Capability of groups, an algebraic topological approach

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Abstract

In this paper, we use one of the famous algebraic topological tools, called Mayer-Vietoris sequence, to find a necessary condition for capability of groups. Also, we try to generalize our main results, for $\mathcal{V}-$capability of groups.

1 Introduction

R. Baer [12] initiated an investigation of the question "which conditions a group G must fulfill in order to be the group of inner automorphisms of a group E?", that is $G \cong E/Z(E)$. Following M. Hall and J. K. Senior [10], such a group G is called *capable*. Baer [12] determined all capable groups which are direct sums of cyclic groups. As P. Hall [9] mentioned, characterizations of capable groups are important in classifying groups of prime-power order.

F. R. Beyl, U. Felgner and P. Schmid [3] proved that every group G possesses a uniquely determined central subgroup $Z^*(G)$ which is minimal subject to being the image in G of the center of some central extension of G. This $Z^*(G)$ is characteristic in G and is the image of the center of every stem cover of G. Moreover, $Z^*(G)$ is the smallest central subgroup of G, $1 \leq Z^*(G) \leq Z(G)$ whose factor group is capable [3]. Hence G is capable if and only if $Z^*(G) = 1$. They proved that if N is a central subgroup of G, then $N \subseteq Z^*(G)$ if and only if the mapping $M(G) \to M(G/N)$ induced by the natural epimorphism, is monomorphism.

Then M. R. R. Moghadam and S. Kayvanfar [15] generalized the concept of capability to $\mathcal{V}-$capability for a group G. They introduced the subgroup $(V^*)^*(G)$ which is associated with the variety $\mathcal{V}$ defined by a set of laws V and a group G in order to establish a necessary and sufficient condition under which G can be $\mathcal{V}-$capable. They exhibited a close relationship between the groups $\mathcal{V}M(G)$ and $\mathcal{V}M(G/N)$, where N is a normal subgroup contained in the marginal subgroup of G with respect to the variety $\mathcal{V}$. Using this relationship, they gave a necessary and sufficient condition for a group G to be $\mathcal{V}-$capable.

In this paper, we use algebraic topological methods, especially Mayer-Vietoris sequence, to establish a necessary condition for capability of groups. Also in section 3, we try to generalize our results for $\mathcal{V}-$capability of groups.

2 Main Results

First, we note that for any group G one can construct functorially a connected CW-complex $K(G)$, called Eilenberg-MacLane space, whose fundamental group is isomorphic to G which has all higher homotopy groups trivial [20]. By considering $H_n(X)$ as the nth singular homology group of a topological space X, with coefficients in the group $\mathbf{Z}$, we recall the relation $H_n(G) \cong H_n(K(G))$, for all $n \geq 0$, [2].

By Hopf formula for any CW-complex K with $\pi_1(K) = G$ and F/R as a free presentation for G we have the following isomorphism

$$\frac{H_2(K)}{h_2(\pi_2(K))} \cong \frac{R \cap F'}{[R, F]}.$$

where h_2 is the corresponding Hurewicz map [11]. Hence a topological definition of the Schur multiplier of a group G can be considered as the second homology group of the Eilenberg-MacLane space $K(G)$, $H_2(K(G))$. Also, we recall that $H_1(K(G)) = H_1(G) = G_{ab}$.

For any two normal subgroups M and N of G, we consider the following homotopy pushout diagram

$$\begin{array}{ccc} K(G) & \xrightarrow{g_1} & K(G/N) \\ \downarrow{\scriptstyle g_2} & & \downarrow \\ K(G/M) & \longrightarrow & X. \end{array}$$

where g_1 and g_2 are induced by natural maps. Using the Mayer-Vietoris sequence for pushout, we have the following exact sequence

$$\begin{aligned} \cdots \to H_3(X) &\to H_2(K(G)) \to H_2(K(G/M)) \oplus H_2(K(G/N)) \to \\ H_2(X) &\to H_1(K(G)) \to H_1(K(G/M)) \oplus H_1(K(G/N)) \to \\ H_1(X) &\to H_0(K(G)) \to H_0(K(G/M)) \oplus H_0(K(G/N)) \to H_0(X) \to 0. \end{aligned} \tag{1.2}$$

By the above notes, we rewrite this sequence as follows

$$\cdots \to H_3(X) \to M(G) \to M(G/M) \oplus M(G/N) \to$$

$$H_2(X) \to G_{ab} \to (G/M)_{ab} \oplus (G/N)_{ab} \to \cdots \quad (i).$$

Using Corollary 3.4 of [4] we have

$$\pi_1(X) \cong \frac{G}{MN} \quad \text{and} \quad \pi_2(X) \cong \frac{M \cap N}{[M, N]} \quad (ii).$$

Now by these notes and the following Lemma, we deduce our main result and some corollaries about unicentral groups. recall that a group G is unicentral, if and only if $Z^*(G) = Z(G)$ [5].

Lemma 2.1. *[3] Let N be a central subgroup of G, then $N \subseteq Z^*(G)$ if and only if the homomorphism induced by the natural map $M(G) \to M(G/N)$ is a monomorphism.*

Theorem 2.2. *Let G be a group with $M(G) = 1$; M and N be its central subgroups with $M \cap N = 1$. Then $MN/M \leq Z^*(G/M)$ and $MN/N \leq Z^*(G/N)$.*

Proof. Using (ii), the assumption $M \cap N = 1$ implies $\pi_2(X) = 1$. By Hopf Formula, $H_2(X) = M(\pi_1(X)) = M(G/MN)$. With respect to the sequence (i), we obtain the following monomorphisms which are both induced by natural maps,

$$M(G/M) \hookrightarrow M(G/MN) \quad \& \quad M(G/N) \hookrightarrow M(G/MN).$$

Since M and N and so MN are central subgroups of G, MN/M and MN/N are also central in G/M and G/N, respectively. Hence by the above monomorphisms and Lemma 2.1, we conclude the result. □

Corollary 2.3. *Any group G with $M(G) = 1$, is unicentral.*

Proof. By the theorem, for any central subgroup M of G, we have $M \leq Z^*(G)$ (put $N = 1$). In particular, $Z(G) \leq Z^*(G)$. □

The above corollary has been proved in group theory [5]. But here we derived it by topological methods. Also the following corollary is a new result in group theory with a proof in algebraic topology.

