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Shear Tensor and Dynamics of Relativistic Accretion Disks around Rotating Black Holes

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Abstract

In this paper we solve the hydrodynamical equations of optically thin, steady state accretion disks around Kerr black holes. Here, fully general-relativistic equations are used. We use a new method to calculate the shear tensor in the LNRF (Locally Non-Rotating Frame), BLF (Boyer–Lindquist Frame), and FRF (Fluid Rest Frame). We show that two components of shear tensor in the FRF are nonzero (in previous works only one nonzero component was assumed). We can use these tensors in usual transonic solutions and usual causal viscosity, but we derive solutions analytically by some simplifications. Then, we can calculate the four-velocity and density in all frames, such as the LNRF, BLF, and FRF.

Key words: accretion, accretion disks — black holes — hydrodynamics — shear tensors

1. Introduction

Accretion disks are important in several astrophysical systems. They can be found around Young Stellar Objects (YSO), around compact stellar objects in our Galaxy, and around several super-massive black holes in Active Galactic Nuclei (AGN).

In super-massive accretion disks the mass of the black hole is $(10^5$–$10^9)M_\odot$. To study such disks we use general relativity with relativistic hydrodynamics in Kerr metric background geometry. In a relativistic Navier–Stokes fluid we have a stress-energy tensor that is related to the viscosity, and is the cause of redistribution of the energy and momentum in fluid. This tensor is defined by the four-velocity and the metric. In previous studies, the relativistic and stationary solutions of standard black hole disks were solved. Lasota (1994) was the first who wrote down slim-disk equations that include relativistic effects. He also assumed that only the $r\phi$ component of the stress tensor is nonzero. He used a special form for this component, which was followed by Abramowicz et al. (1996, 1997). Chakrabarti (1996) derived transonic solutions of thick and thin disks for a weak viscosity. He assumed a similar form for the viscosity as Lasota (1994). Then, Manmoto (2000) derived a global two-temperature structure of advection-dominated accretion flows (ADAFs) numerically by using full relativistic hydrodynamical equations, including the energy equations for the ions and electrons.

Papaloizou and Szuszkiewicz (1994) introduced a phenomenological and non-relativistic equation for the evolution of the viscous stress tensor (causal viscosity), which has been used by many authors. Gammie and Popham (1998) and Popham and Gammie (1998) solved ADAFs with relativistic causal viscosity. They used the Boyer–Lindquist coordinates. Takahashi (2007b) solved the equations of relativistic disks in the Kerr–Schild coordinates by using the relativistic causal viscosity. In the papers of Gammie and Popham and Takahashi they assumed that in the fluid rest frame (FRF), only the $r\phi$ component of shear viscosity is nonzero; then, they used the transformation tensors to derive the components of the shear stress tensor in the Boyer–Lindquist or Kerr–Schild frames.

In the present study, we concentrate on the stationary axisymmetric accretion flow in the equatorial plane. We use a new method to calculate the shear tensor and azimuthal velocity of fluids in the locally non-rotating frame (LNRF) by using the Keplerian angular velocity. We derive two kinds of shear tensors in LNRF and Boyer–Lindquist frame (BLF); in the first one, the direction of fluid rotation is the same as that of the black hole (Ω^+); in the second one, the direction of fluid rotation is opposite to that of the black hole (Ω^-). We calculate the components of the shear tensor for two kinds of fluids in LNRF, BLF, and FRF; these calculations show that in FRF there are two nonzero components (rt and $r\phi$ components). However, in previous papers, the only nonzero component in FRF was the $r\phi$ component. The rt component results from relativistic calculations of the shear tensor, which changes some components of the four-velocity. Then, by using these shear-tensor components we calculate the four-velocity in LNRF, BLF, and density in all frames.

This paper's agenda is as follows. We introduce the metric and reference frame in section 2. In section 3 basic equations are given. In section 4 the shear tensor is calculated in FRF, LNRF, and BLF. We derive the four-velocity in LNRF and BLF in section 5; the influence of two important parameters on the density and the four-velocity can be seen in this section. Also, the influence of the rt component of the shear tensor can be seen in this section. A summary and conclusions are given in section 6.

2. Metric, Reference Frame

2.1. Background Metric

For background geometry, we use the Boyer–Lindquist coordinates of the rotating black hole space time. In the Boyer–Lindquist coordinates, the Kerr metric is

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= -\alpha^2 dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)(dx^j + \beta^j dt), \end{aligned} \tag{1}$$

where $i,\ j = r,\ \theta$, and ϕ. Nonzero components of the lapse function, α, the shift vector, β^i, and the spatial matrix, γ_{ij}, are given in geometric units as:

$$\alpha = \sqrt{\frac{\Sigma\Delta}{A}}, \qquad \beta^\phi = -\omega, \qquad \gamma_{rr} = \frac{\Sigma}{\Delta},$$
$$\gamma_{\theta\theta} = \Sigma, \qquad \gamma_{\phi\phi} = \frac{A\sin^2\theta}{\Sigma}. \tag{2}$$

Here, we use the geometric mass, $m = GM/c^2$, $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, and $A = \Sigma\Delta + 2mr(r^2 + a^2)$. The positions of the outer and inner horizons, $r\pm$, are calculated by inserting $\Delta = 0$ to get $r\pm = m \pm (m^2 - a^2)^{1/2}$. The angular velocity of frame dragging due to the black-hole rotation is $\omega = -g_{t\phi}/g_{\phi\phi} = 2mar/A$. M is the black-hole mass, G is the gravitational constant, and c is the speed of light. The angular momentum of the black hole, J, is described as

$$a = Jc/(GM^2), \tag{3}$$

where $-1 < a < 1$.

Similar to Gammie and Popham (1998), we set $G = M = c = 1$ for basic scalings. The nonzero components of the metric, $g_{\mu\nu}$, and its inverse, $g^{\mu\nu}$, are calculated in appendix 1.

2.2. Reference Frame

In our study, we use three reference frames. The first one is the Boyer–Lindquist frame (BLF) based on the Boyer–Lindquist coordinates describing the metric, in which our calculations are done. The second one is the locally non-rotating reference frame (LNRF), which is formed by observers with a future-directed unit vector orthogonal to t = constant. By using the Boyer–Lindquist coordinates, the LNRF observer is moving with the angular velocity of frame dragging (ω). The third frame is the fluid rest frame (FRF), an orthonormal tetrad basis carried by observers moving along the fluid.

The physical quantities measured in LNRF are described by using the hat, such as $u^{\hat{\mu}}$ and $u_{\hat{\mu}}$, and in FRF by using parentheses, such as $u^{(\mu)}$ and $u_{(\mu)}$. The transformation matrixes of FRF, LNRF, and BLF are given in appendix 2 (Bardeen 1970; Bardeen et al. 1972; Frolov & Novikov 1998).

3. Basic Equations

The basic equations for the relativistic hydrodynamics are the baryon-mass conservation, $(\rho u^\mu)_{;\mu} = 0$, and the energy momentum conservation, $T^{\mu\nu}_{;\nu} = 0$, where ρ is the rest-mass density and $T^{\mu\nu}$ is the energy-momentum tensor. Basic dynamical equations, except for the baryon mass conservation, are calculated from the energy-momentum tensor, $T^{\mu\nu}$. We use the energy-momentum tensor written as

$$T^{\mu\nu} = \rho\eta u^\mu u^\nu + pg^{\mu\nu} + t^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu, \tag{4}$$

where p is the pressure, $\eta = (\rho + u + p)/\rho$ is the relativistic enthalpy, u is the internal energy, $t^{\mu\nu}$ is the viscous stress-energy tensor, and q^μ is the heat-flux four-vector. The relativistic Navier–Stokes shear stress, $t^{\mu\nu}$, is written as (Misner et al. 1973)

$$t^{\mu\nu} = -2\lambda\sigma^{\mu\nu} - \zeta\Theta h^{\mu\nu}, \tag{5}$$

where λ is the coefficient of dynamical viscosity, ζ is the coefficient of bulk viscosity, $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor, $\Theta = u^\gamma_{;\gamma}$ is the expansion of the fluid world line, and $\sigma^{\mu\nu}$ is the shear tensor of the fluid, which is calculated as

$$\sigma_{\mu\nu} = \frac{1}{2}(u_{\mu;\nu} + u_{\nu;\mu} + a_\mu u_\nu + a_\nu u_\mu) - \frac{1}{3}\Theta h_{\mu\nu}, \tag{6}$$

where $a_\mu = u_{\mu;\gamma} u^\gamma$ is the four-acceleration.

We study a stationary, axisymmetric, and equatorially symmetric global accretion flow in the equatorial plane, i.e., we assume $u_\theta = 0$. We also assume that the effects of the bulk viscosity and heat-flux four-vector are negligible. In the following subsections, we derive the basic equations using the vertical averaging procedures around the equatorial plane, which were derived in, e.g., Gammie and Popham (1998).

3.1. Mass Conservation

The equation for the baryon mass conservation is written as

$$(\rho u^\mu)_{;\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\rho u^\mu)_{,\mu} = 0, \tag{7}$$

where u^μ is the four-velocity and $\sqrt{-g} = r^2$. By averaging the physical quantities around the equatorial plane, and assuming a constant $\dot{M}$ (the mass-accretion rate), we can write equation (7) as

$$(4\pi H_\theta r^2 \rho u^r)_{,r} = 0 \Rightarrow -4\pi H_\theta r^2 \rho u^r = \dot{M}, \tag{8}$$

where H_θ is half-thickness of the accretion disk in the θ direction. If we normalize the rest-mass density, ρ, by setting $\dot{M} = 1$, for calculating global structure of accretion flow, we have

$$-4\pi H_\theta \rho u^r r^2 = 1. \tag{9}$$

Also, by differentiating equation (9) we obtain

$$\frac{d\ln H_\theta}{dr} + \frac{2}{r} + \frac{d\ln u^r}{dr} + \frac{d\ln\rho}{dr} = 0. \tag{10}$$

3.2. *Killing Vectors*

Two specific Killing vectors of the Kerr metric are $\varepsilon_t^\mu = (1,0,0,0)$ and $\varepsilon_\phi^\mu = (0,0,0,1)$, which are used to derive the disk equations. At first, angular momentum conservation can be derived by ε_ϕ^μ,

$$(T_\mu^\nu \varepsilon_\phi^\mu)_{;\nu} = 0 \Rightarrow (T_\phi^\nu)_{;\nu} = 0. \tag{11}$$

By vertically averaging equation (11) we have

$$\frac{1}{r^2}(r^2 T_\phi^r)_{,r} = 0 \Rightarrow \dot{M}\eta l - 4\pi H_\theta r^2 t_\phi^r = \dot{M} j, \tag{12}$$

where $\dot{M} j$ is the total inward flux of the angular momentum. This equation is similar to the angular momentum equation of Gammie and Popham (1998).

