p',q') = G(r',p',q') \\ &= G(p',q',r'). \end{aligned}$$

Similarly we get $G(p',q',r') \leq G(p,q,r)$, so $G^\star$ is well defined. Also trivially,

$$G^\star(p^\star, p^\star, p^\star) = G(p,p,p) = \varepsilon_0.$$

and if $p \neq q$, then

$$p \notin q^\star \quad , \quad q \notin p^\star.$$

Hence $p \nsim q$, so $G(p,p,q) \neq \varepsilon_0$. Thus

$$G^\star(p^\star, p^\star, q^\star) = G(p,p,q) \neq \varepsilon_0.$$

By the fact that,

$$G(p,p,q) \geq G(p,q,r)$$

we lead to

$$G^\star(p,p,q) \geq G^\star(p,q,r).$$

It is trivial to verify the other properties of $G^\star$. □

3 Menger probabilistic G-metric space

In this section we introduce Menger probabilistic G-metric spaces. Recall that a mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied

1. $T(a,1) = a$, for every $a \in [0,1]$,
2. $T(a,b) = T(b,a)$, for every $a,b \in [0,1]$,
3. $T(a,c) \geq T(b,d)$, whenever $a \geq b$ and $c \geq d$, $(a,b,c,d \in [0,1])$,
4. $T(a,T(b,c)) = T(T(a,b),c)$, $(a,b,c \in [0,1])$.

The following are the four basic t-norms:
(a) The minimum t-norm, T_M, is defined by $T_M(x,y) = \min\{x,y\}$.
(b) The product t-norm, T_P, is defined by $T_P(x,y) = xy$.
(c) The Lukasiewicz t-norm, T_L, is defined by $T_L(x,y) = \max\{x+y-1,0\}$.
(d) The weakest t-norm, the drastic product, T_D, is defined by

$$\begin{cases} T_D(x,y) = \min\{x,y\}, & \text{if } \max\{x,y\} = 1 \\ 0, & \text{otherwise.} \end{cases}$$

As regards the pointwise ordering, we have the inequalities $T_D < T_L < T_P < T_M$.

Definition 3.1. *Suppose S is a nonempty set and T is a t-norm and $G : S^3 \to \Delta^+$ is a function. The triple (S,G,T) is called a Menger probabilistic G-metric space if for every $p,q,r,s \in S$ and $x,y > 0$,*

1 $G(p,p,p) = \varepsilon_0$,

2 *If $p \neq q$, then $G(p,p,q) \neq \varepsilon_0$,*

3 $G(p,p,q) \geq G(p,q,r)$,

4 $G(p,q,r) = G(p,r,q) = G(q,r,p) = ...$,

5 $G(p,q,r)(x+y) \geq T(G(p,s,s)(x), G(s,q,r)(y))$.

In the Menger probabilistic G-metric space (S,G,T) with

$$\sup_{0<t<1} T(t,t) = 1$$

a sequence $\{u_n\}$ in S,

i) is called convergent to $u \in S$ if for every $\varepsilon, \lambda > 0$, there exists $N \in \mathbb{N}$ such that,

$$\forall n \geq N \ ; \ G_{u_n,u,u}(\varepsilon) > 1 - \lambda.$$

ii) is said to be a Cauchy sequence, if for every $\varepsilon, \lambda > 0$ there exists $N \in \mathbb{N}$ such that,

$$\forall m,n,l \geq N \ ; \ G_{u_m,u_n,u_l}(\varepsilon) > 1 - \lambda.$$

As usual a Menger probabilistic G-metric space is said to be complete if every Cauchy sequence in S converges to a $u \in S$.

Theorem 3.1. *Let (S,G,T_L) be a Menger probabilistic G-metric space and define,*

$$G^{\star}_{p,q,r} = \sup\{t \geq 0 | G_{p,q,r}(t) \leq 1 - t\}.$$

Then,

i) $G^\star$ is a G-metric.

ii) S is G-complete if and only if it is $G^\star$-complete.

