J. Appl. Math. & Informatics Vol. **30**(2012), No. 5 - 6, pp. 829 - 845
Website: http://www.kcam.biz

ON SOLUTIONS AND STABILITY OF A GENERALIZED QUADRATIC EQUATION ON NON-ARCHIMEDEAN NORMED SPACES

MOHAMMAD JANFADA* AND RAHELE SHOURVARZI

Abstract. In this paper we study general solutions and generalized Hyers-Ulam-Rassias stability of the following function equation

$$f(x-\sum_{i=1}^{k}x_i)+(k-1)f(x)+(k-1)\sum_{i=1}^{k}f(x_i)=f(x-x_1)$$
$$+\sum_{i=2}^{k}f(x_i-x)+\sum_{i=1}^{k}\sum_{j=1,\ j>i}^{k}f(x_i+x_j),$$

for $k\geq 2$, on non-Archimedean Banach spaces. It will be proved that this equation is equivalent to the so-called quadratic functional equation.

AMS Mathematics Subject Classification : 39B22, 39B82, 46S10.
Key words and phrases : Hyers–Ulam–Rassias stability, Quadratic equation, Non-Archimedean norm.

1. introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation ϵ must be close to an exact solution of ϵ?"

If the problem accepts a solution, we say that equation ϵ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [37] in 1940.

We are given a group G and a metric group G' with metric $d(.,.)$. Given $\epsilon>0$, dose there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $d(f(xy),f(x)f(y))<\delta$ for all $x,y\in G$, then a homomorphism $h:G\to G'$ exists with $d(f(x),h(x))<\epsilon$, for all $x\in G$?

Received October 10, 2011. Revised March 5, 2012. Accepted March 7, 2012. *Corresponding author.

Ulam's problem was partially solved by Hyers [15] in 1941; Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \ x, y \in E_1,$$

where $\epsilon > 0$ is a constant. Then the limit $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists, for each $x \in E_1$, and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leq \epsilon, \tag{1}$$

for all $x \in E_1$.

In 1987, Th.M. Rassias [31], formulated and proved the following theorem, which implies Hyers'theorem as a special case; Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\epsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \tag{2}$$

for all $x, y \in E$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \frac{\epsilon\|x\|^p}{(1 - 2^{p-1})},$$

for all $x \in E$. The terminology Hyers-Ulam stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [12], [14], [16] and [26]. In 1994, P. Găvruta, [13], provided a further generalization of Th.M. Rassias's theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ in (2) by a general control function $\varphi(x, y)$ for the existence of a unique linear mapping .

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular every solution of the quadratic functional equation is said to be a quadratic mapping, see [30, 32]. It is well know that a function f between real vector spaces X and Y is quadratic if and only if there exists a unique symmetric bi-additive function B from $X \times X$ to Y such that $f(x) = B(x, x)$ for all $x \in X$.(see [2, 16, 27]).

A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [36] for mappings $f : X \to Y$ where X is a normed space and Y is a Banach space. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [5], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation. Borelli and Forti [3] generalized the stability result as follows: Let G be an Abelian group, and X a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x, y),$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that $\phi(x, y) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{4^{i+1}} < \infty$, for all $x, y \in G$. Then there exists a unique quadratic mapping $Q: G \to X$ with the property $\|f(x)-Q(x)\| \le \phi(x,x)$, for all $x \in G$. Stability of the quadratic and cubic functional also studied by many other authors in various cases (see for example [6], [17]-[25], and [33]).

Let X and Y be some given vector non-Archimedean spaces, and let $f : X \to Y$ be a given function. For any $k \ge 2$, define

$$\begin{aligned} Df(x,x_1,...,x_k) &:= f(x-\sum_{i=1}^{k} x_i) + (k-1)f(x) + (k-1)\sum_{i=1}^{k} f(x_i) - f(x-x_1) \\ &- \sum_{i=2}^{k} f(x_i - x) - \sum_{i=1}^{k} \sum_{j=1,\ j>i}^{k} f(x_i + x_j) \end{aligned}$$

where $x, x_i \in X,\ i = 0,...,k$. One can see that the quadratic function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ satisfies not only the following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{3}$$

but also

$$Df(x,x_1,...,x_k) := 0 \tag{4}$$

for all $x_i \in \mathbb{R}$. So it is natural that these functional equations are called quadratic.

Solutions and Hyers-Ulam-Rassias stability of the functional equation (4) has been studied in [23], for $k = 2$. Indeed it is prove that, for $k = 2$, (4) and (3) are equivalent. In Section 2, we shall prove this fact for any $k \ge 2$. Also in Section 3, by following some ideas from [28, 29], we establish the stability of (4) in the setting of non-Archimedean normed spaces. Stability of functional equations in non-Archimedean normed spaces was studied by many authors (see for example [1], [6]-[11], [20], [29] and [34, 35]).