Another equation that can be obtained by using ε_t^μ is

$$(T_\mu^\nu \varepsilon_t^\mu)_{;\nu} = 0 \Rightarrow (T_t^{\ \nu})_{;\nu} = 0. \tag{13}$$

Similarly, by vertically averaging and inserting a constant $\dot{E}$, we obtain

$$4\pi H_\theta r^2 [(p+\rho+u) u_t u^r + t_t^r] = \dot{E}. \tag{14}$$

This equation expresses the constancy of mass-energy flux, $\dot{E}$, in terms of the radius; $\dot{E}$ is the actual rate of change of the black-hole mass. If the fluid is cold and slow at large radii, then $\dot{E} \approx \dot{M}$ (Gammie & Popham 1998).

3.3. *Energy Equation*

The equation of local energy conservation is obtained from $u_\mu T^{\mu\nu}_{;\nu} = 0$, as follows:

$$-u^r \frac{d(\rho+u)}{dr} - (\rho+u+p)\Theta + (\rho+u+p) u^\nu u^\mu_{;\nu} u_\mu + t^{\mu\nu}_{;\nu} u_\mu = 0, \tag{15}$$

where Θ is defined as

$$\Theta = u^\nu_{;\nu}. \tag{16}$$

Also, from equation (7) we have

$$(\rho u^\nu)_{;\nu} = 0 \Rightarrow \rho_{;\nu} u^\nu + \rho u^\nu_{;\nu} = \frac{d\rho}{dr} u^r + \rho u^\nu_{;\nu} = 0. \tag{17}$$

Therefore, the energy equation can be written as

$$u^r \left(\frac{du}{dr} - \frac{u+p}{\rho} \frac{d\rho}{dr} \right) = (\rho+u+p) a^\mu u_\mu + t^{\mu\nu}_{;\nu} u_\mu. \tag{18}$$

In this paper, because we do not derive the temperature, pressure, and internal energy, we will not use this equation. If we want to derive these variables we must use a state equation as well as the relation of the shear tensor components.

4. Shear Tensor

Lasota (1994) used the $r\phi$ component of the shear stress tensor as follows:

$$t_\phi^r = -\nu\rho \frac{A^{3/2}\Delta^{1/2}\gamma_\phi^3}{r^5} \frac{d\Omega}{dr}, \tag{19}$$

$$\gamma_\phi \equiv [1 - (v^{\hat{\phi}})^2]^{-1/2}, \tag{20}$$

where $v^{\hat{\phi}}$ is the azimuthal component of velocity in the LNRF, which is introduced by Manmoto (2000),

$$v^{\hat{\phi}} = \frac{A}{r^2 \Delta^{1/2}} \tilde{\Omega}, \tag{21}$$

where $\tilde{\Omega}$ is defined as

$$\tilde{\Omega} = \Omega - \omega. \tag{22}$$

Abramowicz et al. (1996, 1997), Manmoto (2000), and others also used equation (19) for the stress tensor. In 1994, Papaloizou and Szuszkiewicz introduced a non-relativistic causal viscosity, which was used in relativistic form by Peitz and Appl (1997, 1998), Gammie and Popham (1998), and Popham and Gammie (1998) in Kerr metric and Takahashi (2007b) in Kerr–Schild metric. They assumed that in the FRF all components vanish, except $t_{r\phi} = t_{\phi r}$. The equation $t_{r\phi} = S$ can be introduced in relativistic form as

$$\frac{DS}{D\tau} = -\frac{S - S_0}{\tau_r}, \tag{23}$$

where $D/D\tau = u^\mu(\ )_{;\mu}$ and S_0 is the equilibrium value of the stress tensor and τ_r is the relaxation time scale. Therefore, in steady state we have

$$u^r \frac{dS}{dr} = -\frac{S - S_0}{\tau_r}. \tag{24}$$

We use a new method to derive the shear tensor approximately. We calculate $v^{\hat{\phi}}$ from equation (21) in LNRF, assuming $u^r = 0$ (the azimuthal velocity is much greater than accretion velocity or radial velocity, except very close to the inner edge, and for fluids with a small viscosity). In order to simplify, we assume $\Omega = \Omega_\mathrm{k}$, where the Keplerian angular velocity, Ω_k, is defined as

$$\Omega_\mathrm{k}^\pm = \pm \frac{M}{r^{3/2} \pm a M^{1/2}}. \tag{25}$$

Therefore, for the azimuthal velocity we have

$$v^{\hat{\phi}} = \frac{A}{r^2 \Delta^{1/2}} \left(\frac{\pm 1}{r^{3/2} \pm a} - \frac{2ar}{A} \right). \tag{26}$$

We first calculate $u^{\hat{t}} = \hat{\gamma} = \sqrt{\dfrac{1}{1 - (v^{\hat{\phi}})^2}}$.

We can calculate the four-velocity in LNRF (appendix 3). The four-velocity for Ω^- (direction of fluid rotation is opposite to the black hole rotation) is

$$u_{\hat{\mu}} = \left[\frac{-r\sqrt{\Delta}(r^{\frac{3}{2}} - a)}{\sqrt{r^4 (r^{\frac{3}{2}} - a)^2 \Delta - A^2 - 4a^2 r^2 (r^{\frac{3}{2}} - a)^2 - 4arA(r^{\frac{3}{2}} - a)}},\ 0, 0,\ \frac{-(r^3 + ra^2 + 2ar^{\frac{3}{2}})}{\sqrt{r^4 (r^{\frac{3}{2}} - a)^2 \Delta - A^2 - 4a^2 r^2 (r^{\frac{3}{2}} - a)^{-} 4arA(r^{\frac{3}{2}} - a)}} \right]. \tag{27}$$

Calculations show that in LNRF $\Gamma^{\hat{\alpha}}_{\hat{\mu}\hat{\nu}} = 0$ and $\Theta = 0$. The shear tensor can be calculated by inserting the four-velocity of LNRF in equation (46). The nonzero components of $u^-_{\hat{\mu};\hat{\nu}}$, $a^-_{\hat{\mu}}(a_{\hat{\mu}} = u_{\hat{\mu};\hat{\gamma}}u^{\hat{\gamma}})$, and $\sigma_{\hat{\mu}\hat{\nu}}$ for Ω^- in LNRF are:

$$u^-_{\hat{t};\hat{r}} = \frac{1}{2\sqrt{\Delta}B^-}(+24r^6a^3 + 12ar^8 - 36r^5a^3 - 18ar^7 +12a^5r^4 + 6a^5r^3 + r^{\frac{19}{2}} + 3a^6r^{\frac{7}{2}} + 2a^6r^{\frac{5}{2}} - 36r^{\frac{11}{2}}a^2 +5r^{\frac{15}{2}}a^2 + 10r^{\frac{13}{2}}a^2 + 7r^{\frac{11}{2}}a^4 + 4r^{\frac{9}{2}}a^4 + 4r^{\frac{7}{2}}a^4),$$
$$u^-_{\hat{\phi};\hat{r}} = \frac{1}{2\sqrt{\Delta}B^-}(+2r^{\frac{11}{2}}a^3 + 9ar^{\frac{15}{2}} + 12r^{\frac{9}{2}}a^3 - 18ar^{\frac{13}{2}} -3a^5r^{\frac{7}{2}} - 2a^5r^{\frac{5}{2}} + r^9 + 18r^5a^2 + 4r^7a^2 - 20r^6a^2 +3r^5a^4 - 4r^4a^4 - 2r^3a^4), \tag{28}$$

$$a^-_{\hat{t}} = u^-_{\hat{t};\hat{r}}u^{\hat{r}} = 0, \qquad a^-_{\hat{r}} = u^-_{\hat{r};\hat{t}}u^{\hat{t}} + u^-_{\hat{r};\hat{\phi}}u^{\hat{\phi}} = 0,$$
$$a^-_{\hat{\phi}} = u^-_{\hat{\phi};\hat{r}}u^{\hat{r}} = 0, \tag{29}$$

$$\sigma^-_{\hat{t}\hat{r}} = \sigma^-_{\hat{r}\hat{t}} = \frac{1}{4\sqrt{\Delta}B^-}(+24r^6a^3 + 12ar^8 - 36r^5a^3 - 18ar^7 +12a^5r^4 + 6a^5r^3 + r^{\frac{19}{2}} + 3a^6r^{\frac{7}{2}} + 2a^6r^{\frac{5}{2}} - 36r^{\frac{11}{2}}a^2 +5r^{\frac{15}{2}}a^2 + 10r^{\frac{13}{2}}a^2 + 7r^{\frac{11}{2}}a^4 + 4r^{\frac{9}{2}}a^4 + 4r^{\frac{7}{2}}a^4),$$
$$\sigma^-_{\hat{r}\hat{\phi}} = \sigma^-_{\hat{\phi}\hat{r}} = \frac{1}{4\sqrt{\Delta}B^-}(+2r^{\frac{11}{2}}a^3 + 9ar^{\frac{15}{2}} + 12r^{\frac{9}{2}}a^3 + r^9 -18ar^{\frac{13}{2}} - 3a^5r^{\frac{7}{2}} - 2a^5r^{\frac{5}{2}} + 18r^5a^2 + 4r^7a^2 - 20r^6a^2 +3r^5a^4 - 4r^4a^4 - 2r^3a^4), \tag{30}$$

where $B^- = (r^7 - 2r^{\frac{11}{2}}a - r^4a^2 - 3r^6 - 6r^3a^2 + r^5a^2 - 2r^{\frac{7}{2}}a^3 - 4a^3r^{\frac{5}{2}})^{\frac{3}{2}}$. Also, for Ω^+, following appendix 3, the four-velocity is

$$u_{\hat{\mu}} = \left[\frac{-r\sqrt{\Delta}(r^{\frac{3}{2}}+a)}{\sqrt{r^4(r^{\frac{3}{2}}+a)^2\Delta - A^2 - 4a^2r^2(r^{\frac{3}{2}}+a)^2 + 4arA(r^{\frac{3}{2}}+a)}},\ 0, 0,\ \frac{r^3 + ra^2 - 2ar^{\frac{3}{2}}}{\sqrt{r^4(r^{\frac{3}{2}}+a)^2\Delta - A^2 - 4a^2r^2(r^{\frac{3}{2}}+a)^2 + 4arA(r^{\frac{3}{2}}+a)}}\right]. \tag{31}$$

