Finding the optimized lower bound for the variance of unbiased estimators in some well-known families of distributions

S. Nayeban, A. H. Rezaei Roknabadi and G. R. Mohtashami Borzadaran

Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.

Abstract. One of the most fundamental things in estimation theory about accuracy of an unbiased estimator is computing or approximating its variance. Most of the time, the variance has complicated form or cannot be computed. In this paper, we consider two well-known lower bounds for the variance of unbiased estimators, which are Bhattacharyya (1946, 1947) and Kshirsagar (2000) bounds for some versatile families of distributions in statistics and especially in reliability such as, generalized gamma (GG), inverse Gaussian, Burr type XII and Burr type III distributions. In these distributions, the general forms of Bhattacharyya and Kshirsagar matrices are obtained. In addition, we evaluate different Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of some parameter functions and conclude that in each case, which bound has higher convergence and is better to use.

Keywords. Bhattacharyya bound, Cramer-Rao bound, generalized gamma distribution, Hammersley-Chapman-Robins bound, inverse Gaussian distribution, Kshirsagar bound

1 Introduction

In the estimation of a function $g(\theta)$ of a real parameter θ, the Bhattacharyya bounds for the variance of an unbiased estimator are known as a generalization of the Cramer-Rao bound (see Lehmann and Casella (1998)) and the Kshirsagar bounds are known as a generalization of the Hammersley-Chapman-Robins bound (see Kshirsagar (2000), Koike (2002)). In the study of life testing and reliability analysis of a component or system, one important approach is to consider an underlying 'life' distribution and to find suitable estimates of the parameters of that distribution. Among statistical distributions, there are a number of distributions that have been proved to be useful in life testing such as natural exponential family (NEF), generalized gamma (GG), inverse Gaussian, Burr type XII and Burr type III distributions. This paper is organized as follows. Section 2 presents definitions of Bhattacharyya and Kshirsagar bounds. In Section 3, we obtain the general forms of Bhattacharyya and Kshirsagar matrices in some families of useful distributions in reliability and modeling data. We evaluate and compare these two bounds for some parametric functions in Section 4.

2 Bhattacharyya and Kshirsagar bounds

In this section, we briefly introduce the structure of Bhattacharyya and Kshirsagar lower bounds for variance of unbiased estimator of a function $g(\theta)$ of a real parameter θ.

2.1 Bhattacharyya lower bounds

Bhattacharyya (1946, 1947) obtained a generalized form of the Cramer-Rao inequality which is related to the Bhattacharyya matrix. The Bhattacharyya matrix is the covariance matrix of the random vector,

$$\frac{1}{f(X|\theta)}(f^{(1)}(X|\theta), f^{(2)}(X|\theta), \ldots, f^{(r)}(X|\theta)), \tag{1}$$

where $f^{(j)}(.|\theta)$ is the j^{th} derivative of the probability density function $f(.|\theta)$ w.r.t. the parameter θ. The covariance matrix of the above random vector is referred to as the $r \times r$ Bhattacharyya matrix and r is the order of it. It is obvious that $(1,1)^{th}$ element of the Bhattacharyya matrix is the Fisher information.

The Bhattacharyya bound for any unbiased estimator $T(X)$ of the $g(\theta)$, under the regularity conditions, is defined as follows,

$$Var_\theta(T(X)) \geq J_\theta \mathbf{M}^{-1} J_\theta^t := B_r(\theta), \tag{2}$$

where t refers to the transpose, $J_\theta = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(r)}(\theta))$, $g^{(j)}(\theta) = \frac{\partial^j g(\theta)}{\partial \theta^j}$ for $j = 1, 2, \ldots, r$ and $\mathbf{M}^{-1}$ is the inverse of the Bhattacharyya matrix, where

$$\mathbf{M} = (M_{ij}) = (Cov_\theta\{\frac{f^{(i)}(X|\theta)}{f(X|\theta)}, \frac{f^{(j)}(X|\theta)}{f(X|\theta)}\}),$$

such that $E_\theta(\frac{f^{(i)}(X|\theta)}{f(X|\theta)}) = 0$ for $i, j = 1, 2, \ldots, r$.