For our purpose in Section3, we need some preliminaries on non-Archimedean normed spaces which is presented here.

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r+s| \le \max\{|r|,|s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) for any $r \in K, x \in X$, $\|rx\| = |r|\|x\|$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \le \max\{\|x\|,\|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} \qquad (n > m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

2. Solution of equation (4)

Throughout this section, X and Y are non-Archimedean vector space and non-Archimedean Banach space, respectively. The following theorem prove that the functional equation (4) is equivalent to the equation (3). That is every solution of the equation (4) is a quadratic function.

Theorem 1. *Let X and Y be common domain and range of the f's in the equations (3) and (4). Then the equation (4) is equivalent to (3).*

Proof. Suppose that the equation (4) hold for every $k \geq 2$. Then for $k = 2$ and $x, x_1, x_2 \in X$, we have

$$f(x - x_1 - x_2) + f(x) + f(x_1) + f(x_2) = f(x - x_1) + f(x_2 + x_3) + f(x_2 - x).$$

Then by Theorem 2.1 in [23], f satisfy in (3). Now, suppose a function $f : X \to Y$ satisfies (3). Then trivially f is even. Now, using mathematical induction, we are going to show that

$$Df(x, x_1, ..., x_k) = 0 \tag{5}$$

for any $k \geq 2$ and $x, x_1, x_2, ..., x_n \in X$. For $k = 2$, see the proof of Theorem 2.1 [23]. Suppose (5) is holds for $k - 1$, we prove that (5) is valid for any k. Let $x, x_1, x_2, ..., x_n \in X$ be given. For convenience let us consider

$$A_1 := (k-3)f(x) + (k-3)f(x_1) + (k-4)\sum_{i=2}^{k-1} f(x_i) + f(x_1 + x),$$

$$A_2 := \frac{f(x - \sum_{i=1}^{k} x_i)}{2} + \frac{f(x - \sum_{i=1}^{k-1} x_i + x_k)}{2},$$

$$A_3 := \sum_{i=2}^{k-1} f(x_i + x_k),$$

$$A_4 := \sum_{i=1}^{\frac{k-2}{2}} f(x_{2i} - x_{2i+1}) \qquad \text{when } k \text{ is even}$$

$$A_5 := \sum_{i=1}^{\frac{k-3}{2}} f(x_{2i} - x_{2i+1}) \qquad \text{when } k \text{ is odd}$$

By the assumption of induction and the fact that f is even, we have

$$f(x - x_1) + \sum_{i=2}^{k} f(x_i - x) + \sum_{i=1}^{k} \sum_{j=1,\ j>i}^{k} f(x_i + x_j)$$

$$
\begin{aligned}
&= f(x-\sum_{i=1}^{k-1}x_i)+(k-2)\sum_{i=1}^{k-1}f(x_i)+(k-2)f(x)\\
&+f(x_k-x)+f(x_1+x_k)+A_3\\
&= A_3+f(x-\sum_{i=1}^{k-1}x_i)+(k-2)\sum_{i=1}^{k-1}f(x_i)+(k-2)f(x)+f(x-x_k-x_1)\\
&+f(x)+f(x_k)+f(x_1)-f(x_1-x)\\
&= A_3+f(x-\sum_{i=1}^{k-1}x_i)+(k-2)\sum_{i=2}^{k-1}f(x_i)+(k-1)f(x)+(k-1)f(x_1)\\
&+f(x_k)+f(x-x_k-x_1)-f(x_1-x)\\
&= A_3+\frac{f(x-\sum_{i=1}^{k}x_i)}{2}+\frac{f(x-\sum_{i=1}^{k-1}x_i+x_k)}{2}+(k-3)(f(x)+f(x_1))\\
&+(k-4)\sum_{i=2}^{k-1}f(x_i)+2f(x)+2f(x_1)+2f(x_2)+...+2f(x_{k-1})+2f(x_k)\\
&+f(x-x_k-x_1)-f(x_1-x)\\
&= A_1+A_2+A_3+f(x_1-x)+f(x_2+x_3)+f(x_2-x_3)\\
&+2f(x_4)+...+2f(x_{k-1})+2f(x_k)+f(x-x_k-x_1)-f(x_1-x)\\
&= A_1+A_2+A_3+\frac{f(x-x_k-x_1-x_2-x_3)}{2}+\frac{f(x-x_k-x_1+x_2+x_3)}{2}\\
&+f(x_2-x_3)+2f(x_4)+...+2f(x_{k-1})+2f(x_k)\\
&= A_1+A_2+A_3+f(x-x_1-x_2-x_3-x_k)-\frac{f(x-x_k-x_1-x_2-x_3)}{2}\\
&+\frac{f(x-x_k-x_1+x_2+x_3)}{2}+f(x_2-x_3)+f(x_4+x_5)+f(x_4-x_5)\\
&2f(x_6)+2f(x_7)...+2f(x_{k-2})+2f(x_{k-1})+2f(x_k)\\
&= A_1+A_2+A_3+\frac{f(x-x_1-x_2-x_3-x_k-x_4-x_5)}{2}\\
&+\frac{f(x-x_1-x_2-x_3-x_k+x_4+x_5)}{2}-\frac{f(x-x_k-x_1-x_2-x_3)}{2}\\
&+\frac{f(x-x_k-x_1+x_2+x_3)}{2}+f(x_2-x_3)+f(x_4-x_5)+f(x_6+x_7)\\
&+f(x_6-x_7)+2f(x_8)+2f(x_9)...+2f(x_{k-2})+2f(x_{k-1})+2f(x_k)\\
&= A_1+A_2+A_3+f(x-x_1-x_2-x_3-x_4-x_5-x_k)\\
&-\frac{f(x-x_1-x_2-x_3-x_k-x_4-x_5)}{2}+\frac{f(x-x_1-x_2-x_3-x_k+x_4+x_5)}{2}\\
&-\frac{f(x-x_k-x_1-x_2-x_3)}{2}+\frac{f(x-x_k-x_1+x_2+x_3)}{2}+f(x_2-x_3)\\
&+f(x_4-x_5)+f(x_6+x_7)+f(x_6-x_7)+2f(x_8)+2f(x_9)+...+2f(x_{k-2})
\end{aligned}
$$