Therefore, the nonzero components of $u_{\hat{\mu};\hat{\nu}}$ and $\sigma_{\hat{\mu}\hat{\nu}}$ for Ω^+ are (similar to Ω^-, $a_{\hat{\mu}} = 0$):

$$u^+_{\hat{t};\hat{r}} = \frac{1}{2\sqrt{\Delta}B^+}(-24r^6a^3 - 12ar^8 + 36r^5a^3 + 18ar^7 -12a^5r^4 - 6a^5r^3 + r^{\frac{19}{2}} + 3a^6r^{\frac{7}{2}} + 2a^6r^{\frac{5}{2}} - 36r^{\frac{11}{2}}a^2 +5r^{\frac{15}{2}}a^2 + 10r^{\frac{13}{2}}a^2 + 7r^{\frac{11}{2}}a^4 + 4r^{\frac{9}{2}}a^4 + 4r^{\frac{7}{2}}a^4),$$
$$u^+_{\hat{\phi};\hat{r}} = \frac{-1}{2\sqrt{\Delta}B^+}(-2r^{\frac{11}{2}}a^3 - 9ar^{\frac{15}{2}} - 12r^{\frac{9}{2}}a^3 + 18ar^{\frac{13}{2}} +3a^5r^{\frac{7}{2}} + 2a^5r^{\frac{5}{2}} + r^9 + 18r^5a^2 + 4r^7a^2 - 20r^6a^2 +3r^5a^4 - 4r^4a^4 - 2r^3a^4), \tag{32}$$

$$\sigma^+_{\hat{t}\hat{r}} = \sigma^+_{\hat{r}\hat{t}} = \frac{1}{4\sqrt{\Delta}B^+}(-24r^6a^3 - 12ar^8 + 36r^5a^3 + 18ar^7 -12a^5r^4 - 6a^5r^3 + r^{\frac{19}{2}} + 3a^6r^{\frac{7}{2}} + 2a^6r^{\frac{5}{2}} - 36r^{\frac{11}{2}}a^2 +5r^{\frac{15}{2}}a^2 + 10r^{\frac{13}{2}}a^2 + 7r^{\frac{11}{2}}a^4 + 4r^{\frac{9}{2}}a^4 + 4r^{\frac{7}{2}}a^4),$$
$$\sigma^+_{\hat{r}\hat{\phi}} = \sigma^+_{\hat{\phi}\hat{r}} = \frac{-1}{4\sqrt{\Delta}B^+}(-2r^{\frac{11}{2}}a^3 - 9ar^{\frac{15}{2}} - 12r^{\frac{9}{2}}a^3 + r^9 +18ar^{\frac{13}{2}} + 3a^5r^{\frac{7}{2}} + 2a^5r^{\frac{5}{2}} + 18r^5a^2 + 4r^7a^2 -20r^6a^2 + 3r^5a^4 - 4r^4a^4 - 2r^3a^4), \tag{33}$$

where $B^+ = (r^7 + 2r^{\frac{11}{2}}a - r^4a^2 - 3r^6 - 6r^3a^2 + r^5a^2 + 2r^{\frac{7}{2}}a^3 + 4a^3r^{\frac{5}{2}})^{\frac{3}{2}}$. The shear tensor in BLF can be derived from $\sigma_{\alpha\beta} = e^{\hat{\mu}}_{\alpha}e^{\hat{\nu}}_{\beta}\sigma_{\hat{\mu}\hat{\nu}}$ in which $e^{\hat{\mu}}_{\alpha}$ and $e^{\hat{\nu}}_{\beta}$ are the transformation tensors (appendix 2). There are two nonzero components of shear tensor in BLF, σ_{tr} and $\sigma_{r\phi}$, which are given in appendix 4. Also using transformation matrices, two nonzero components of the shear tensor in FRF can be calculated including $\sigma_{(t)(r)}$ and $\sigma_{(r)(\phi)}$, which are given in appendix 4 as well.

5. Deriving Four-Velocity

Assuming special values for λ (for example in Takahashi 2007b, λ = 1.5, 1.7, 2.1, 2.2, 2.3, 2.4, and 2.5), we can solve equations (10), (12), (14), and (18) to derive the four-velocity, density, pressure, etc, numerically. In numerical solutions we must insert the values of physical variables on boundary conditions, especially on the horizon. Some of these boundary conditions are non-physical; therefore, we want to derive analytical solutions without any boundary conditions. We use some non-relativistic relations to calculate the four-velocity and the density in LNRF and BLF analytically. At first, the $t_{r\phi}$ relation of the stress tensor in Takahashi (2007a) is used in a relativistic disk. This relation is

$$t_{r\phi} = -\nu\rho r^2\frac{d\Omega}{dr}. \tag{34}$$

Similar to Abramowicz et al.(1996), we assume $\eta = 1$; therefore, from equation (5) by zero bulk viscosity and $\lambda = \rho\eta\nu$, we have

$$\sigma^{\pm}_{r\phi} = \frac{r^2}{2}\frac{d\Omega}{dr}. \tag{35}$$

Therefore, $\Omega^{\pm}$ is

$$\Omega^{\pm} = \int \frac{2}{r^2}\sigma^{\pm}_{r\phi}dr. \tag{36}$$

For calculating $\Omega^{\pm}$ we use equations (30) and (33) with a simplification in $1/B^+$ and $1/B^-$ as:

$$\frac{1}{B^+} = (r^7 + 2r^{\frac{11}{2}}a - r^4a^2 - 3r^6 - 6r^3a^2 + r^5a^2 +2r^{\frac{7}{2}}a^3 + 4a^3r^{\frac{5}{2}})^{-\frac{3}{2}} \approx \frac{1}{\sqrt{r^{21}}}\left[1 + \frac{3}{2r^7}(3r^6 - 2ar^{\frac{11}{2}} - r^5a^2 + r^4a^2 + 6r^3a^2 -r^5a^2 - 2r^{\frac{7}{2}}a^3 - 4a^3r^{\frac{5}{2}}) + \frac{15}{8r^{14}}(3r^6 + 2r^{\frac{11}{2}}a + r^4a^2 + 6r^3a^2 - r^5a^2 + 2r^{\frac{7}{2}}a^3 + 4a^3r^{\frac{5}{2}})^2\right], \tag{37}$$

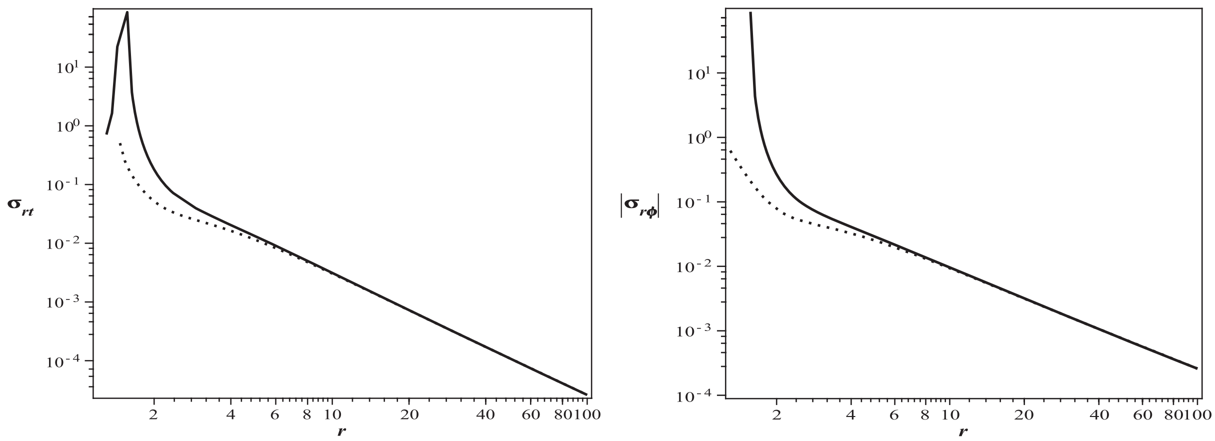

Fig. 1. Influence of simplifying in shear tensor components for Ω^+ and $a = 0.9$. Solid curves are without simplification and the dotted curves are with simplification.

