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EXACT SOLUTIONS OF SOME TYPES OF FREDHOLM INTEGRAL EQUATIONS: HE'S VIM COMBINED WITH FINITE SERIES

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Abstract

It is well known that one of the advantages of He's variational iteration method is the free choice of initial approximation. From this advantage, in this paper, we construct a finite series solution with unknown parameters. Some types of the Fredholm integral equations are used to illustrate effectiveness and convenience of the method. A comparison is made between the He's original VIM and the presented one. The results reveal that the proposed method is very effective and simple and gives the exact solution.

1. Introduction

In 1999, the variational iteration method (VIM) [1-9] was proposed by He.

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This method is now widely used by many researchers to study linear and nonlinear problems. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications. It is based on Lagrange multiplier and it has the merits of simplicity and easy execution. Unlike the traditional numerical methods, the VIM needs no discretization, linearization, transformation or perturbation. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The VIM was successfully applied to autonomous ordinary and partial differential equations [1-28]. He [14] was the first to apply the variational iteration method to fractional differential equations. Application of the variational iteration method to various integral equations has become a hot topic [29, 30]. For a relatively comprehensive survey on the method and new interpretation and development, the reader is referred to the review articles [15, 31]. To illustrate its basic idea of the method, He [15, 31] considered the following general nonlinear equation

$$Lu(t) + Nu(t) = f(t), \tag{1}$$

where L is a linear operator, N is a nonlinear operator and $f(t)$ is a given continuous function. The basic character of the method is to construct a correction functional for the system, which reads

$$u_{n+1}(t) = u_n(t) + \int_0^x \lambda(s)\{Lu_n(s) + N\tilde{u}_n - f(s)\}ds, \tag{2}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [9], u_n is the n-th approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta u_n = 0$. It has been shown that this method is very effective and easy for linear problems. Its exact solution can be obtained by only one iteration, because λ can be exactly identified. But for nonlinear problems, there are secular terms, which should be considered [5]. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected by function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximations $u_n(t)$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n\to\infty} u_n(t).$$

He's VIM has been shown to solve effectively, easily and accurately a large class of nonlinear and linear problems with approximations that converge rapidly to accurate solutions. Now consider the Fredholm integral equation (FIE) of the second kind, which read

$$u(x) = f(x) + \int_a^b k(x, t)u(t)dt, \quad c \le x \le d, \tag{3}$$

where $k(x, t)$ is the kernel of the integral equation. According to Reference [30], the variational iteration formula for equation (3) can be constructed in the form:

$$u_{n+1}(x) = f(x) + \int_a^b k(x, t)u_n(t)dt. \tag{4}$$

If we start with the initial approximation $u_0(x) = f(x)$ then the first few approximations are given by

$$u_0(x) = f(x),$$

$$u_1(x) = f(x) + \int_a^b k(x, t)f(t)dt,$$

$$u_2(x) = f(x) + \int_a^b k(x, t)\left[f(t) + \int_a^b k(t, s)f(s)ds\right]dt, \tag{5}$$

and so on.

In most cases the integrations of equation (5) is not easily evaluated or it requires tedious computing. Therefore, in the following section, we introduce He's VIM combined with finite series based on the advantage of He's VIM, which reduces the size of calculations and causes a rapid convergence and gives the exact solution.

2. He's VIM Combined with Finite Series

In this section we propose a scheme to accelerate the rate of convergence of VIM applied to linear Fredholm integral equations with kernels of the form $\sum_{i=1}^{N} a_i(x) b_i(t)$. According to He's variational iteration method; the initial guess can involve some unknown parameters. We, therefore, define a new variational iteration as follows:

$$\begin{cases} u_{n+1}(x) = f(x) + \int_a^b k(x, t) u_n(t) dt, \\ u_0(x) = f(x) + \sum_{j=1}^{N} c_j a_j(x), \end{cases} \tag{6}$$

where c_j, $j = 1, 2, \ldots, N$ are called the *accelerating parameters*, and for $c_j = 0$, $j = 1, \ldots, N$, we have

$$\begin{cases} u_{n+1}(x) = f(x) + \int_a^b k(x, t) u_n(t) dt, \\ u_0(x) = f(x). \end{cases}$$

This is the original VIM.