$$
\begin{aligned}
&+2f(x_{k-1})+2f(x_k)\\
&=A_1+A_2+A_3+\frac{f(x-\sum_{i=1}^{7}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{5}x_i+x_6+x_7-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{5}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}\\
&+f(x_2-x_3)+f(x_4-x_5)+f(x_6-x_7)+f(x_8+x_9)+f(x_8-x_9)\\
&+2f(x_{10})+...+2f(x_{k-2})+2f(x_{k-1})+2f(x_k)=\mathbf{I}. \qquad (6)
\end{aligned}
$$

Now if k is even then continuing this process, we have

$$
\begin{aligned}
&\mathbf{I}=A_1+A_2+A_3+\frac{f(x-\sum_{i=1}^{7}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{5}x_i+x_6+x_7-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{5}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}\\
&+f(x_2-x_3)+f(x_4-x_5)+f(x_6-x_7)+f(x_8+x_9)+f(x_8-x_9)+...\\
&+f(x_{k-2}+x_{k-1})+f(x_{k-2}-x_{k-1})+2f(x_k)\\
&=A_1+A_2+A_3+\frac{f(x-\sum_{i=1}^{k-1}x_i-x_k)}{2}\\
&+\frac{f(x-\sum_{i=1}^{k-3}x_i+x_{k-2}+x_{k-1}-x_k)}{2}-\frac{f(x-\sum_{i=1}^{k-3}x_i-x_k)}{2}\\
&+\frac{f(x-\sum_{i=1}^{k-5}x_i+x_{k-4}+x_{k-3}-x_k)}{2}-\frac{f(x-\sum_{i=1}^{k-5}x_i-x_k)}{2}\\
&+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}\\
&+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+A_4+2f(x_k)\\
&=A_1+A_3+A_4+\frac{f(x-\sum_{i=1}^{k}x_i)}{2}+\frac{f(x-\sum_{i=1}^{k-3}x_i-x_{k-1}-x_{k-2}+x_k)}{2}\\
&+\frac{f(x-\sum_{i=1}^{k-1}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-3}x_i+x_{k-2}+x_{k-1}-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{k-3}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-5}x_i+x_{k-4}+x_{k-3}-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{k-5}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+2f(x_k)\\
&=A_1+A_3+A_4+f(x-\sum_{i=1}^{k}x_i)+f(x-\sum_{i=1}^{k-3}x_i)+f(x_k-x_{k-2}-x_{k-1})
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{f(x-\sum_{i=1}^{k-3}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-5}x_i+x_{k-4}+x_{k-3}-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{k-5}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+2f(x_k)\\
&=A_1+A_3+A_4+f(x-\sum_{i=1}^{k}x_i)+\frac{f(x-\sum_{i=1}^{k-3}x_i-x_k)}{2}\\
&+\frac{f(x-\sum_{i=1}^{k-5}x_i-x_{k-4}-x_{k-3}+x_k)}{2}-f(x_k)+f(x_k-x_{k-2}-x_{k-1})\\
&-\frac{f(x-\sum_{i=1}^{k-3}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-5}x_i+x_{k-4}+x_{k-3}-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{k-5}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}\\
&-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+2f(x_k)\\
&=A_1+A_3+A_4+f(x-\sum_{i=1}^{k}x_i)+f(x-\sum_{i=1}^{k-5}x_i)+f(x_k-x_{k-4}-x_{k-3})\\
&-f(x_k)+f(x_k-x_{k-2}-x_{k-1})-\frac{f(x-\sum_{i=1}^{k-5}x_i-x_k)}{2}+...\\