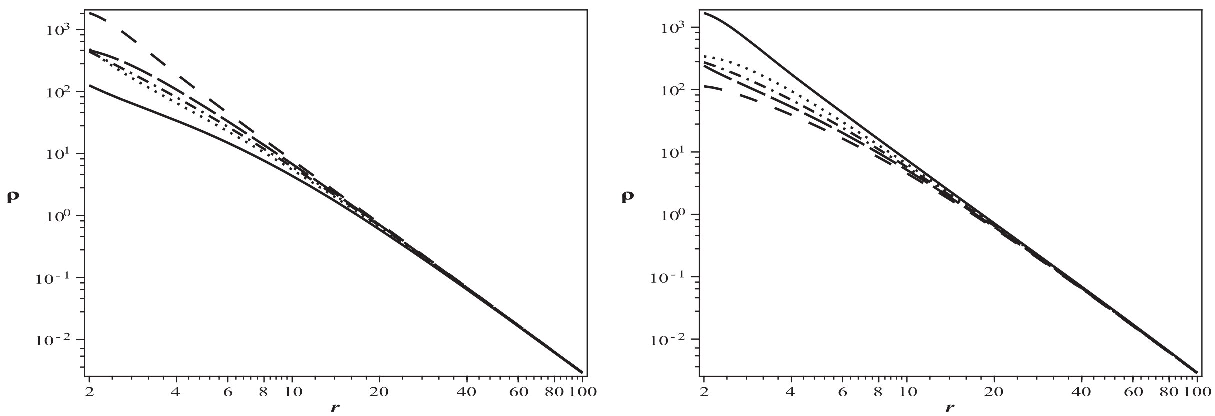

Fig. 2. Surface density, left: Ω^-, right: Ω^+ ($\alpha = 0.01$, $j = 3$, solid: $a = 0.9$, dotted: $a = 0.4$, dash-dotted: $a = 0$, long-dashed: $a = -0.4$, and dash-spaced: $a = -0.9$).

$$\frac{1}{B^-} = (r^7 - 2r^{\frac{11}{2}}a - r^4a^2 - 3r^6 - 6r^3a^2 + r^5a^2 - 2r^{\frac{7}{2}}a^3 - 4a^3r^{\frac{5}{2}})^{-\frac{3}{2}}$$
$$\approx \frac{1}{\sqrt{r^{21}}}\left[1 + \frac{3}{2r^7}(3r^6 + 2ar^{\frac{11}{2}} - r^5a^2 + r^4a^2 + 6r^3a^2 - r^5a^2 + 2r^{\frac{7}{2}}a^3 + 4a^3r^{\frac{5}{2}}) + \frac{15}{8r^{14}}(3r^6 - 2r^{\frac{11}{2}}a + r^4a^2 + 6r^3a^2 - r^5a^2 - 2r^{\frac{7}{2}}a^3 - 4a^3r^{\frac{5}{2}})^2\right]. \quad (38)$$

The influences of this approximation in the shear tensor components are seen in figure 1 for Ω^+ and $a = 0.9$. The solid curves have no simplifying, and the dotted curves are with this simplifying.

Setting $l = \Omega r^2$ (Takahashi 2007a), the angular momentum can be calculated. From equations (36) and (12), ρ can be derived as

$$\rho = \frac{j - l}{8\pi H_\theta r^2 \nu \sigma_{\phi}^{\pm r}}, \quad (39)$$

where ν is (Takahashi 2007a)

$$\nu = \alpha a_{\mathrm{s}}^2/\Omega_{\mathrm{k}}, \quad (40)$$

which is the usual α prescription for the viscosity (Shakura & Sunyaev 1973). In some papers H_θ is calculated by studying the vertical structure, such as done by Riffert and Herold (1995), Abramowicz et al. (1997), and Takahashi (2007b). In these papers H_θ was calculated by vertical averaging, then introduced by other variables, such as l, u_t, p, ρ, etc. If we use each of those relations of H_θ, we must solve all equations numerically. However, we want to derive an analytical solution. Therefore, similar to Takahashi (2007a), we use a simple form of H_θ,

$$H_\theta = \sqrt{\frac{5}{2}}\frac{a_{\mathrm{s}}}{\Omega_{\mathrm{k}}}. \quad (41)$$

Definitely, ρ is the same in LNRF and BLF, and is shown in figure 2 for (Ω^-) and (Ω^+) by assuming $a_{\mathrm{s}} = 0.1$.

Also, $u^{\hat{r}}$ can be calculated from equation (9) and $u_{\hat{t}}$ can be found from equation (14). Assuming $\dot{E} \simeq \dot{M} = 1$, we have

$$u^{\hat{r}} = \frac{1}{4\pi H_\theta r^2 \rho}, \qquad u_{\hat{t}} = -\dot{E} - \frac{t_{\hat{t}}^{\hat{r}}}{\rho u^{\hat{r}}}. \quad (42)$$

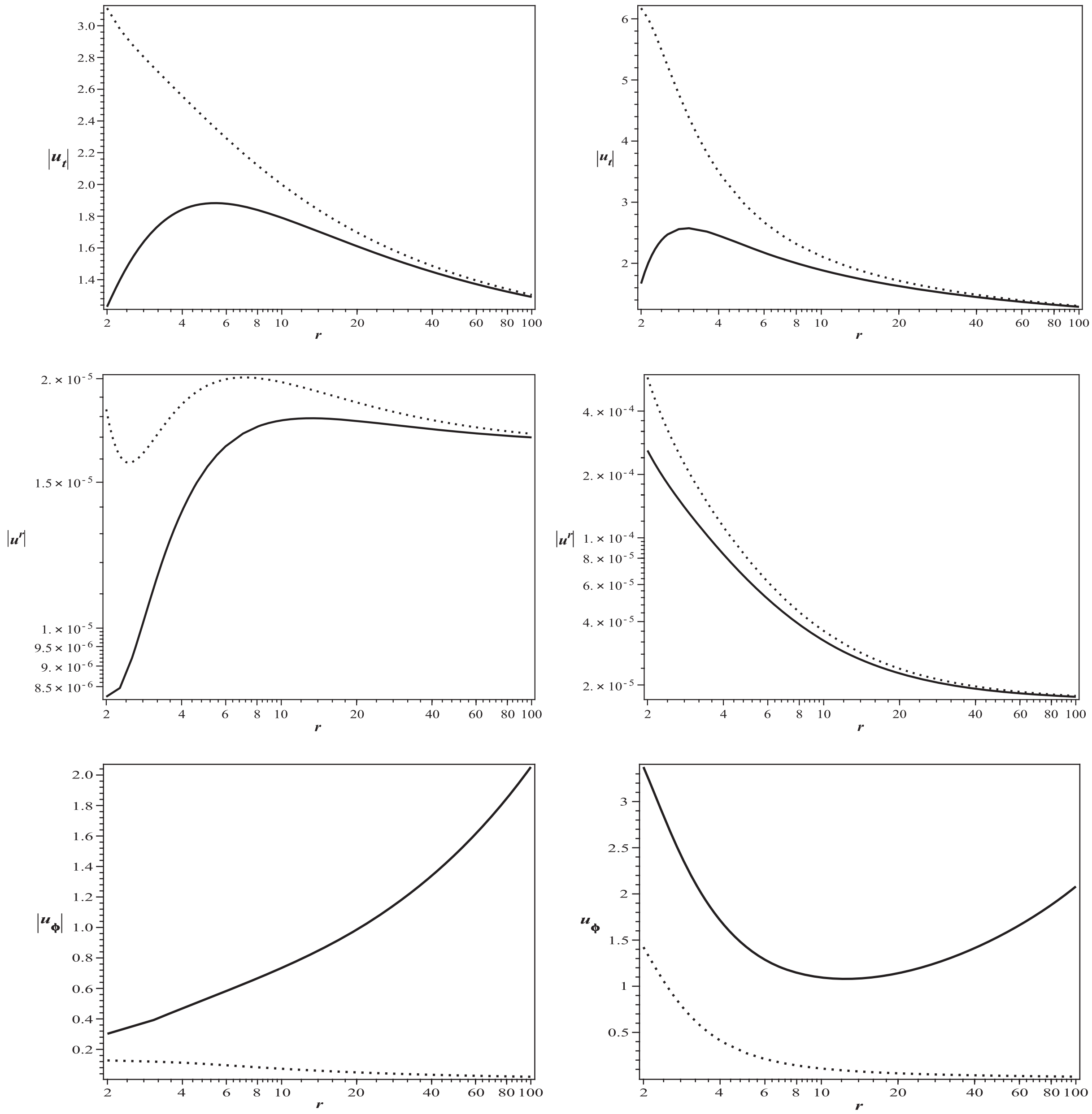

Fig. 3. Four-velocity in LNRF and BLF. Left: Ω^-, right: Ω^+ ($\alpha = 0.01$, $j = 3$, $a = -0.9$, and $a_s = 0.1$, solid: in BLF, and dotted: in LNRF).

We can calculate the four-velocity in BLF using the transformation equations in appendix 2, follows:

$$u_t = e_t^{\hat{\mu}} u_{\hat{\mu}} = \sqrt{\frac{\Delta\Sigma}{A}} u_{\hat{t}} - \frac{2ar}{\sqrt{A\Sigma}} u_{\hat{\phi}}, \tag{43}$$

$$u_r = e_r^{\hat{\mu}} u_{\hat{\mu}} = \sqrt{\frac{\Sigma}{\Delta}} u_{\hat{r}}, \tag{44}$$

$$u_\phi = e_\phi^{\hat{\mu}} u_{\hat{\mu}} = \sqrt{\frac{A}{\Sigma}} u_{\hat{\phi}}. \tag{45}$$

Now, we may see ($|u_t|$, $|u_{\hat{t}}|$), ($|u_r|$, $|u_{\hat{r}}|$), and ($|u_\phi|$,$|u_{\hat{\phi}}|$) in figure 3.

The four-velocity and $|\Omega|$ in BLF($|u_\theta = 0|$) are shown in figure 4, setting $a_s = 0.1$. In equation (40), $0 < \alpha < 1$ is the viscosity coefficient in the α prescription, which influences the redistribution of energy and the momentum. The influence of these parameters are shown in figure 5 with $a_s = 0.1$. The four-velocity and the density can be derived for various total inward flux of angular momentum (for example $j = 1$, 2, and 3), which are shown in figure 6.