If we substitute $r = 1$ in (2), it reduces to the Cramer-Rao inequality. By using the properties of the multiple correlation coefficient, it is easy to show that as the order of the Bhattacharyya matrix (r) increases, the Bhattacharyya bound becomes sharper and sharper.

One can see more details and information about Bhattacharyya bound in the papers such as, Blight and Rao (1974), Tanaka and Akahira (2003), Tanaka (2006), Mohtashami Borzadaran (2001, 2006), Khorashadizadeh and Mohtashami (2007), Mohtashami Borzadaran et al. (2010).

2.2 Kshirsagar lower bounds

It is well-known that, the Hammersley-Chapman-Robbins is a sharper lower bound than Cramer-Rao which needs no regularity conditions. This lower bound has been introduced independently by Hammersley (1950) and Chapman and Robbins (1951) as follow,

If there exists ϕ, such that $\phi \in \Theta$ and $S(\phi) \subset S(\theta)$, where $S(\theta) = \{x|f(x|\theta) > 0\}$, then,

$$Var_\theta(T(X)) \geq \sup_\phi \frac{[g(\phi) - g(\theta)]^2}{E_\theta\left(\frac{f(X|\phi) - f(X|\theta)}{f(X|\theta)}\right)^2}. \tag{3}$$

One can see the recent researches about this lower bound in Akahira and Ohyauchi (2007) and Ohyauchi (2004).

Recently, Kshirsagar (2000) extended the Hammersley-Chapman-Robbins lower bound in the same manner of the Bhattacharyya inequality. This bound does not need the assumptions of the common support and the existence of the derivative of the density function. The Kshirsagar bound states that for any unbiased estimator $T(X)$ of $g(\theta)$,

$$Var_\theta(T(X)) \geq \sup_\phi \lambda_\theta^t \Sigma^{-1} \lambda_\theta := K_r(\theta), \tag{4}$$

where t refers to the transpose, $\lambda_\theta = (g(\phi_1) - g(\theta), g(\phi_2) - g(\theta), \ldots, g(\phi_r) - g(\theta))^t$ and Σ^{-1} is the inverse of matrix with elements as follow,

$$\Sigma_{ij} = Cov_\theta(\omega_i, \omega_j), \quad i, j = 1, 2, \ldots, r,$$

where, $\omega_i = \frac{f(X|\phi_i) - f(X|\theta)}{f(X|\theta)}$ and the supremum is taken over the set of all $\phi_i \in \Theta, (i = 1, 2, \ldots, r)$, satisfying,

$$S(\phi_r) \subset S(\phi_{r-1}) \subset \ldots \subset S(\phi_1) \subset S(\theta).$$

He showed that for fixed k, this bound is sharper than the Bhattacharyya bound of order k. Although, computing the Kshirsagar bound and taking the supremums are difficult, but, nowadays, using computers make it a little easier to compute. Qin and Nayak (2008) obtained the Kshirsagar lower bounds for mean squared error of prediction.

3 The general form of Bhattacharyya and Kshirsagar matrices in some families of distributions

In this section, we obtain the general forms of Bhattacharyya and Kshirsagar matrices in some well-known families of distributions in reliability modeling.

3.1 Burr XII and Burr III distributions

Let X and Y have Burr XII and Burr III distributions respectively with probability density function (pdf) as,

$$f(x) = \frac{\alpha\theta x^{\alpha-1}}{(1 + x^\alpha)^{\theta+1}}; \quad x > 0, \alpha > 0, \theta > 0, \tag{5}$$

$$f(y) = \frac{\alpha\theta y^{-\alpha-1}}{(1 + y^{-\alpha})^{\theta+1}}; \quad y > 0, \alpha > 0, \theta > 0. \tag{6}$$