Proof. For any $t > 0$, $G_{p,p,p} = \varepsilon_0(t) = 1$, so

$$G^\star_{p,p,p} = \sup\{t \geq 0 | G_{p,p,p}(t) = 1 \leq 1-t\} = 0.$$

Also if $p \neq q$, then $G_{p,p,q} \neq \varepsilon_0$. Hence

$$\exists t \in (0,1) \ s.t. \ G_{p,p,q}(t) < 1,$$

so

$$G^\star_{p,p,q} = \sup\{t \geq 0 | G_{p,p,q}(t) \leq 1-t\} > 0.$$

Now for any $p,q,r \in S$ we know, $G_{p,p,q} \geq G_{p,q,r}$, so

$$\{t | G_{p,p,q}(t) \leq 1-t\} \subseteq \{t | G_{p,q,r}(t) \leq 1-t\}.$$

Hence $G^\star_{p,p,q} \leq G^\star_{p,q,r}$. These prove first, second and the third part of definition of G-metric for $G^\star$. Commutativity of $G^\star$ is trivial.
We are going to prove that,

$$G^\star_{p,q,r} \leq G^\star_{p,s,s} + G^\star_{s,q,r}, \tag{3.2}$$

for all $p,q,r,s \in S$.
To do this, put

$$\begin{aligned} A &= \{t | G_{p,q,r} \leq 1-t\} \\ B &= \{\lambda | G_{p,s,s}(\lambda) \leq 1-\lambda\} \\ C &= \{\mu | G_{s,q,r \leq 1-\mu}\}. \end{aligned}$$

Suppose $t_1 > G^\star(p,s,s)$ and $t_2 > G^\star_{s,q,r}$ are upper bounds for B and C, respectively. Then

$$G(p,s,s)(t_1) > 1-t_1 \text{ and } G_{s,q,r}(t_2) > 1-t_2.$$

Therefore

$$\begin{aligned} G_{p,q,r}(t_1+t_2) &\geq T_L(G_{p,s,s}(t_1), G_{s,q,r}(t_2)) \\ &\geq G_{p,s,s}(t_1) + G_{s,q,r}(t_2) - 1 \\ &> 1-(t_1+t_2). \end{aligned}$$

Thus t_1+t_2 is an upper bound for A. Hence $G^\star$ satisfies (3.2). Consequently G^* is a G-metric.

For proving *ii*), let (S,G,T_L) be G-complete and (u_n) be a Cauchy sequence in the $G^\star$-metric. We prove that (u_n) is Cauchy with the probabilistic G-metric G. Let $\varepsilon, \lambda > 0$ be given. If $\varepsilon < \lambda$ then for ε,

$$\exists N \in \mathbb{N} \ s.t. \ \forall m,n,l \geq N \ ; \qquad G^*_{u_m,u_n,u_l} < \varepsilon,$$

since (u_n) is $G^\star$-Cauchy. By definition of G^*, for every $m,n,l \geq N$

$$G_{u_m,u_n,u_l}(\varepsilon) > 1-\varepsilon > 1-\lambda.$$

Now if $\lambda < \varepsilon$ then for λ,

$$\exists N \in \mathbb{N}, \ \forall m,n,l \geq N \quad ; \quad G^\star_{u_m,u_n,u_l} < \lambda,$$

since (u_n) is $G^\star$-Cauchy. By definition of $G^\star$, the fact that G_{u_m,u_n,u_l} is nondecreasing implies that

$$G_{u_m,u_n,u_l}(\varepsilon) \geq G_{u_m,u_n,u_l}(\lambda) > 1-\lambda.$$

Thus (u_n) is G-Cauchy. Now by G-completeness of S with G, there exists $u \in S$ such that (u_n) is G-convergent to u. So for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, for every $m, n \geq N$,

$$G_{u_m,u_n,u}(\frac{\varepsilon}{2}) > 1 - \frac{\varepsilon}{2}.$$

This means that $\frac{\varepsilon}{2}$ is an upper bound for the segment $\{t | G_{u_m,u_n,u} \leq 1-t\}$. Thus $G^{\star}_{u_n,u_m,u} \leq \frac{\varepsilon}{2}\varepsilon <$, i.e. (u_n) converges to u with $G^{\star}$ and so S is $G^{\star}$-complete.
Conversely suppose that S is $G^{\star}$-complete and (u_n) is a G-Cauchy sequence in S. Thus for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that,

$$\forall m, n, l \geq N; \quad G_{u_m,u_n,u_l}(\frac{\varepsilon}{2}) > 1 - \frac{\varepsilon}{2}.$$