Corollary 2.4. *Let G be a group with $M(G) = 1$ and $Z(G)$ be inner direct product of M and N. Then G/N and G/M are unicentral.*

Proof. Using Theorem 2.2, $Z(G)/M \leq Z^*(G/M)$ and $Z(G)/N \leq Z^*(G/N)$. Also we have $Z(G)/M = Z(G/M)$ and $Z(G)/N = Z(G/N)$ which completes the proof. □

3 A Note on $\mathcal{V}$−capability of Groups

Let $\mathcal{V}$ be a variety of groups defined by a set of laws V and consider the functor $V(-)$ between groups which takes a group to its verbal subgroup. By a pair of groups (G, N) we mean a group G with normal subgroup N. A homomorphism of pairs $(G_1, N_1) \to (G_2, N_2)$ is a group homomorphism $G_1 \to G_2$ that sends N_1 into N_2.

As we asserted in section 2, finding a topological interpretation for algebraic concepts can be useful to solve some algebraic problems. Similar to the topological definition of Schur multiplier, the Baer invariant of G, $\mathcal{V}M(G)$, has also a topological interpretation. For this, consider $\mathcal{V}M(G)$ to be the first homotopy group, $\pi_1(K_.\!/V(K_.))$, of the factor of a free simplicial resolution of the group G [8], [17].

Fifth International Group Theory Conference
Islamic Azad University, Mashhad Branch
13-15 March 2013

G. Ellis [7] defined the Schur multiplier of a pair of groups (G, N) denoted by $M(G, N)$, as a functorial abelian group whose principal feature is a natural exact sequence

$$\begin{aligned} \cdots \quad &\to M(G,N) \to M(G) \to M(G/N) \\ &\to N/[N,G] \to G^{ab} \to (G/N)^{ab} \to 0. \end{aligned}$$

The natural epimorphism $G \to G/N$ implies the following exact sequence of free simplicial groups

$$1 \to \ker(\alpha) \to K_. \xrightarrow{\alpha} L_. \to 1$$

where $K_.$ and $L_.$ are free simplicial resolutions of G and G/N, respectively [8]. The author, Z. Vasagh and B. Mashayekhy extended the above notation to the Baer invariant of a pair [18] and a triple [19] of groups. They defined $\mathcal{V}M(G, N)$, as the first homotopy group of the kernel of a map α_v, $\pi_1(\ker \alpha_v)$, where $\alpha_v : K_./V(K_.) \to L_./V(L_.)$ is induced from the simplicial map α. Also they obtained the following long exact sequence

$$\begin{aligned} \cdots \quad &\to \mathcal{V}M(G,N) \to \mathcal{V}M(G) \to \mathcal{V}M(G/N) \\ &\to N/[NV^*G] \to G/V(G) \to (G/N)/V(G/N) \to 0. \end{aligned}$$

Moreover, in Corollary 3.4 of [18], they showed that $\mathcal{V}M(G, N) \cong \frac{R \cap [SV^*F]}{RV^*F}$, where F/R is a free presentation of G, $N = S/R$ is a normal subgroup of G, and G is semidirect product of N by a group Q. In particular, using this corollary they concluded that their definition of $\mathcal{V}M(G, N)$ is a vast generalization of the one in [16].

In continue, we recall the definition of $\mathcal{V}$−capability of groups [15]; a group G is said to be $\mathcal{V}$−capable if there exists a group E such that $G \cong E/V^*(E)$, where $V^*(E)$ is the marginal subgroup of E, which is defined as follows [12]:

$$\{g \in E \mid v(x_1, x_2, ..., x_n) = v(x_1, x_2, ..., gx_i, x_{i+1}, ..., x_n)$$
$$\forall x_1, x_2, ..., x_n \in E, \ \forall i \in \{1, 2, ..., n\}\}.$$

If $\psi : E \to G$ is a surjective homomorphism with $ker\psi \subseteq V^*(E)$, then the intersection of all subgroups of the form $\psi(V^*(E))$ is denoted by $(V^*)^*(G)$. It is obvious that $(V^*)^*(G)$ is a characteristic subgroup of G contained in $V^*(G)$. If $\mathcal{V}$ is the variety of abelian groups, then the subgroup $(V^*)^*(G)$ is the same as $Z^*(G)$ and in this case $\mathcal{V}$−capability is equal to capability.

Lemma 3.1. *[15] Let $\mathcal{V}$ be a variety of groups with a set of laws V. Let G be a group and N be a normal subgroup with the property $N \subseteq V^*(G)$. Then $N \subseteq (V^*)^*(G)$ if and only if the homomorphism induced by the natural map $\mathcal{V}M(G) \to \mathcal{V}M(G/N)$ is a monomorphism.*

Remark 3.2.
i) By the notations of Lemma 3.1, if $\mathcal{V}M(G, N) = 1$, then $N \subseteq (V^*)^*(G)$.
ii) As a consequence of (i), if $\mathcal{V}M(G, V^*(G)) = 1$, then G is $\mathcal{V}$−unicentral.
(By $\mathcal{V}$−unicentral, we mean $(V^*)^*(G) = V^*(G)$.)

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13-15 March 2013

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