&+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}\\
&+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+2f(x_k)\\
&=A_1+A_3+A_4+f(x-\sum_{i=1}^{k}x_i)+\frac{f(x-\sum_{i=1}^{k-5}x_i+x_k)}{2}\\
&+\frac{f(x-\sum_{i=1}^{k-5}x_i-x_k)}{2}-f(x_k)+f(x_k-x_{k-4}-x_{k-3})\\
&-f(x_k)+f(x_k-x_{k-2}-x_{k-1})-\frac{f(x-\sum_{i=1}^{k-5}x_i-x_k)}{2}\\
&+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}\\
&+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+2f(x_k)=\ldots=A_1+A_3+A_4\\
&+f(x-\sum_{i=1}^{k}x_i)+\frac{f(x-\sum_{i=1}^{3}x_i+x_k)}{2}-\frac{(k-4)}{2}f(x_k)\\
&+f(x_k-x_4-x_5)+f(x_k-x_6-x_7)+...+f(x_k-x_{k-2}-x_{k-1})\\
&+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+2f(x_k)\\
&=A_1+A_3+A_4+f(x-\sum_{i=1}^{k}x_i)+f(x-x_1)-\frac{(k-4)}{2}f(x_k)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{\frac{k-2}{2}} f(x_k - x_{2i} - x_{2i+1}) + 2f(x_k) \\
&= A_1 + A_3 + A_4 + f(x - \sum_{i=1}^{k} x_i) + f(x - x_1) - \frac{(k-4)}{2} f(x_k) \\
&+\sum_{i=1}^{\frac{k-2}{2}} \big[f(x_k - x_{2i}) + f(x_k - x_{2i+1}) + f(x_{2i} + x_{2i+1}) \\
&-f(x_k) - f(x_{2i}) - f(x_{2i+1})\big] + 2f(x_k) \\
&= A_1 + A_3 + f(x - \sum_{i=1}^{k} x_i) - \frac{(k-4)}{2} f(x_k) + f(x - x_1) + \sum_{i=2}^{k-1} f(x_k - x_i) \\
&-\frac{(k-2)}{2} f(x_k) - \sum_{i=2}^{k-1} f(x_i) + \sum_{i=1}^{\frac{k-2}{2}} f(x_{2i} - x_{2i+1}) + \sum_{i=1}^{\frac{k-2}{2}} f(x_{2i} - x_{2i+1}) + 2f(x_k) \\
&= f(x - \sum_{i=1}^{k} x_i) + f(x - x_1) + \sum_{i=1}^{\frac{k-2}{2}} \big[f(x_{2i} + x_{2i+1}) + f(x_{2i} - x_{2i+1})\big] \\
&-\sum_{i=2}^{k-1} f(x_i) - (k-3) f(x_k) + (k-3) f(x) + (k-3) f(x_1) \\
&+(k-4)\sum_{i=2}^{k-1} f(x_i) + f(x_1 + x) + 2f(x_k) + \sum_{i=2}^{k-1} \big[f(x_i + x_k) + f(x_k - x_i)\big] \\
&= f(x - \sum_{i=1}^{k} x_i) + 2f(x) + 2f(x_1) + 2\sum_{i=2}^{k-1} (f(x_k) + f(x_i)) \\
&+2\sum_{i=1}^{\frac{k-2}{2}} \big[f(x_2 i) + f(x_{2i+1})\big] - \sum_{i=2}^{k-1} f(x_i) - (k-3) f(x_k) + (k-3) f(x) \\
&+(k-3) f(x_1) + (k-4)\sum_{i=2}^{k-1} f(x_i) + 2f(x_k) \\
&= f(x - \sum_{i=1}^{k} x_i) + (k-1) f(x) + (k-1)\sum_{i=1}^{k-1} f(x_i).
\end{aligned}
$$

If k is odd continuing the process in (6), we have

$$
\begin{aligned}
&\mathbf{I} = A_1 + A_2 + A_3 + \frac{f(x - \sum_{i=1}^{7} x_i - x_k)}{2} + \frac{f(x - \sum_{i=1}^{5} x_i + x_6 + x_7 - x_k)}{2} \\
&-\frac{f(x - \sum_{i=1}^{5} x_i - x_k)}{2} + \frac{f(x - \sum_{i=1}^{3} x_i + x_4 + x_5 - x_k)}{2} \\
&-\frac{f(x - \sum_{i=1}^{3} x_i - x_k)}{2} + \frac{f(x - x_1 + x_2 + x_3 - x_k)}{2} + f(x_2 - x_3) \\
&+f(x_4 - x_5) + f(x_6 - x_7) + f(x_8 + x_9) + f(x_8 - x_9) + ... + f(x_{k-3} + x_{k-2}) \\
&+f(x_{k-3} - x_{k-2}) + 2f(x_{k-1}) + 2f(x_k) = \ldots = A_1 + A_2 + A_3 + A_5
\end{aligned}
$$