Hence

$$\forall m, n, l \geq N; \quad G^{*}_{u_m,u_n,u_l} < \frac{\varepsilon}{2} < \varepsilon.$$

This implies that (u_n) is a $G^{\star}$-Cauchy sequence sequence and so is $G^{\star}$-convergent to some u in S. Hence for given ε, λ, with $\varepsilon < \lambda$, there exists $N \in \mathbb{N}$ such that

$$\forall m, n \geq N; \quad G^{\star}_{u_m,u_n,u} < \varepsilon < \lambda.$$

By definition of $G^{\star}$,

$$\forall m, n \geq N; \quad G_{u_m,u_n,u}(\varepsilon) > 1 - \varepsilon > 1 - \lambda.$$

Now if $\lambda \leq \varepsilon$ then

$$\exists N > 0 \quad s.t. \quad \forall m, n, l \geq N \quad ; \quad G_{u_m,u_n,u_l}(\varepsilon) > G_{u_m,u_n,u_l}(\lambda) > 1 - \lambda \geq 1 - \varepsilon,$$

since (u_n) is G-Cauchy. By definition of $G^{\star}$

$$\forall m, n, l \geq N; \quad G^{\star}_{u_m,u_n,u_l} < \varepsilon$$

But S is $G^{\star}$-complete, so there exists $u \in S$ such that (u_n) is $G^{\star}$-convergent to u. This implies that there exists $N \in \mathbb{N}$ such that

$$\forall m, n \geq N; \quad G^{\star}_{u_m,u_n,u} < \lambda < \varepsilon.$$

Finally by definition of $G^{\star}$

$$\forall m, n \geq N \quad ; \quad G_{u_m,u_n,u}(\varepsilon) \geq G_{u_m,u_n,u}(\lambda) > 1 - \lambda.$$

Hence (u_n) is G-convergent and so S is G-complete. □

4 Fixed points of contractive maps in Menger probabilistic G-metric space

In this section, first we introduce the concept of G-contractive mapping in Menger probabilistic G-metric space and then its relation with G-contractive map in its dependent G-metric space is studied. This result shows that the existence of a convergent subsequence of an iterate sequence (of a contractive map) implies the existence of a fixed point.

In order to do this, we introduce the following definition;

Definition 4.1. *Let (S, G, T) be a Menger probabilistic G-metric space. a mapping $f : S \to S$ is said to be a G-contraction if for any $t \in (0, \infty)$,*

$$G_{p,q,r}(t) > 1 - t$$

implies that

$$G_{f(p),f(q),f(r)}(kt) > 1 - kt$$

for some fixed $k \in (0, 1)$.

One can easily see that if $f : S \to S$ is a G-contraction and (u_n) is a convergent sequence to some u in the Menger probabilistic G-metric space S, then $(f(u_n))$ converges to $f(u)$.

We recall that a function f on a G-metric space with a G-metric $G^\star$ is called G-contraction if for any $t \in (0,\infty)$, the relation $G_{p,q,r} < t$ implies that $G_{f(p),f(q),f(r)} < kt$, for some $k \in (0,1)$.

Lemma 4.1. *Let (S,G,T_L) be a Menger probabilistic G-metric space and*

$$G^\star_{p,q,r} = \sup\{t | G_{p,q,r}(t) \leq 1-t\}$$

then a function $f : S \to S$ is a G-contraction mapping if and only if it is $G^\star$-contraction.

Proof. By Theorem 3.1, we know that $G^\star$ is a G-metric on S.
Let f be a G-contraction in the Menger probabilistic G-metric space and for $t \in (0,\infty)$

$$G^\star_{p,q,r} < t.$$

By definition of $G^\star$, we get

$$G_{p,q,r}(t) > 1-t.$$

But f is G-contraction, so

$$G_{f(p),f(q),f(r)}(kt) > 1-kt,$$

for some fixed $k \in (0,1)$. Now definition of $G^\star$ implies that

$$G^\star_{f(p),f(q),f(r)} < kt,$$

which means that f is $G^\star$-contraction. The converse of this lemma can be proved similarly. □

Theorem 4.1. *Let (S,G,T_L) be a Menger probabilistic G-metric space. Suppose A is G-contraction on S and for some u in S, $A^{n_i}(u)$ is a convergent subsequence of $A^n(u)$, then $\xi = A(\lim_{i\to\infty} A^{n_i}(u))$ is the unique fixed point of A.*