where α and θ are the shape parameters. It is easily seen that the Burr III is the simple transformation, $Y = \frac{1}{X}$, of Burr XII and therefore it retains most of the properties of (5). The Burr XII distribution has been used in quality control and reliability by many authors such as, Zimmer et al. (1998), Soliman (2005). Here, the thing, that is very important, is the variances of the estimators. In what follows, we try to evaluate the some sharp bounds for the variance of all unbiased estimators of $g(\theta)$ in Burr XII and Burr III distributions. The interesting thing is that, the forms of Bhattacharyya matrix for both Burr XII and Burr III are the same, but the forms of Kshirsagar matrix are different in their signs. We see that for the matrices of order more than 5, the differences of the bounds are about less than 0.0001, so, we calculate the 5×5 Bhattacharyya and Kshirsagar matrices. Suppose α be a known value, then the general form of the 5×5 Bhattacharyya matrix in both Burr XII and Bur III, is as follows,

$$\begin{pmatrix} \frac{1}{\theta^2} & \frac{-2}{\theta^3} & \frac{6}{\theta^4} & \frac{-24}{\theta^5} & \frac{120}{\theta^6} \\ & \frac{8}{\theta^4} & \frac{-36}{\theta^5} & \frac{192}{\theta^6} & \frac{-1200}{\theta^7} \\ & & \frac{216}{\theta^6} & \frac{-1440}{\theta^7} & \frac{10800}{\theta^8} \\ & & & \frac{11520}{\theta^8} & \frac{-100800}{\theta^9} \\ & & & & \frac{1008000}{\theta^{10}} \end{pmatrix}. \tag{7}$$

Also, in Kshirsagar bound, by supposing $\phi_i = \theta + i\delta$ for $i = 1, 2, \ldots, k$, where $\delta > -\frac{\theta}{i}$, one can see that in Burr XII,

$$\Sigma_{rs} = E_\theta(\psi_r.\psi_s) = \frac{rs\delta^2}{\theta[(r+s)\delta + \theta]}; \quad r, s = 1, 2, \ldots, k, \tag{8}$$

and in Burr III,

$$\Sigma_{rs} = E_\theta(\psi_r.\psi_s) = -\frac{rs\delta^2}{\theta[(r+s)\delta + \theta]}; \quad r, s = 1, 2, \ldots, k. \tag{9}$$

3.2 Generalized gamma (GG) distribution

The probability density function (pdf) of the generalized gamma (GG) distribution is as follows:

$$f(x) = \frac{\alpha x^{\alpha p - 1} \mathrm{e}^{-\left(\frac{x}{\beta}\right)^\alpha}}{\beta^{\alpha p} \Gamma(p)}; \quad x \geq 0, \alpha, \beta, p > 0, \tag{10}$$

where $\Gamma(\cdot)$ is the gamma function, α and p are the shape parameters, and β is the scale parameter. This distribution has several subfamilies. Figure 1 present the interrelations between the distributions.

By supposing β as an unknown parameter and α and p be known, the general form of the diagonal elements of Bhattacharyya matrix in GG distribution are as follow,

$$W_{rr} = \frac{\alpha^2 p}{\beta^{2r}} H_{r-1,2r-2}(p, \alpha), \quad r = 1, 2, ...,$$

where $H_{r-1,2r-2}(p, \alpha)$ is a polynomial function of p and α with degree of $r-1$ for p and degree of $2r-2$ for α. Also, the general form of the non-diagonal elements of Bhattacharyya matrix in GG distribution is,

$$W_{rs} = \frac{(\alpha - 1)\alpha^2 p}{\beta^{r+s}} H_{min(r,s)-1,r+s-3}(p, \alpha), \quad r, s = 1, 2, ...,$$

By supposing $\phi_i = \theta + i\delta$ for $i = 1, 2, \ldots, k$, where $\delta > -\frac{\theta}{i}$, the Kshirsagar bound in GG distribution have not a closed form and thus we calculate it with numerical methods.