$$+\frac{f(x-\sum_{i=1}^{k-2}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-4}x_i+x_{k-3}+x_{k-2}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-6}x_i+x_{k-5}+x_{k-4}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-6}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-8}x_i+x_{k-7}+x_{k-6}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-8}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+2f(x_{k-1})+2f(x_k)$$
$$=A_1+A_2+A_3+A_5+\frac{f(x-\sum_{i=1}^{k}x_i)}{2}+\frac{f(x-\sum_{i=1}^{k-2}x_i+x_{k-1}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-2}-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-4}x_i+x_{k-3}+x_{k-2}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-6}x_i+x_{k-5}+x_{k-4}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-6}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-8}x_i+x_{k-7}+x_{k-6}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-8}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+f(x_{k-1})+2f(x_k)$$
$$=A_1+A_3+A_5+\frac{f(x-\sum_{i=1}^{k}x_i)}{2}+\frac{f(x-\sum_{i=1}^{k-2}x_i-x_{k-1}+x_k)}{2}$$
$$+\frac{f(x-\sum_{i=1}^{k}x_i)}{2}+\frac{f(x-\sum_{i=1}^{k-2}x_i+x_{k-1}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-2}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-4}x_i+x_{k-3}+x_{k-2}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+f(x_{k-1})+2f(x_k)$$
$$=A_1+A_3+A_5+f(x-\sum_{i=1}^{k}x_i)+f(x-\sum_{i=1}^{k-2}x_i)+f(x_{k-1}-x_k)$$
$$-\frac{f(x-\sum_{i=1}^{k-2}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-4}x_i+x_{k-3}+x_{k-2}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+f(x_{k-1})+2f(x_k)$$
$$=A_1+A_3+A_5+f(x-\sum_{i=1}^{k}x_i)+\frac{f(x-\sum_{i=1}^{k-2}x_i+x_k)}{2}$$

$$+\frac{f(x-\sum_{i=1}^{k-2}x_i-x_k)}{2}-f(x_k)+f(x_{k-1}-x_k)-\frac{f(x-\sum_{i=1}^{k-2}x_i-x_k)}{2}$$
$$+\frac{f(x-\sum_{i=1}^{k-4}x_i+x_{k-3}+x_{k-2}-x_k)}{2}-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+...$$
$$+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}$$
$$+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+f(x_{k-1})+2f(x_k)$$
$$=A_1+A_3+A_5+f(x-\sum_{i=1}^{k}x_i)+\frac{f(x-\sum_{i=1}^{k-4}x_i-x_{k-3}-x_{k-2}+x_k)}{2}$$
$$-f(x_k)+f(x_{k-1}-x_k)+\frac{f(x-\sum_{i=1}^{k-4}x_i+x_{k-3}+x_{k-2}-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+f(x_{k-1})+2f(x_k)$$
$$=A_1+A_3+A_5+f(x-\sum_{i=1}^{k}x_i)+f(x-\sum_{i=1}^{k-4}x_i)+f(x_{k-3}+x_{k-2}-x_k)-f(x_k)$$
$$+f(x_{k-1}-x_k)-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}$$
$$-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+f(x_{k-1})+2f(x_k)$$
$$=A_1+A_3+A_5+f(x-\sum_{i=1}^{k}x_i)+\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}+\frac{f(x-\sum_{i=1}^{k-4}x_i+x_k)}{2}$$
$$-f(x_k)+f(x_{k-3}+x_{k-2}-x_k)-f(x_k)+f(x_{k-1}-x_k)-\frac{f(x-\sum_{i=1}^{k-4}x_i-x_k)}{2}$$
$$+...+\frac{f(x-\sum_{i=1}^{3}x_i+x_4+x_5-x_k)}{2}-\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}$$
$$+\frac{f(x-x_1+x_2+x_3-x_k)}{2}+f(x_{k-1})+2f(x_k)=\ldots=A_1+A_3+A_5$$
$$+f(x-\sum_{i=1}^{k}x_i)+\frac{f(x-\sum_{i=1}^{3}x_i-x_k)}{2}-\frac{k-3}{2}f(x_k)$$
$$+\sum_{i=2}^{\frac{k-3}{2}}f(x_{2i}+x_{2i+1}-x_k)+f(x_{k-1}-x_k)+\frac{f(x-x_1+x_2+x_3-x_k)}{2}$$
$$+f(x_{k-1})+2f(x_k)$$
$$=A_1+A_3+A_5+f(x-\sum_{i=1}^{k}x_i)+f(x-x_1)+f(x_2+x_3-x_k)$$
$$-\frac{k-3}{2}f(x_k)+\sum_{i=2}^{\frac{k-3}{2}}f(x_{2i}+x_{2i+1}-x_k)+f(x_{k-1}-x_k)+f(x_{k-1})+2f(x_k)$$