5.1. *Influence of rt Component on the Four-Velocity*

In previous papers, such as Abramowicz et al. (1996, 1997), Gammie and Popham (1998), Manmoto (2000), and Takahashi (2007a, 2007b), only the $r\phi$ component of the shear tensor was assumed to be nonzero in FRF, but in section 4 we showed that the rt component is also nonzero. The rt component of the shear tensor in LNRF results from the covariant derivative

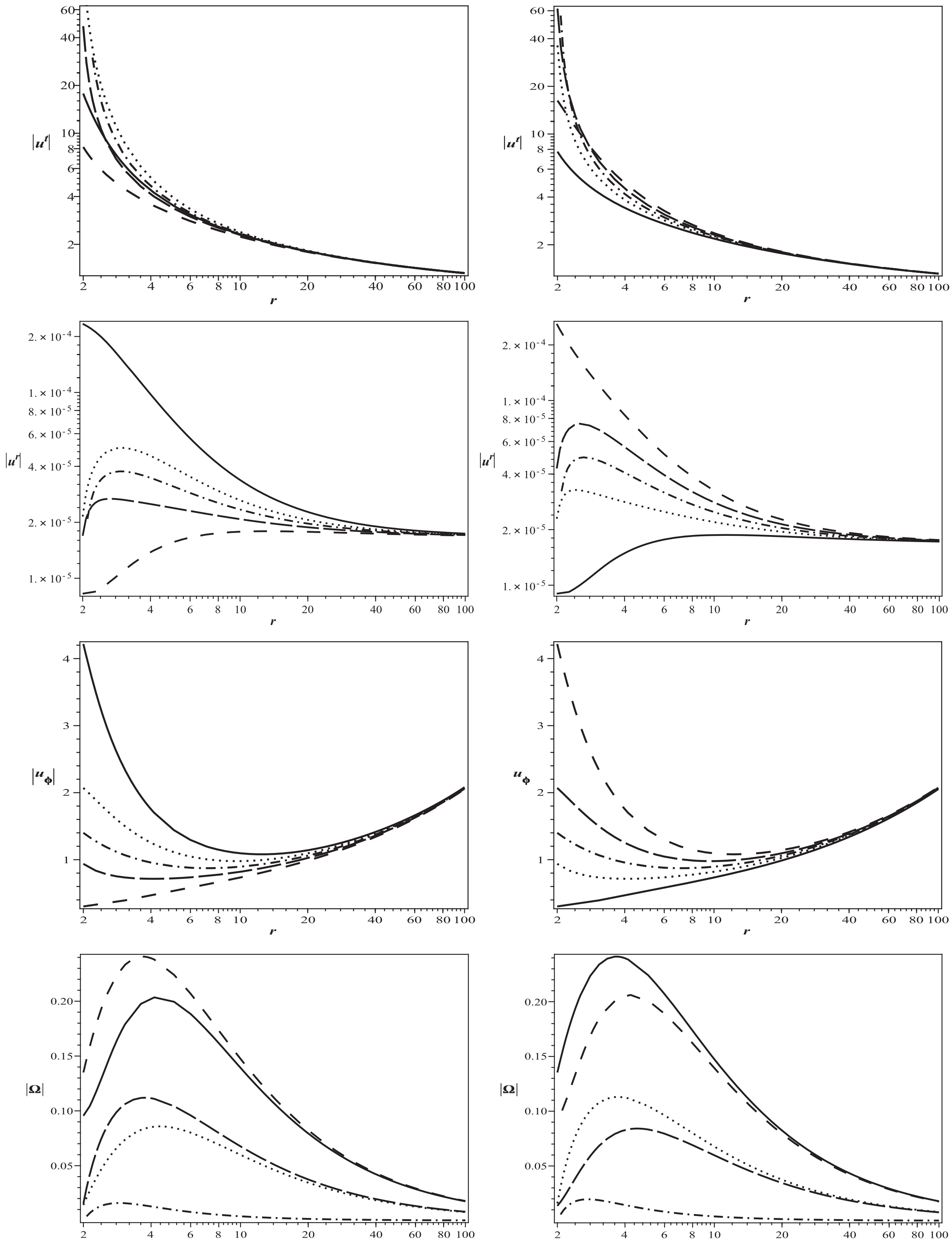

Fig. 4. Four-velocity and $|\Omega|$ in the BLF. Left: Ω^-, right: Ω^+($\alpha = 0.01$, $j = 3$, solid: $a = 0.9$, dotted: $a = 0.4$, dash-dotted: $a = 0$, long-dashed: $a = -0.4$, and dash-spaced: $a = -0.9$).

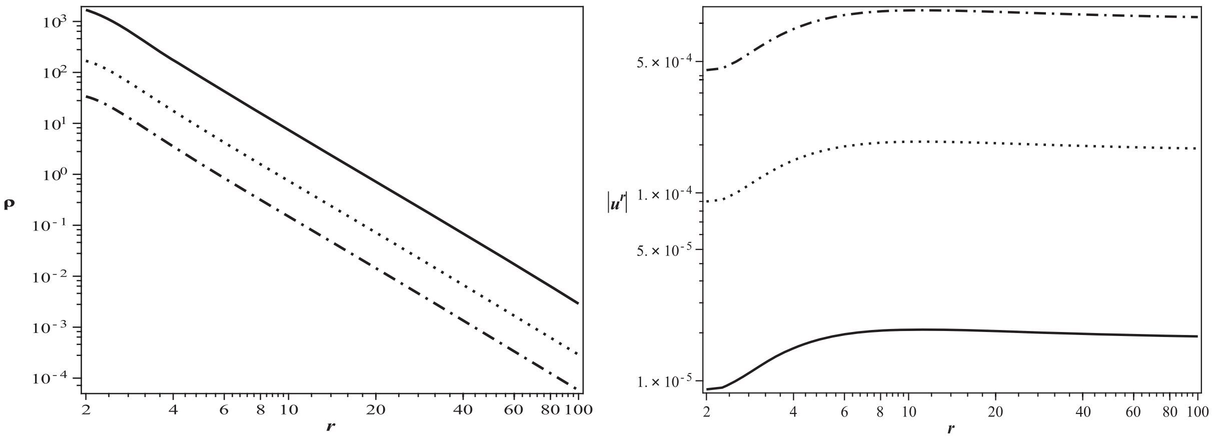

Fig. 5. Influence of α on the density and the radial four-velocity for $\Omega^+(a = 0.9$, $j = 3$, and $a_s = 0.1$. Solid: $\alpha = 0.01$, dotted: $\alpha = 0.1$, and dash-dotted: $\alpha = 0.5$).

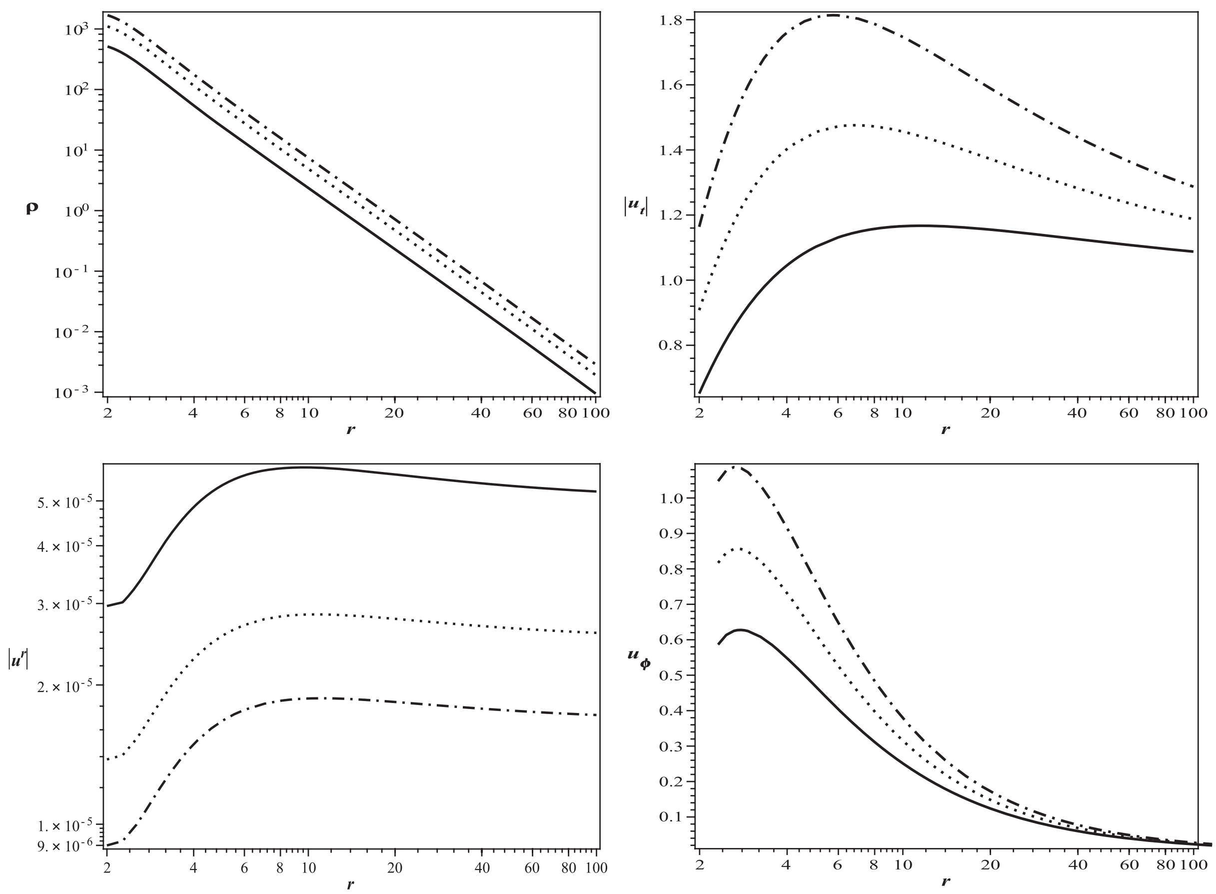

Fig. 6. Influence of j in density and the four-velocity of Ω^+ in BLF ($\alpha = 0.01$, $a = 0.9$, and $a_s = 0.1$. Solid: $j = 1$, dotted: $j = 2$, and dash-spaced: $j = 3$).