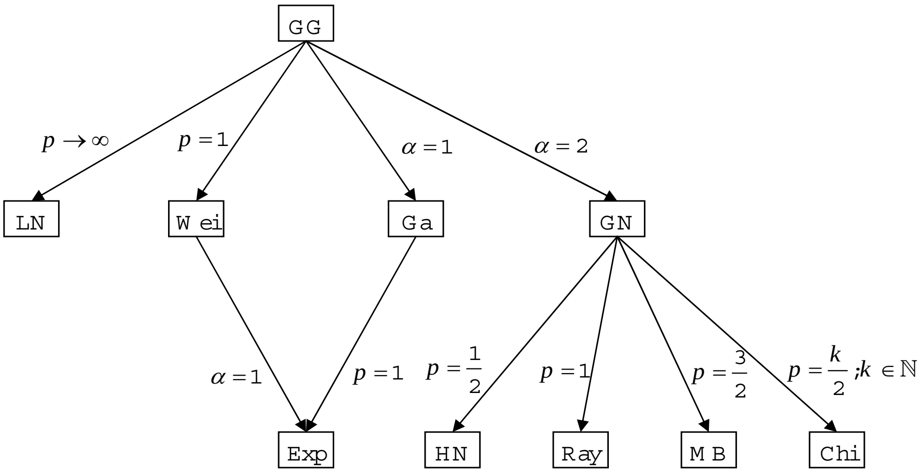

Fig. 1. *Generalized gamma distribution (GG) and its interrelations: gamma distribution (Ga), Weibull distribution (Wei), Log-normal distribution (LN), generalized normal distribution (GN), Half-normal distribution (HN), Rayleigh distribution (Ray), Maxwell-Boltzmann (MB), Chi square (Chi), and exponential distribution (Exp).*

3.3 Inverse Gaussian distribution

The inverse Gaussian distribution is a very versatile positive-domain two-parametric probabilistic model having numerous applications in diverse fields, lifetime models in connection with repairs, accelerated life testing, reliability problems and frailty models. The standard or canonical two-parameter inverse Gaussian distribution has probability density function given by:

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{ \frac{-\lambda}{2\theta^2 x}(x-\theta)^2 \right\}; \quad x > 0,$$

where $\theta, \lambda > 0$, and we denote it by IG(θ, λ). In this distribution, we have $E(X) = \theta, Var(X) = \frac{\theta^3}{\lambda}$.

Shanbhag (1972, 1979) showed that the diagonality of Bhattacharyya matrix characterizes the natural exponential family with quadratic variance function of parameter (NEF-QVF). Recently, Khorashadizadeh and Mohtashami Borzadaran (2007) and Mohtashami Borzaradan et al. (2010) have showed that in natural exponential family with cubic variance function of θ (NEF-CVF), the Bhattacharyya matrix is not diagonal and they obtained the general form of 5×5 Bhattacharya matrix. Also, they computed the Bhattacharyya matrix in inverse Gaussian distribution which is belonging to the NEF-CVF. The term $\lambda_\theta^t \Sigma^{-1} \lambda_\theta$ in Kshirsagar bound for inverse Gaussian distribution is an increasing function of δ, and therefore the supremum does not exist.

4 Some examples of parameter functions

In this section, we compute and compare the Bhattacharyya and Kshirsagar bounds for different functions of parameter.

4.1 Reliability function

In Tables 1, 2 and 3, we present the Bhattacharyya (denoted by B_i) and Kshirsagar (denoted by K_i) lower bounds for the variance of any unbiased estimator of reliability function in generalized gamma, Burr III and Burr XII distributions respectively.

Table 1. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the reliability function $g(\theta) = \frac{\Gamma(p,(\frac{x}{\theta})^{\alpha})}{\Gamma(p)}$ in generalized gamma distribution in which all parameters are positive and $\Gamma(a,b) = \int_b^{\infty} x^{\alpha-1}e^{-x}dx$.