2.1. Application to FIE of the second kind

We first assume that $k(x, t) = a(x) b(t)$, thus for the following equation

$$u(x) = f(x) + \int_a^b k(x, t) u(t) dt, \quad c \le x \le d.$$

We consider (6) as follows:

$$\begin{cases} u_{n+1}(x) = f(x) + \int_a^b k(x, t) u_n(t) dt, \\ u_0(x) = f(x) + ca(x). \end{cases} \tag{7}$$

Therefore, we have the first-order approximation

$$u_0(x) = f(x) + ca(x),$$

$$u_1(x) = f(x) + \int_a^b k(x, t) u_0(t) dt,$$

$$\Rightarrow u_1(x) = f(x) + pa(x) + cqa(x),$$

where

$$p = \int_a^b b(t)f(t)dt \text{ and } q = \int_a^b a(t)b(t)dt.$$

Now, we find c so that $u_0(x) = u_1(x)$, since $u_0 = u_1$ then we will have $u_0 = u_1 = u_2 = \dots,$ and the exact solution will be obtained as $u(x) = u_0(x)$. Therefore, for all values of x we should have

$$(1 - q)c = p,$$

or

$$c = \frac{p}{1-q} = \frac{\int_a^b b(t)f(t)dt}{1 - \int_a^b a(t)b(t)dt} = \frac{1}{1 - \int_a^b k(t, t)dt} \int_a^b b(t)f(t)dt, \tag{8}$$

provided that

$$\int_a^b k(t, t)dt \neq 1.$$

Let us now consider the general case:

$$\sum_{i=1}^{N} a_i(x)b_i(t).$$

Now, choosing the variational iteration formula as follows:

$$\begin{cases} u_{n+1}(x) = f(x) + \int_a^b k(x, t)u_n(t)dt, \\ u_0(x) = f(x) + \sum_{j=1}^{N} c_j a_j(x). \end{cases}$$

By doing similar manipulations, we obtain

$$u_0(x) = f(x) + \sum_{j=1}^{N} c_j a_j(x),$$

$$u_1(x) = f(x) + \int_a^b k(x, t)u_0(t)dt,$$

$$\Rightarrow u_1(x) = f(x) + \sum_{i=1}^{N} a_i(x)\left[\int_a^b b_i(t)f(t)dt + \sum_{j=1}^{N} c_j \int_a^b b_i(t)a_j(x)dt\right], \qquad (9)$$

$$\vdots$$

As before, we try to find the parameters $c_j,\ j = 1, \ldots, N$ so that $u_0 = u_1$, therefore, in view of (9) we should have

$$c_j = \int_a^b b_i(t)f(t)dt + \sum_{j=1}^{N} c_j \int_a^b b_i(t)a_j(x)dt, \ \ \forall\, x \in [c, d]. \qquad (10)$$

Let

$$d_i = \int_a^b b_i(t)f(t)dt, \text{ and } e_{ij} = \int_a^b b_i(t)a_j(x)dt.$$

Then

$$c_i(x) = d_i + \sum_{j=1}^{N} c_j e_{ij},\ c_i,\ \ i = 1, \ldots, N. \qquad (11)$$

Under certain conditions, the values of $c_i,\ i = 1, \ldots, N$ can be obtained from the system of linear equations in (11). Let the matrix E and the vectors C and D be defined as follows:

$$E = [e_{ij}],\ \ C = [c_i],\ \ D = [d_i],$$

from (11), therefore, we can write

$$(I - E)C = D,$$

and if $(I - E)$ is nonsingular then

$$C = (I - E)^{-1}D. \qquad (12)$$

Remark 2.1.1. In the case of non-degenerate kernels, by using Taylor expansion for functions of two variables, we can write $k(x, t)$ (if possible) as follows:

$$k(x, t) = \sum_{i=1}^{N} a_i(x)b_i(t),$$

and by applying the presented method we can approximate the solution of the given integral equation.

2.2. Application to FIE of the first kind

Consider the following integral equation:

$$f(x) = \int_{\alpha}^{\beta} k(x, t)u(t)dt, \ \gamma \le x \le \mu. \tag{13}$$

Let $k(x, t) = a(x)b(t)$ and $f(x) = pa(x)$. Using (6) we can write

$$\begin{cases} u_{n+1}(x) = u_n(x) + f(x) - \int_{\alpha}^{\beta} k(x, t)u_n(t)dt, \\ u_0(x) = ca(x). \end{cases}$$