of $u_{\hat{t}}$. In general relativity, the gravitating field causes time dilation. Due to this time dilation, the coordinate time (t) and proper time (τ) are not the same ($dt > d\tau$); therefor, $u^{\hat{t}}$ in LNRF can be derived as

$$u^{\hat{t}} = \frac{d\hat{t}}{d\tau}. \tag{46}$$

This $u^{\hat{t}} = |u_{\hat{t}}|$ is different in each radius, and also the tangent of $u_{\hat{t}}$ creates the rt component of the shear tensor. To see the influence of this component, we derive the four-velocity with both the rt component of the shear tensor and without it. If we set $\sigma^r_t = 0$ in the previous section, the figures are similar to that of the previous papers. From equations (39) and (42) it is clear that the rt component affects u_t in all frames, because

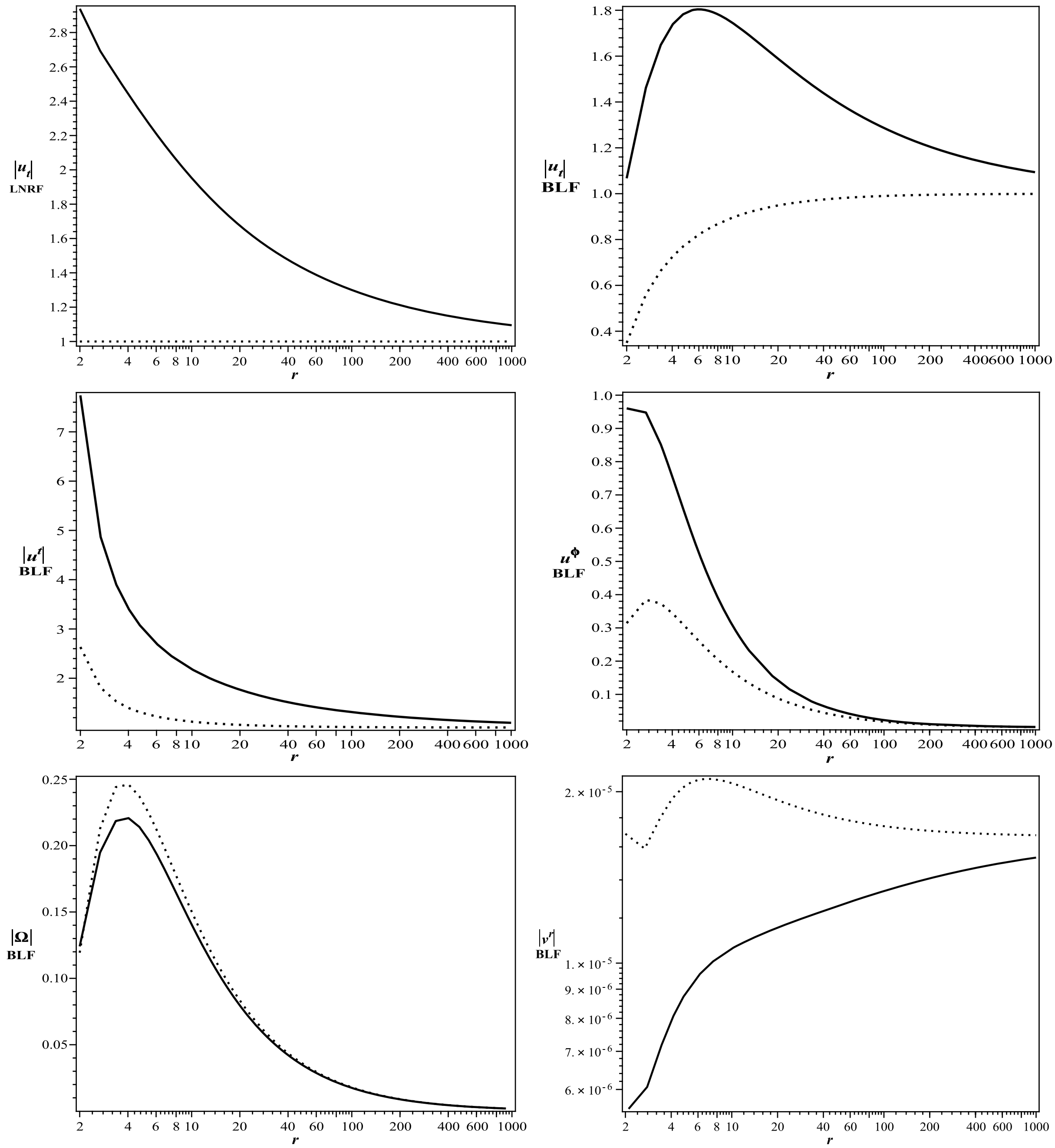

Fig. 7. Influence of the rt component of the shear tensor on $|u_{\hat{t}}|$ (in LNRF) and $|u_t|$, $|u^t|$, u^ϕ (in BLF), $|\Omega|$ and $|v^r|$ for Ω^+. The solid with the rt component and dotted without it ($\alpha = 0.01$, $a = 0.9$, $a_s = 0.1$, and $j = 3$).

without this component, the second term of u_t in equation (42) vanishes. If we calculate $u^\mu = g^{\mu\nu}u_\nu$, the rt component also affects u^ϕ in BLF. The influences of the rt component of the shear tensor are shown in figure 7. The solid curves are with this component, and dotted curves are without it. Obviously, the rt component of the shear tensor has greater influence on u_t and u^t. Therefore, for deriving u_t and u^t we can not eliminate the rt component.

6. Summery and Conclusion

In the causal viscosity method, only the $r\phi$ component of the shear tensor is assumed to be nonzero in FRF. We do not use this method for viscosity, because our calculations show that there are two nonzero components of the shear tensor in FRF. In our method, we use the azimuthal velocity of LNRF and the Keplerian angular velocity; we then calculate all components of the shear tensor in LNRF for two kinds of fluids [rotation in the direction of black hole (Ω^+) and rotation in the opposite direction of black hole (Ω^-)]. Using the transformation tensor we can calculate all components of the shear tensor in BLF and FRF.

The solid curves of figure 1 show the non-physical treatments in the shear tensor components close to the inner edge. This non-physical treatment in $\sigma_{\hat{t}\hat{r}}$ may be concerned to assuming $u^{\hat{r}} \ll u^{\hat{\phi}}$; therefore, if we find a suitable $v^{\hat{r}}$, this treatment may be resolved. We may have a suitable $v^{\hat{r}}$ if we put $u^{\hat{t}} = \hat{\gamma} = \sqrt{\frac{1}{1-(v^{\hat{\phi}})^2-(v^{\hat{r}})^2}} > 0 \Rightarrow 1 - (v^{\hat{\phi}})^2 > (v^{\hat{r}})^2$. According to equation (21), the most important origin of non-physical treatments in $\sigma_{\hat{r}\hat{\phi}}$ is the $1/\Delta$ term ($1/\Delta$ has a singularity in the inner edge). Therefore, by using the Taylor expansion, the non-physical treatments of $\sigma_{\hat{t}\hat{r}}$ and $\sigma_{\hat{r}\hat{\phi}}$ were solved, as can be seen by the dotted curves of figure 1.

We solved the hydrodynamical equations in LNRF with assuming $\eta = 1$; we derived the density and four-velocity in BLF and LNRF. The density that is calculated analytically in LNRF is the same in all frames, such as BLF. However, a comparison of four-velocity in LNRF and BLF (figure 3) shows that u_r and u_t have a small difference in the two frames. In equation (44), $\sqrt{\Sigma/\Delta} \approx \sqrt{r^2/(r^2-2r+a^2)}$ is near 1, especially at larger radii; therefore u_r is similar in the two frames. Following equation (43), for u_t we have two parts: the first part is ($\sqrt{\Delta\Sigma/A} \approx \sqrt{(r-2)/r}$) $u_{\hat{t}}$ and the second part is ($-2ar/\sqrt{A\Sigma} \approx -2a/r^2$) $u_{\hat{\phi}}$. Because value of $u_{\hat{\phi}}$ is greater than $u_{\hat{t}}$ (dotted curves in figure 3), the first term has more influence on u_t than the second term (the second term is important just in inner radii).

Note that $u_{\hat{\phi}}$ and u_ϕ are different in the two frames, because in equation (45), $\sqrt{A/\Sigma} \approx r$. The previous equation is logical because the LNRF rotates with the angular velocity of the frame dragging (the frame dragging velocity can not be seen in LNRF but it can be seen in BLF); therefore u_ϕ is greater than $u_{\hat{\phi}}$ at all radii.

In section 5, the four-velocity was calculated analytically in LNRF and BLF. The sign of Ω introduces two types of fluids, Ω^+ and Ω^-, but it can be seen that for example, in figures 2 or 4, in $a = 0.9$, Ω^+ is similar to Ω^- in $a = -0.9$ (and in $a = 0.4$ Ω^+ is similar to Ω^- in $a = -0.4$ and so on) therefore we use (Ω^+) in figures 5, 6, and 7.

The effects of the α coefficient are shown in figure 5. It shows that if α increases, then the shear viscosity grows, the radial four-velocity will increase, and then the density decreases.

The rt component of the shear tensor in LNRF, which results from the relativistic calculations [equation (46)], is equal to the covariant derivative of $u_{\hat{t}}$. If we ignore the rt component of the shear tensor (the dotted curves in figure 7), the figures are similar to the figures of previous work, such as Gammie and Popham (1998) and Takahashi (2007b). However, with rt component of the shear tensor (the solid curves in figure 7) a few number of four-velocities change in all frames; especially $u^{\hat{t}}$, u^t, u^ϕ, and v^r will change more than the others. The influences of the rt component on all physical variables can be derived if we solve all equations of the disk with a state equation numerically.