α	θ	p	x	B_1	B_2	B_3	B_4
1	1	1	2	0.073262	0.073268	0.081408	0.089550
1	2	2	3	0.126020	0.173281	0.182141	0.182161
2	2	1	5	0.000145	0.000801	0.001030	0.001175
2	3	2	0.5	0.0000002	0.0000006	0.0000012	0.0000019
2	2	0.5	1	0.096529	0.146810	0.174980	0.190880
α	θ	p	x	K_1	K_2	K_3	K_4
1	1	1	2	0.073268	0.087000	0.093810	0.098740
1	2	2	3	0.175610	0.187512	0.190012	0.194025
2	2	1	5	0.000589	0.000935	0.001075	0.001244
2	3	2	0.5	0.000270	0.000271	0.000271	0.000271
2	2	0.5	1	0.202894	0.445210	0.548710	0.599810

Table 2. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the reliability function $g(\theta) = 1-(1+b)^{-\theta}$ in the Burr III distribution, in which b is positive and constant.

θ	b	B_1	B_2	B_3	B_4	B_5	$K_1 \approx K_2 \approx \ldots \approx K_5$
1	1	0.120113	0.171397	0.189380	0.193563	0.193761	0.250000
1	2	0.134105	0.161345	0.162755	0.164088	0.169406	0.444444
2	5	0.009908	0.016120	0.018066	0.018229	0.019653	0.945216
3	1	0.067563	0.067670	0.076366	0.083803	0.085934	0.765625

We see that, as the order of Bhattacharyya and Kshirsagar matrices increase, the bounds get sharper and sharper. Here, the important point is that, although evaluating the Kshirsagar bounds are more difficult and time consuming, because of taking supremums, but they are sharper than their corresponding Bhattacharyya bounds. So, in this case, we propose to use the first Kshirsagar lower bound.

4.2 Hazard rate function

The Bhattacharyya and Kshirsagar bounds for the hazard rate function in Burr III and generalized gamma distributions are presented in Tables 4 and 5 respectively.

As we see for the hazard rate function, the Kshirsagar bounds cannot be computed and also they do not significantly differ from Bhattacharyya bounds. So, in this case, we propose to use the Bhattacharyya bounds.

Table 3. Bhattacharyya and Kshirsagarbounds for the variance of any unbiased estimator of the reliability function $g(\theta) = (1 + c)^{-\theta}$ in the Burr XII distribution, in which c is positive and constant.

θ	c	B_1	B_2	B_3	B_4	B_5
0.1	0.2	0.000320	0.000635	0.000944	0.001247	0.001545
0.5	2	0.1005791	0.15349	0.17874	0.18903	0.19216
1	4	0.10361	0.10756	0.11083	0.11971	0.12712
2	6	0.00630	0.01195	0.01280	0.01329	0.01457
3	2	0.014900	0.02115	0.024666	0.024667	0.025896
θ	a	K_1	K_2	K_3	K_4	K_5
0.1	0.2	0.01442	0.014655	0.014734	0.014771	0.0164607
0.5	2	0.187019	0.207298	0.211920	0.22005	0.224273
1	4	0.106320	0.125115	0.132446	0.136643	0.140774
2	6	0.009103	0.012488	0.0140133	0.0148518	0.0153838
3	2	0.018009	0.023722	0.026178	0.027357	0.0288086

Table 4. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the hazard rate function $g(\theta) = \frac{\alpha\theta(x^{\alpha+1}+x)}{(1+x^{-\alpha})^{\theta}-1}$ in the Burr III distribution, in which all parameters are positive and constant.

θ	x	α	B_1	B_2	B_3	B_4	B_5
0.1	0.2	0.3	0.002015	0.003964	0.005848	0.007668	0.009425
0.5	2	1	0.001509	0.002912	0.004212	0.005413	0.006520
3	2	4	0.027484	0.053229	0.077295	0.099743	0.120635
θ	y	a	K_1	K_2	K_3	K_4	K_5
0.1	0.2	0.3	0.002125	0.003998	0.006051	0.009151	0.009989
0.5	2	1	0.001652	0.003129	0.005915	0.006464	0.007089
3	2	4	0.027584	0.069857	0.080504	0.130115	0.151514

Table 5. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the hazard rate function $h(t) = \frac{\alpha t^{\alpha p-1} e^{-\left(\frac{t}{\beta}\right)^{\alpha}}}{\beta^{\alpha p}\Gamma(p,(\frac{t}{\beta})^{\alpha})}$ in GG distribution, in which all parameters are positive. The term $\lambda_{\theta}^{t}\Sigma^{-1}\lambda_{\theta}$ in Kshirsagar bound for the hazard rate in GG distribution is increasing function of δ and hence its supermum does not exist.