Proof. Let $A(\xi) \neq \xi$, then there exists $t_0 \in (0,\infty)$ such that,

$$G_{A(\xi),\xi,\xi}(t_0) \neq 1.$$

So there exists $\lambda \in (0,1)$, such that

$$1-\lambda < G_{A(\xi),\xi,\xi}(t_0) < 1.$$

By letting $t = \max\{t_0, \lambda\}$ we get

$$G_{A(\xi),\xi,\xi}(t) \geq G_{A(\xi),\xi,\xi}(t_0) > 1-\lambda > 1-t.$$

But A is a G-contraction so for some $k \in (0,1)$,

$$G_{A^2(\xi),A(\xi),A(\xi)}(kt) > 1-kt.$$

Using induction argument one can see that

$$G_{A^{n+1}(\xi),A^n(\xi),A^n(\xi)}(k^n t) > 1-k^n t. \tag{4.3}$$

Taking n and n_i large enough such that

$$k^n t < 1 \quad \text{and} \quad k^{n_i} t < 1$$

and putting $p = A^n(\xi)$, we obtain

$$p = A^n(\xi) = A^n(\lim_{i\to\infty} A^{n_i}(u)) = \lim_{i\to\infty} A^{n_i+n}(u).$$

Let $s = \max\{k^n t, k^{n_i} t\}$. By (4.3)

$$G_{A(p),p,p}(s) > G_{A(p),p,p}(k^n t) > 1-k^n t > 1-s.$$

If $G^\star$ is the G-metric introduced in Lemma 4.1, then

$$G^\star_{A(p),p,p} = \sup\{t | G_{A(p),p,p}(t) \leq 1-t\}.$$

So

$$G^\star_{A(p),p,p} < s < 1.$$

By the fact that $A^{n_i}(u) \to \xi$ and $A^{n_i+1}(u) \to A(\xi)$, for every $t, \lambda > 0$, there exists $N \in \mathbb{N}$, such that for every $n_i > N$,

$$G_{A^{n_i}(u),\xi,\xi}(t) > 1-\lambda \ , \ G_{A^{n_i+1}(u),A(\xi),A(\xi)}(t) > 1-\lambda.$$

Let $l > j > n+n_i$. We are going to prove that,

$$G^\star_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)} \leq k^{l-j} G^\star_{A^{n_j}(u),A^{n_j+1}(u),A^{n_j+1}(u)}. \tag{4.4}$$

If we prove this inequality, then the this together with the facts that $k \in (0,1)$ and $G^\star_{A^{n_j}(u),A^{n_j+1}(u),A^{n_j+1}(u)} < 1$ imply that $\lim_{l\to\infty} A^{n_l}(u) = \lim_{l\to\infty} A^{n_l+1}(u)$ in the generalized metric $G^\star$ and so is valid in the Menger probabilistic G. This leads to the equality $\xi = A(\xi)$ which is a contradiction.
First we prove that,

$$G^\star_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)} \leq k G^\star_{A^{n_l-1}(u),A^{n_l}(u),A^{n_l}(u)}. \tag{4.5}$$

To do this, let

$$s \in \{t | G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(t) \leq 1-t\}$$

then

$$G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(s) \leq 1-s.$$

Put $t = s/k$. We find that

$$G_{A^{n_{l-1}}(u),A^{n_l}(u),A^{n_l}(u)}(t) \leq 1-t,$$

since otherwise by contractivity of A it should be

$$G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(kt) = G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(s) > 1-kt = 1-s,$$

which is not the case. Therefore

$$t = s/k \in \{t | G_{A^{n_{l-1}}(u),A^{n_l}(u),A^{n_l}(u)}(t) \leq 1-t\}$$

or equivalently

$$s \in k\{t | G_{A^{n_{l-1}}(u),A^{n_l}(u),A^{n_l}(u)}(t) \leq 1-t\}.$$

So

$$\sup\{t | G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(t) \leq 1-t\} \leq k \sup\{t | G_{A^{n_{l-1}}(u),A^{n_l}(u),A^{n_l}(u)}(t) \leq 1-t\}$$

and consequently (4.5) is valid. Now by induction argument, one leads to (4.5) which completes the proof. □

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