When we have time dilation, u^t must be greater than $1\left(dt > d\tau \Rightarrow u^t = \frac{dt}{d\tau} > 1\right)$. According to figure 4, the effect of time dilation is much greater at the inner edge than at other places. If we ignore the rt component of the shear tensor, or put $|u_{\hat{t}}| = 1$ (dotted curve in the first panel of first column of figure 7), it is equivalent to ignoring time dilation. Equations (42) and (44) show that the rt component of the shear tensor has no influence on $u^{\hat{r}}$ and u^r. Near to the black hole, u^t increases, which causes a decrease in v^r $\left(= \frac{u^r}{u^t}\right)$.

The energy can be calculated from $T^{tt} = \rho\eta u^t u^t + pg^{tt}$, where the first term of energy is related to u_t. Figure 2 shows that ρ is greater at the inner edge than at other places, and is also the same as the rt component of the shear tensor and without it. The u_t that uses the rt component of the shear tensor is greater than the u_t without it. Therefore, the first term of the energy with the rt component of the shear tensor is greater than the this term of energy without the rt component.

Far from the black hole, the general-relativistic influences are vanished; therefore, in outer edge, the influences of the rt component of the shear tensor are too small (as it is seen in outer radii of figure 7).

If we want to calculate the temperature, pressure, internal energy, cooling and heating or radiation, we must use a suitable state equation with equations (9), (12), (14), and (18) (energy equation), then, to solve the equations numerically suitable boundary conditions are needed.

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Appendix 1. Metric Components

Nonzero components of Kerr metric are given as:

$$g_{tt} = -\alpha^2 + \beta_\phi\beta^\phi = -\left(1 - \frac{2mr}{\Sigma}\right), \qquad g_{rr} = \gamma_{rr} = \frac{\Sigma}{\Delta},$$
$$g_{t\phi} = \beta_\phi = -\frac{2mar\sin^2\theta}{\Sigma}, \qquad g_{\theta\theta} = \gamma_{\theta\theta} = \Sigma,$$
$$g_{\phi\phi} = \gamma_{\phi\phi} = \frac{A\sin^2\theta}{\Sigma}, \tag{A1}$$

and inverse components of metric $g^{\mu\nu}$ are:

$$g^{tt} = -\frac{1}{\alpha^2} = -\frac{A}{\Sigma\Delta}, \qquad g^{t\phi} = \frac{\beta^\phi}{\alpha^2} = -\frac{2mar}{\Sigma\Delta},$$
$$g^{rr} = \gamma^{rr} = \frac{\Delta}{\Sigma}, \qquad g^{\theta\theta} = \gamma^{\theta\theta} = \frac{1}{\Sigma},$$
$$g^{\phi\phi} = \gamma^{\phi\phi} - \frac{(\beta^\phi)^2}{\alpha^2} = \frac{1}{\Delta\sin^2\theta}\left(1 - \frac{2mr}{\Sigma}\right). \tag{A2}$$

Appendix 2. Transformation between BLF, LNRF, and FRF

Components of $e^{\hat{\nu}}_{\mu}$ connecting between BLF (Boyer–Lindquist Frame) and LNRF (Locally Non-Rotating Frame) are calculated as:

$$\begin{bmatrix} e_t^{\hat{t}} & e_t^{\hat{r}} & e_t^{\hat{\theta}} & e_t^{\hat{\phi}} \\ e_r^{\hat{t}} & e_r^{\hat{r}} & e_r^{\hat{\theta}} & e_r^{\hat{\phi}} \\ e_\theta^{\hat{t}} & e_\theta^{\hat{r}} & e_\theta^{\hat{\theta}} & e_\theta^{\hat{\phi}} \\ e_\phi^{\hat{t}} & e_\phi^{\hat{r}} & e_\phi^{\hat{\theta}} & e_\phi^{\hat{\phi}} \end{bmatrix} = \begin{bmatrix} \left(\frac{\Sigma\Delta}{A}\right)^{\frac{1}{2}} & 0 & 0 & -\frac{2Mar\sin\theta}{(\Sigma A)^{\frac{1}{2}}} \\ 0 & \left(\frac{\Sigma}{\Delta}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \Sigma^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \left(\frac{A}{\Sigma}\right)^{\frac{1}{2}}\sin\theta \end{bmatrix}, \quad \text{(A3)}$$

$$\begin{bmatrix} e_{\hat{t}}^{t} & e_{\hat{r}}^{t} & e_{\hat{\theta}}^{t} & e_{\hat{\phi}}^{t} \\ e_{\hat{t}}^{r} & e_{\hat{r}}^{r} & e_{\hat{\theta}}^{r} & e_{\hat{\phi}}^{r} \\ e_{\hat{t}}^{\theta} & e_{\hat{r}}^{\theta} & e_{\hat{\theta}}^{\theta} & e_{\hat{\phi}}^{\theta} \\ e_{\hat{t}}^{\phi} & e_{\hat{r}}^{\phi} & e_{\hat{\theta}}^{\phi} & e_{\hat{\phi}}^{\phi} \end{bmatrix} = \begin{bmatrix} \left(\frac{A}{\Sigma\Delta}\right)^{\frac{1}{2}} & 0 & 0 & \frac{2Mar}{(\Sigma A\Delta)^{\frac{1}{2}}} \\ 0 & \left(\frac{\Delta}{\Sigma}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & \left(\frac{\Sigma}{A}\right)^{\frac{1}{2}}\frac{1}{\sin\theta} \end{bmatrix}. \quad \text{(A4)}$$

The transformation between LNRF and FRF (Fluid Rest Frame) are as follows:

$$\begin{bmatrix} e_{(t)}^{\hat{t}} & e_{(r)}^{\hat{t}} & e_{(\theta)}^{\hat{t}} & e_{(\phi)}^{\hat{t}} \\ e_{(t)}^{\hat{r}} & e_{(r)}^{\hat{r}} & e_{(\theta)}^{\hat{r}} & e_{(\phi)}^{\hat{r}} \\ e_{(t)}^{\hat{\theta}} & e_{(r)}^{\hat{\theta}} & e_{(\theta)}^{\hat{\theta}} & e_{(\phi)}^{\hat{\theta}} \\ e_{(t)}^{\hat{\phi}} & e_{(r)}^{\hat{\phi}} & e_{(\theta)}^{\hat{\phi}} & e_{(\phi)}^{\hat{\phi}} \end{bmatrix} = \begin{bmatrix} \hat{\gamma} & \hat{\gamma}v_{\hat{r}} & 0 & \hat{\gamma}v_{\hat{\phi}} \\ \hat{\gamma}v_{\hat{r}} & 1+\frac{\hat{\gamma}^2 v_{\hat{r}}^2}{1+\hat{\gamma}} & 0 & \frac{\hat{\gamma}^2 v_{\hat{r}} v_{\hat{\phi}}}{1+\hat{\gamma}} \\ 0 & 0 & 1 & 0 \\ \hat{\gamma}v_{\hat{\phi}} & \frac{\hat{\gamma}^2 v_{\hat{r}} v_{\hat{\phi}}}{1+\hat{\gamma}} & 0 & 1+\frac{\hat{\gamma}^2 v_{\hat{\phi}}^2}{1+\hat{\gamma}} \end{bmatrix}, \quad \text{(A5)}$$

$$\begin{bmatrix} e_{\hat{t}}^{(t)} & e_{\hat{t}}^{(r)} & e_{\hat{t}}^{(\theta)} & e_{\hat{t}}^{(\phi)} \\ e_{\hat{r}}^{(t)} & e_{\hat{r}}^{(r)} & e_{\hat{r}}^{(\theta)} & e_{\hat{r}}^{(\phi)} \\ e_{\hat{\theta}}^{(t)} & e_{\hat{\theta}}^{(r)} & e_{\hat{\theta}}^{(\theta)} & e_{\hat{\theta}}^{(\phi)} \\ e_{\hat{\phi}}^{(t)} & e_{\hat{\phi}}^{(r)} & e_{\hat{\phi}}^{(\theta)} & e_{\hat{\phi}}^{(\phi)} \end{bmatrix} = \begin{bmatrix} e_{(t)}^{\hat{t}} & -e_{(r)}^{\hat{t}} & -e_{(\theta)}^{\hat{t}} & -e_{(\phi)}^{\hat{t}} \\ -e_{(t)}^{\hat{r}} & e_{(r)}^{\hat{r}} & e_{(\theta)}^{\hat{r}} & e_{(\phi)}^{\hat{r}} \\ -e_{(t)}^{\hat{\theta}} & e_{(r)}^{\hat{\theta}} & e_{(\theta)}^{\hat{\theta}} & e_{(\phi)}^{\hat{\theta}} \\ -e_{(t)}^{\hat{\phi}} & e_{(r)}^{\hat{\phi}} & e_{(\theta)}^{\hat{\phi}} & e_{(\phi)}^{\hat{\phi}} \end{bmatrix}. \quad \text{(A6)}$$

Where we use $u^{\hat{t}} = -u_{\hat{t}} = \alpha u^t$, the Lorentz factor $\hat{\gamma}$ is calculated as

$$\hat{\gamma} \equiv (1-\hat{v}^2)^{-\frac{1}{2}} = \alpha u^t, \qquad (\hat{v^2} = v_{\hat{i}} v^{\hat{i}} = v_{\hat{r}}^2 + v_{\hat{\theta}}^2 + v_{\hat{\phi}}^2). \quad \text{(A7)}$$

In LNRF, three velocity components are calculated as

$$v^{\hat{i}} = \frac{u^{\hat{i}}}{u^{\hat{t}}}, \qquad (i = r, \theta, \phi). \quad \text{(A8)}$$