α	β	p	t	B_1	B_2	B_3	B_4
1	1	2	1	0.28125	0.53646	0.75618	0.94388
2	2	1	3	2.25000	4.5000	6.75000	9.00000
2	3	2	0.5	0.00001	0.00004	0.00007	0.00011
2	2	0.5	1	0.84224	1.76540	2.77160	3.85750

References

Akahira, M. and Ohyauchi, N. (2007), A Bayesian view of the Hammersley-Chapman-Robbins-type inequality. Statistics, 41, 2, 137-144.

Bhattacharyya, A. (1946), On some analogues of the amount of information and their use in statistical estimation, Sankhya A, 8, 1-14.

Bhattacharyya, A. (1947), On some analogues of the amount of information and their use in statistical estimation II, Sankhya A, 8, 201-218.

Blight, B. J. N. and Rao, R. V. (1974), The convergence of Bhattacharyya bounds, Biometrika, 61(1), 137-142.

Chapman, D. G. and Robbins, H. (1951), Minimum variance estimation without regularity assumptions, Ann. Math. Statist., 22, 581-586.

Hammersley, J. M. (1950), On estimating restricted parameters, J. Roy. Statist. Soc. Ser. B, 12, 192-240.

Khorashadizadeh, M. and Mohtashami Borzadaran, G. R. (2007), The structure of Bhattacharyya matrix in natural exponential family and its role in approximating the variance of a statistics, J. Statistical Research of Iran (JSRI), 4(1), 29-46.

Koike, K. (2002), On the inequality of Kshirsagar, Commun. Statist. Theory Meth. 31, 1617-1627.

Kshirsagar, A. M. (2000), An extension of the Chapman-Robbins inequality, J. Indian Statist. Assoc., 38, 355-362.

Lehmann, E. L. and Casella, G. (1998), Theory of Point Estimation, 2nded., Springer, New York.

Mohtashami Borzadaran, G. R. (2001), Results related to the Bhattacharyya matrices, Sankhya A, 63, 1, 113-117.

Mohtashami Borzadaran, G. R. (2006), A note via diagonality of the 2×2 Bhattacharyya matrices, J. Math. Sci. Inf., 1(2), 73-78.

Mohtashami Borzadaran, G. R., Rezaei Roknabadi, A. H. and Khorashadizadeh, M. (2010), A view on Bhattacharyya bounds for inverse Gaussian distributions, Metrika, 72, 2, 151-161.

Ohyauchi, N. (2004), The vincze inequality for the Bayes risk, Journal of the Japan Statistical Society, 34, 65-74.

Qin, M. and Nayak, T. K. (2008), Kshirsagar-type lower bounds for mean squared error of prediction, Communications in Statistics - Theory and Methods, 37, 6, 861-872.

Shanbhag, D. N. (1972), Some characterizations based on the Bhattacharyya matrix, J. Appl. Probab., 9, 580-587.

Shanbhag, D. N. (1979), Diagonality of the Bhattacharyya matrix as a characterization, Theory Probab. Appl., 24, 430-433.

Soliman, A. A. (2005), Estimation of parameters of life from Progressively censored data using Burr-XII model, IEEE Transactions on Reliability, 54, 1, pp. 34-42.

Tanaka, H. (2006), Location and scale parameter family of distributions attaining the Bhattacharyya bound, Communications in Statistics - Theory and Methods, 35, 9, 1611-1628.

Tanaka, H. and Akahira, M. (2003), On a family of distributions attaining the Bhattacharyya bound, Ann. Inst. Stat. Math., 55, 309-317.

Zimmer, W. J., Keats, J. B. and Wang, F. K. (1998), The Burr XII distribution in reliability analysis, J. Quality Technology, 30, 4, pp. 386-394.