$$= A_1 + A_3 + A_5 + f(x - \sum_{i=1}^{k} x_i) + f(x - x_1) - \frac{k-3}{2} f(x_k)$$

$$+ \sum_{i=1}^{\frac{k-3}{2}} f(x_k - x_{2i} - x_{2i+1}) + f(x_{k-1} - x_k) + f(x_{k-1}) + 2f(x_k)$$

$$= A_1 + A_3 + A_5 + f(x - \sum_{i=1}^{k} x_i) + f(x - x_1) - \frac{k-3}{2} f(x_k)$$

$$+ \sum_{i=1}^{\frac{k-3}{2}} \big[f(x_k - x_{2i}) + f(x_k - x_{2i+1}) + f(x_{2i} + x_{2i+1}) - f(x_k)$$

$$- f(x_{2i}) - f(x_{2i+1}) \big] + f(x_{k-1} - x_k) + f(x_{k-1}) + 2f(x_k)$$

$$= f(x - \sum_{i=1}^{k} x_i) + f(x - x_1) - \frac{k-3}{2} f(x_k) + \sum_{i=2}^{k-2} f(x_k - x_i)$$

$$+ \sum_{i=1}^{\frac{k-3}{2}} f(x_{2i} + x_{2i+1}) - \frac{k-3}{2} f(x_k) - \sum_{i=2}^{k-2} f(x_i) + (k-3) f(x) + (k-3) f(x_1)$$

$$+ (k-4) \sum_{i=2}^{k-1} f(x_i) + f(x_1 + x) + \sum_{i=2}^{k-2} f(x_i + x_k) + f(x_{k-1} + x_k)$$

$$+ \sum_{i=1}^{\frac{k-3}{2}} f(x_{2i} - x_{2i+1}) + f(x_{k-1} - x_k) + f(x_{k-1}) + 2f(x_k)$$

$$= f(x - \sum_{i=1}^{k} x_i) + f(x - x_1) - (k-3) f(x_k) + \sum_{i=2}^{k-2} \big[f(x_k - x_i) + f(x_k + x_i) \big]$$

$$+ \sum_{i=1}^{\frac{k-3}{2}} \big[f(x_{2i} + x_{2i+1}) + f(x_{2i} - x_{2i+1}) \big] - \sum_{i=2}^{k-2} f(x_i) + (k-3) f(x)$$

$$+ (k-3) f(x_1) + (k-4) \sum_{i=2}^{k-1} f(x_i) + f(x_1 + x) + f(x_{k-1} + x_k) + f(x_{k-1} - x_k)$$

$$+ f(x_{k-1}) + 2f(x_k) = f(x - \sum_{i=1}^{k} x_i) - (k-3) f(x_k) + 2 \sum_{i=2}^{k-2} \big[f(x_k) + f(x_i) \big]$$

$$+ 2 \sum_{i=1}^{\frac{k-3}{2}} \big[f(x_{2i}) + f(x_{2i+1}) \big] + 2(f(x) + f(x_1)) + 2(f(x_{k-1}) + f(x_k))$$

$$- \sum_{i=1}^{k-2} f(x_i) + (k-3) f(x) + (k-3) f(x_1) + (k-4) \sum_{i=2}^{k-1} f(x_i)$$

$$+ f(x_{k-1}) + 2f(x_k) = f(x - \sum_{i=1}^{k} x_i) + (k-1) f(x) + (k-1) \sum_{i=1}^{k-1} f(x_i).$$

Thus (3) and (4) are equivalent. □

3. Hyers-Ulam-Rassias stability of the equation (4)

In this section, we assume that X and Y are a vector space and a complete non-Archimedean normed space, respectively.

Theorem 2. *Let $k \geq 2$ and let $\varphi : \underbrace{X \times X \times ... \times X}_{k+1-times} \to [0, \infty)$ be a mapping such that with $\psi(x, y) := \varphi(x, y, -y, 0, ..., 0)$,*

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{|4|^n} = 0$$

$$\tilde{\psi}(x, y) := \lim_{n\to\infty} \max\{ \frac{\psi(2^i x, 2^i y)}{|4|^i}, \ 0 \leq i < n\} < \infty \tag{7}$$

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\{ \frac{\psi(2^i x, 2^i y)}{|4|^i}, \ k \leq i < n + k\} = 0$$

or

$$\lim_{n\to\infty} |4|^{n+1} \psi(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}) = 0$$

$$\tilde{\psi}(x, y) = \lim_{n\to\infty} \max\{ |4|^{i+1} \psi(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}), \ 0 \leq i < n\} < \infty \tag{8}$$

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\{ |4|^{i+1} \psi(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}), \ k \leq i < n + k\} = 0.$$