By similar operations in Subsection 2.1 we obtain

$$u_0(x) = ca(x),$$

$$u_1(x) = u_0(x) + f(x) - \int_{\alpha}^{\beta} k(x, t)u_0(t)dt,$$

$$\Rightarrow u_1(x) = ca(x) + pa(x) - cqa(x), \text{ where } q = \int_{\alpha}^{\beta} a(t)b(t)dt = \int_{\alpha}^{\beta} k(t, t)dt, \tag{14}$$

$$\vdots$$

For $u_0 = u_1$, as stated above, from (14) we should have

$$p - cq = 0, \ \forall x \in [\gamma, \mu q].$$

Consequently,

$$c = \frac{p}{q}, \ (q \ne 0).$$

If we assume $k(x, t) = \sum_{i=1}^{N} a_i(x)b_i(t)$, then it is easy to verify that

$$f(x) = \sum_{i=1}^{N} d_i a_i(x).$$

In this case as in Subsection 2.1, we choose the following variational iteration

$$\begin{cases} u_{n+1}(x) = u_n(x) + f(x) - \lambda \int_\alpha^\beta k(x, t) u_n(t) dt, \\ u_0(x) = \sum_{i=1}^N c_i a_i(x) \end{cases} \tag{15}$$

By simple operations we obtain the following approximations

$$u_0(x) = \sum_{i=1}^N c_i a_i(x),$$

$$u_1(x) = \sum_{i=1}^N \left[c_i + d_i - \sum_{i=1}^N c_j a_{ij} \right] a_i(x) \text{ where } e_{ij} = \int_\alpha^\beta b_i(t) a_j(t) dt, \tag{16}$$

$$\vdots$$

Proceeding as before, for $u_0 = u_1$, the parameters c_i, $i = 1, \ldots, N$, should satisfy the following linear system of equations

$$d_i - \sum_{i=1}^N c_i e_{ij} = 0, \; i = 1, \ldots, N \tag{17}$$

Or in matrix form we have

$$CE = D,$$

where $C = [c_i]$, $D = [d_i]$ and $E = [e_{ij}]$ for i, $j = 1, \ldots, N$, or

$$C = E^{-1} D. \tag{18}$$

Provided the matrix E is not singular.

Remark 2.2.1. For non-degenerate kernels, using Taylor expansion, we can write

$$k(x, t) = \sum_{i=1}^N a_i(x) b_i(t), \;\; f(x) = \sum_{i=1}^N d_i a_i(x),$$

and then by applying the presented method, we can approximate the solution of the given problem.

3. Applications

Example 3.1. Consider the following integral equation

$$u(x) = \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x u(t) dt. \tag{19}$$

We apply the original and the presented methods to approximate the solution as follows:

The original method. We, according to the original VIM, have

$$u_0(x) = \cos x,$$

$$u_1(x) = \cos x + \frac{1}{2}\sin x,$$

$$u_2(x) = \cos x + \frac{2^2 - 1}{2^2}\sin x,$$

$$u_3(x) = \cos x + \frac{2^3 - 1}{2^3}\sin x,$$

$$\vdots$$

By continuing this procedure, we finally obtain

$$u(x) = \cos x + \lim_{n\to\infty}\left(\frac{2^n - 1}{2^n}\right)\sin x \to \cos x + \sin x.$$

The modified method. In this case we have:

$$u_0(x) = f(x) + ca(x),$$

$$c = \frac{p}{1-q} = \frac{1}{\int_0^{\pi/2} k(t, t) dt.}\int_0^{\pi/2} b(t) f(t) dt,$$

and

$$u(x) = u_0(x) = f(x) + ca(x).$$

For this example $a(x) = \sin x$ and $b(x) = 1/2$, thus we obtain

$$c = \frac{1}{1 - \int_0^{\pi/2}\frac{1}{2}\sin t dt}\int_0^{\pi/2}\frac{1}{2}\cos t dt \Rightarrow c = 1.$$

We, therefore, obtain

$$u(x) = \cos x + \sin x$$

which is the exact solution. As it can be seen, after one term the exact solution is obtained.

Example 3.2. Approximate the solution of

$$u(x) = x + \frac{1}{2}\int_0^1 (xt^2 + x^2 t)u(t)dt. \tag{20}$$

The original method. We can obtain the following according to the original VIM:

$$u_0(x) = x,$$

$$u_1(x) = \frac{5}{4}x + \frac{1}{3}x^2,$$

$$u_2(x) = \frac{331}{240}x + \frac{1}{2}x^2,$$

$$u_3(x) = \frac{1387}{960}x + \frac{421}{720}x^2,$$

$$\vdots$$

When n tends to infinity, the obtained solution inclines to the exact solution [30], which is $u(x) = \frac{20}{119}(9x + 4x^2)$.