After some calculation, we have $\left(\Omega = \frac{u^\phi}{u^t}\right)$:

$$v^{\hat{r}} = \frac{A^{\frac{1}{2}}}{\Delta}\frac{u^r}{u^t}, v^{\hat{\theta}} = 0, \qquad v^{\hat{\phi}} = \frac{\sqrt{\gamma_{\phi\phi}}}{\alpha}(\beta^\phi + \Omega). \quad \text{(A9)}$$

Appendix 3. Deriving Four-Velocity in LNRF

From equation (26) we have

$$v^{\hat{\phi}} \pm = \frac{A}{r^2\sqrt{\Delta}}\left(\frac{\pm 1}{r^{3/2} \pm a} - \frac{2ar}{A}\right). \quad \text{(A10)}$$

First, we derive the four-velocity for $\Omega_k + (\Omega^+) = 1/(r^{\frac{3}{2}} + a)$ as

$$(v^{\hat{\phi}})^2 = \frac{A^2 + 4a^2r^2(r^{\frac{3}{2}}+a)^2 - 4arA(r^{\frac{3}{2}}+a)}{r^4(r^{\frac{3}{2}}+a)^2\Delta}. \quad \text{(A11)}$$

In LNRF $\hat{\gamma} = u^{\hat{t}} = \sqrt{\frac{1}{1-\hat{v}^2}}$ and in our study we put $u_\theta = 0$. If we suppose $u_{\hat{\phi}} >> u_{\hat{r}}$ [as it was in previous papers such as Popham and Gammie (1998); Takahashi (2007a, 2007b)], then $\hat{v^2} \approx (v^{\hat{\phi}})^2$ and we have

$$\hat{\gamma} = u^{\hat{t}} = -u_{\hat{t}} = r\sqrt{\Delta}(r^{\frac{3}{2}}+a) \times [r^4(r^{\frac{3}{2}}+a)^2\Delta - A^2 - 4a^2r^2(r^{\frac{3}{2}}+a)^2 + 4arA(r^{\frac{3}{2}}+a)]^{-\frac{1}{2}}. \quad \text{(A12)}$$

Also, we know that $v^{\hat{\phi}} = u^{\hat{\phi}}/u^{\hat{t}}$. Therefore,

$$u^{\hat{\phi}} = u^{\hat{t}} v^{\hat{\phi}} = (r^3 + ra^2 - 2ar^{\frac{3}{2}}) \times [r^4(r^{\frac{3}{2}}+a)^2\Delta - A^2 - 4a^2r^2(r^{\frac{3}{2}}+a)^2 + 4arA(r^{\frac{3}{2}}+a)]^{-\frac{1}{2}}. \quad \text{(A13)}$$

For $\Omega_k^-(\Omega^-) = -1/(r^{\frac{3}{2}} - a)$ we have

$$\hat{\gamma} = u^{\hat{t}} = -u_{\hat{t}} = \frac{1}{\sqrt{1-(v^{\hat{\phi}})^2}} = r\sqrt{\Delta}(r^{\frac{3}{2}}-a) \times [r^4(r^{\frac{3}{2}}-a)^2\Delta - A^2 - 4a^2r^2(r^{\frac{3}{2}}-a)^2 - 4arA(r^{\frac{3}{2}}-a)]^{-\frac{1}{2}}, \quad \text{(A14)}$$

$$u^{\hat{\phi}} = u^{\hat{t}} v^{\hat{\phi}} = (r^3 + ra^2 + 2ar^{\frac{3}{2}}) \times [r^4(r^{\frac{3}{2}}-a)^2\Delta - A^2 - 4a^2r^2(r^{\frac{3}{2}}-a)^2 - 4arA(r^{\frac{3}{2}}-a)]^{-\frac{1}{2}}. \quad \text{(A15)}$$

Appendix 4. Nonzero Components of Shear Tensor in BLF and FRF

Nonzero components of the shear tensor in BLF can be calculated by $\sigma_{\alpha\beta} = e_\alpha^{\hat{\mu}} e_\beta^{\hat{\nu}} \sigma_{\hat{\mu}\hat{\nu}}$, where $e^{\hat{\mu}}$ and $e^{\hat{\nu}}$ are given in appendix 2. Therefore, these components for Ω^+ are:

$$\sigma^+_{tr} = \sigma^+_{rt} = \frac{1}{4B^+\Delta^{\frac{1}{2}}}(6r^{\frac{9}{2}}a^6 + 8r^8a^3 + 36r^{\frac{15}{2}}a^2 + 4r^{\frac{7}{2}}a^6 - 24r^{\frac{11}{2}}a^4 - 4r^4a^5 - 40r^7a^3 - 8r^5a^5 + 2r^{10}a + 6r^6a^5 - 18r^{\frac{17}{2}}a^2 - 4r^{\frac{13}{2}}a^4 + 36r^6a^3) + \frac{1}{4B^+\Delta^{\frac{3}{2}}}(+3a^8r^{\frac{11}{2}} + r^{\frac{27}{2}} + 42r^{11}a - 12a^7r^6 + 2r^{\frac{9}{2}}a^8 + 6r^{\frac{23}{2}}a^2 + 12r^{\frac{19}{2}}a^4 - 12r^{12}a - 36a^5r^8 + 54r^7a^5 - 6r^5a^7 + 12a^5r^6 - 36r^{10}a - 40a^4r^{\frac{15}{2}} - 72r^8a^3 - 36r^{10}a^3 - 2r^{\frac{25}{2}} - 56r^{\frac{19}{2}}a^2 - 8r^{\frac{13}{2}}a^4 + 10r^{\frac{15}{2}}a^6 + 72r^{\frac{17}{2}}a^2 + 102r^9a^3), \quad \text{(A16)}$$

$$\sigma^+_{r\phi} = \sigma^+_{\phi r} = -\frac{\sqrt{A}}{4B^+\sqrt{\Delta}}(3a^5r^{\frac{7}{2}} + 2a^5r^{\frac{5}{2}} + 3a^4r^5 - 9r^{\frac{15}{2}}a + 18r^5a^2 - 12r^{\frac{9}{2}}a^3 + 18r^{\frac{13}{2}}a - 2r^{\frac{11}{2}}a^3 - 2r^3a^4 - 20r^6a^2 - 4r^4a^4 + 4r^7a^2 + r^9). \quad \text{(A17)}$$

Non-zero components of the shear tensor in FRF can be calculated by $\sigma_{(\alpha)(\beta)} = e^{\hat{\mu}}_{(\alpha)} e^{\hat{\nu}}_{(\beta)} \sigma_{\hat{\mu}\hat{\nu}}$, where $e^{\hat{\mu}}_{(\alpha)}$ and $e^{\hat{\nu}}_{(\beta)}$ are transformation matrixes of appendix 2.

$$\sigma_{(\alpha)(\beta)} = \begin{bmatrix} \hat{\gamma} & 0 & 0 & u^{\hat{\phi}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ u^{\hat{\phi}} & 0 & 0 & 1 + \frac{u^2_{\hat{\phi}}}{1+\hat{\gamma}} \end{bmatrix} \times \begin{bmatrix} \hat{\gamma} & 0 & 0 & u_{\hat{\phi}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ u_{\hat{\phi}} & 0 & 0 & 1 + \frac{u^2_{\hat{\phi}}}{1+\hat{\gamma}} \end{bmatrix} \begin{bmatrix} 0 & \sigma_{\hat{r}\hat{t}} & 0 & 0 \\ \sigma_{\hat{r}\hat{t}} & 0 & 0 & \sigma_{\hat{r}\hat{\phi}} \\ 0 & 0 & 0 & 0 \\ 0 & \sigma_{\hat{r}\hat{\phi}} & 0 & 0 \end{bmatrix}. \quad \text{(A18)}$$

Also, we have:

$$\begin{bmatrix} \sigma_{(t)(t)} & \sigma_{(t)(r)} & \sigma_{(t)(\theta)} & \sigma_{(t)(\phi)} \\ \sigma_{(r)(t)} & \sigma_{(r)(r)} & \sigma_{(r)(\theta)} & \sigma_{(r)(\phi)} \\ \sigma_{(\theta)(t)} & \sigma_{(\theta)(r)} & \sigma_{(\theta)(\theta)} & \sigma_{(\theta)(\phi)} \\ \sigma_{(\phi)(t)} & \sigma_{(\phi)(r)} & \sigma_{(\phi)(\theta)} & \sigma_{(\phi)(\phi)} \end{bmatrix} = \begin{bmatrix} 0 & u_{\hat{\phi}}\sigma_{\hat{r}\hat{\phi}} & 0 & 0 \\ u_{\hat{\phi}}\sigma_{\hat{r}\hat{\phi}} & 0 & 0 & \sigma_{\hat{r}\hat{t}}u_{\hat{\phi}} + \sigma_{\hat{r}\hat{\phi}} + \frac{\sigma_{\hat{r}\hat{\phi}}u^2_{\hat{\phi}}}{1+\hat{\gamma}} \\ 0 & 0 & 0 & 0 \\ 0 & \sigma_{\hat{r}\hat{t}}u_{\hat{\phi}} + \sigma_{\hat{r}\hat{\phi}} + \frac{\sigma_{\hat{r}\hat{\phi}}u^2_{\hat{\phi}}}{1+\hat{\gamma}} & 0 & 0 \end{bmatrix}, \quad \text{(A19)}$$

$$\sigma_{(t)(r)} = \sigma_{(r)(t)} = u_{\hat{\phi}}\sigma_{\hat{r}\hat{\phi}}, \quad \text{(A20)}$$

$$\sigma_{(r)(\phi)} = \sigma_{(\phi)(r)} = \sigma_{\hat{r}\hat{t}}u_{\hat{\phi}} + \sigma_{\hat{r}\hat{\phi}} + \frac{\sigma_{\hat{r}\hat{\phi}}u^2_{\hat{\phi}}}{1+\hat{\gamma}}. \quad \text{(A21)}$$

Therefore, in FRF, two components of the shear tensor are non-zero.

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