Suppose that the function $f : X \to Y$ satisfy

$$\|Df(x, x_1, x_2, ..., x_k)\| \leq \varphi(x, x_1, ..., x_k) \tag{9}$$

for all $x, x_i \in X$, $i = 0, ..., k$. Then there exists exactly one function $Q : X \to Y$ that satisfies the equation (4) and with conditions , for any $x \in X$,

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|} \max\{M' \tilde{\psi}(0, 0), \frac{1}{|k-1|} \tilde{\psi}(x, 0), \frac{1}{|k-1|} \tilde{\psi}(2x, 0), \tilde{\psi}(x, x)\}, \tag{10}$$

where $M = \max\{1, \frac{|k-2|}{|k-1|}, \frac{|4|}{|k(k-1)|}, \frac{1}{|k-1|}\}$, $M' = \max\{M, \frac{|2|}{|k(k-1)|}\}$ and the function Q is given by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, & \text{if } \varphi \text{ satisfies (2)} \\ \lim_{n\to\infty} 4^n f(\frac{x}{2^n}), & \text{if } \varphi \text{ satisfies (8).} \end{cases}$$

Proof. Suppose that φ satisfies (8). Let x, y be elements of X. From (9) we have

$$\|\frac{k(k-1)}{2} f(0)\| \leq \varphi(0, ..., 0) \tag{11}$$

$$\|(k-1)f(x) - (k-1)f(-x) + \frac{k(k-1)}{2} f(0)\| \leq \varphi(x, 0, ..., 0) \tag{12}$$

These relations imply that

$$\|f(x) - f(-x)\| \leq \max\{\frac{\varphi(0, ..., 0)}{|k-1|} + \frac{\varphi(x, 0, ..., 0)}{|k-1|}\}. \tag{13}$$

Then by equation (13) we have

$$\|f(x+x_1)-f(-x-x_1)\| \le \max\{\frac{\varphi(0,...,0)}{|k-1|}+\frac{\varphi(x+x_1,0,...,0)}{|k-1|}\}. \tag{14}$$

Also by (9) we get

$$\begin{aligned}&\|f(x-x_1-x_2)+f(x)+f(x_1)+f(x_2)+(k-2)f(x)-(k-2)f(-x)\\&-f(x-x_1)-f(x_2-x)-f(x_1+x_2)+\frac{k(k-1)}{2}f(0)-f(0)\|\\&\le \varphi(x,x_1,x_2,0,...,0).\end{aligned} \tag{15}$$

From (11),(13) and (15), we get

$$\begin{aligned}&\|f(x-x_1-x_2)+f(x)+f(x_1)+f(x_2)-f(x-x_1)-f(x_2-x)\\&-f(x_1+x_2)-f(0)\| \le \max\{\varphi(x,x_1,x_2,0,...,0),\varphi(0,...,0),\\&\frac{|k-2|}{|k-1|}\ \varphi(0,...,0),\frac{|k-2|}{|k-1|}\ \varphi(x,0,...,0)\}.\end{aligned} \tag{16}$$

By putting $x_2=-x_1$ in (16) and applying (11) we get

$$\begin{aligned}&\|2f(x)+f(x_1)+f(-x_1)-f(x-x_1)-f(-x_1-x)\|\\&\le \max\{\varphi(x,x_1,-x_1,0,...,0),\varphi(0,...,0),\frac{|k-2|}{|k-1|}\ \varphi(0,...,0),\\&\frac{|k-2|}{|k-1|}\ \varphi(x,0,...,0),\frac{|4|}{|k(k-1)|}\ \varphi(0,...,0)\}.\end{aligned} \tag{17}$$

From (13) and (17) we have,

$$\begin{aligned}&\|2f(x)+2f(x_1)-f(x-x_1)-f(-x_1-x)\| \le \max\{\varphi(x,x_1,-x_1,0,...,0),\\&\varphi(0,...,0),\frac{|k-2|}{|k-1|}\ \varphi(0,...,0),\frac{|k-2|}{|k-1|}\ \varphi(x,0,...,0),\\&\frac{|4|}{|k(k-1)|}\ \varphi(0,...,0),\frac{\varphi(x_1,0,...,0)}{|k-1|},\frac{\varphi(0,...,0)}{|k-1|}\}.\end{aligned} \tag{18}$$

From (14) and (18) we get

$$\begin{aligned}&\|2f(x)+2f(x_1)-f(x+x_1)-f(x-x_1)\| \le \max\{\varphi(x,x_1,-x_1,0,...,0),\\&M\varphi(0,...,0),\frac{|k-2|}{|k-1|}\ \varphi(x,0,...,0),\frac{\varphi(x_1,0,...,0)}{|k-1|},\frac{\varphi(x+x_1,0,...,0)}{|k-1|}\}.\end{aligned} \tag{19}$$