The modified method. Using the presented method we have

$$a_1(x) = x, \;\; a_2(x) = x^2, \;\; b_1(x) = t^2, \;\; b_2(x) = t, \;\; f(x) = x,$$

$$u(x) = u_0(x) = x + c_1 x + c_2 x^2. \tag{21}$$

By applying (12) and the information of the problem, we obtain

$$E = \begin{bmatrix} 1/4 & 1/5 \\ 1/3 & 1/4 \end{bmatrix}, \;\; D = \begin{bmatrix} 1/4 \\ 1/3 \end{bmatrix},$$

and

$$\begin{bmatrix} 1/4 & 1/5 \\ 1/3 & 1/4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/3 \end{bmatrix}.$$

Hence the values of c_1 and c_2 are obtained as follows:

$$c_1 = \frac{61}{119}, \ c_2 = \frac{80}{119}.$$

Putting the values of c_1 and c_2 in equation (21), we get

$$u(x) = \frac{20}{119}(9x + 4x^2),$$

This is the same as the exact solution.

4. Application of the Presented Method to Nonlinear FIE

In this section, to show the effectiveness and convenience of the presented method to solve nonlinear Fredholm integral equations, some examples are given.

Example 4.1. We consider the following nonlinear Fredholm integral equation of the second kind

$$u(x) = x + \lambda \int_0^1 xtu^2(t)dt, \ 0 \le \lambda \le 1. \tag{24}$$

The original VIM. According to the original VIM, we have

$$u_0(x) = x,$$

$$u_1(x) = \left(1 + \frac{1}{4}\lambda\right)x,$$

$$u_2(x) = \left(1 + \frac{1}{4}\lambda + \frac{1}{8}\lambda^2 + \frac{1}{64}\lambda^3\right)x,$$

$$u_3(x) = \left(1 + \frac{1}{4}\lambda + \frac{1}{8}\lambda^2 + \frac{5}{64}\lambda^3 + \frac{3}{128}\lambda^4 + \frac{3}{512}\lambda^5 + \frac{1}{1024}\lambda^6 + \frac{1}{16384}\lambda^7\right)x,$$

$$\vdots$$

As n tends to infinity, the obtained solution inclines to the exact solution [30], which is $u(x) = \frac{2}{\lambda}(1 - \sqrt{1 - \lambda})x.$

The modified method. According to the procedure presented above, we consider the following variational iteration formula

$$\begin{cases} u_{n+1}(x) = x + \lambda \int_0^1 x t u_n^2(t) dt, \\ u_0(x) = x + cx. \end{cases} \tag{25}$$

We have the first-order approximation

$$u_1(x) = x + \frac{\lambda}{4}(1+c)^2 x, \tag{26}$$

for $u_0 = u_1$, we should have

$$c = \frac{1}{\lambda}(2 - \lambda \pm 2\sqrt{1-\lambda}). \tag{27}$$

We, therefore, obtain

$$u(x) = u_0(x) = \frac{2}{\lambda}(1 \pm \sqrt{1-\lambda})x. \tag{28}$$

These obtained solutions are the same as exact solutions. That is, the new procedure also is suitable for nonlinear FIE.

Example 4.2. Now we consider the following nonlinear Fredholm integral equation of the first kind

$$\int_0^1 x t[u(t) + u^2(t)] dt = \frac{7}{12} x. \tag{29}$$

We apply the presented method to solve (29). After following the same previous steps, we have the following variational iteration formula for equation (29):

$$\begin{cases} u_{n+1}(x) = u_n(x) + \frac{7}{12} x - \int_0^1 x t [u_n(t) + u_n^2(t)] dt, \\ u_0(x) = cx. \end{cases} \tag{30}$$

By simple operations, we can obtain the first-order approximation as follows

$$u_1(x) = \left(\frac{2}{3} c + \frac{7}{12} - \frac{c^2}{4}\right) x. \tag{31}$$

By setting $u_0 = u_1$, this implies that

$$3c^2 + 4c - 7 = 0, \tag{32}$$

This gives the values 1 and $-7/3$ for c. Substituting these values of c in $u_0(x)$, we obtain the solutions as follows

$$u(x) = x \text{ and } u(x) = -\frac{7}{3}x \tag{33}$$

which are the same as exact solutions.

5. Conclusion

In this paper, we have utilized He's variational iteration method combined with finite series to study some types of nonlinear and linear Fredholm integral equations. As a result, exact solutions of the Fredholm integral equations have been obtained. We also found that the method is of remarkable effectiveness and convenience, and the solution procedure is of complete simplicity as well. Moreover, the method can be easily extended to the other integral equations of the Fredholm type such as Hammerstein integral equations.

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