With $\psi(x,y):=\frac{\varphi(x,y,-y,0,...,0)}{2}$, from (19), we have

$$\begin{aligned}&\|2f(x)+2f(x_1)-f(x+x_1)-f(x-x_1)\|\\&\le \max\{\psi(x,x_1),M\ \psi(0,0),\frac{|k-2|}{|k-1|}\ \psi(x,0),\frac{\psi(x_1,0)}{|k-1|},\frac{\psi(x+x_1,0)}{|k-1|}\}.\end{aligned} \tag{20}$$

Now by letting $x_1=x$ in (20) and from (11) we get

$$\|f(2x)-4f(x)\| \le \max\{\psi(x,x),M\ \psi(0,0),$$

$$\frac{1}{|k-1|}\ \psi(x,0), \frac{1}{|k-1|}\psi(2x,0), \frac{|2|}{|k(k-1)|}\psi(0,0)\}. \tag{21}$$

Thus

$$\|f(x)-4f(\frac{x}{2})\| \leq \max\{\psi(\frac{x}{2},\frac{x}{2}), M'\ \psi(0,0), \frac{1}{|k-1|}\ \psi(\frac{x}{2},0), \frac{1}{|k-1|}\psi(x,0)\}. \tag{22}$$

Replacing x by $\frac{x}{2^n}$ in (22) and multiplying it by $|4|^n$ in (22), we have

$$\|4^n f(\frac{x}{2^n}) - 4^{n+1} f(\frac{x}{2^{n+1}})\| \leq \max\{|4|^n \psi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}), |4|^n M'\ \psi(0,0),$$
$$\frac{|4|^n}{|k-1|}\ \psi(\frac{x}{2^{n+1}},0), \frac{|4|^n}{|k-1|}\psi(\frac{x}{2^n},0)\} = \frac{1}{|4|}\max\{|4|^{n+1}\psi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}),$$
$$|4|^{n+1}M'\ \psi(0,0), \frac{|4|^{n+1}}{|k-1|}\ \psi(\frac{x}{2^{n+1}},0), \frac{|4|^{n+1}}{|k-1|}\psi(\frac{x}{2^n},0)\}.$$

From condition (8), we conclude that the sequence $\{4^n f(\frac{x}{2^n})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y,and so converges in Y for all $x \in X$, since Y is complete. So we can define the mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$.
Similarly for any $m, n \in \mathbb{N}$, $m < n$, we get

$$\|4^m f(\frac{x}{2^m}) - 4^n f(\frac{x}{2^n})\| \leq \frac{1}{|4|}\max\{|4|^{i+1}\psi(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}),$$
$$|4|^{i+1}M'\ \psi(0,0), \frac{|4|^{i+1}}{|k-1|}\ \psi(\frac{x}{2^{i+1}},0), \frac{|4|^{i+1}}{|k-1|}\psi(\frac{2x}{2^{i+1}},0),\ m \leq i < n\}.$$

Letting $m = 0$ and passing the limit $n \to \infty$ in this equation, we get

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|}\max\{M'\tilde{\psi}(0,0), \frac{1}{|k-1|}\tilde{\psi}(x,0), \frac{1}{|k-1|}\tilde{\psi}(2x,0), \tilde{\psi}(x,x)\}.$$

This prove (10) with condition (8).

Now using definition of Q and condition (8) and putting $x_1 = y$ in (20), one can easily show that

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y),\ x, y \in X.$$

On the other hand it follows from Theorem (1) that for any $k \geq 2$,

$$Q(x - \sum_{i=1}^{k} x_i) + (k-1)Q(x) + (k-1)\sum_{i=1}^{k} Q(x_i) = Q(x - x_1) + \sum_{i=2}^{k} Q(x_i - x)$$
$$+ \sum_{i=1}^{k}\sum_{j=1,\ j>i}^{k} Q(x_i + x_j),$$

To prove the uniqueness of Q, let $T : X \to Y$ be another quadratic mapping satisfying (10). From the condition (8) we get

$$\lim_{k\to\infty} |4|^k \tilde{\psi}(\frac{x}{2^k},\frac{x}{2^k}) = \lim_{k\to\infty}\lim_{n\to\infty} \max\{\ |4|^{i+1}\psi(\frac{x}{2^{i+1}},\frac{y}{2^{i+1}}),\ k \leq i < n+k\} = 0$$

Since Q and T are quadratic mappings, (20) implies that

$$\|Q(x)-T(x)\| = \lim_{k\to\infty} |4|^k \|f(\frac{x}{2^k}) - T(\frac{x}{2^k})\|$$
$$\leq \lim_{k\to\infty} \frac{|4|^k}{|4|} \max\{M'\tilde{\psi}(0,0), \frac{1}{|k-1|}\tilde{\psi}(\frac{x}{2^k},0), \frac{1}{|k-1|}\tilde{\psi}(\frac{2x}{2^k},0), \tilde{\psi}(\frac{x}{2^k},\frac{x}{2^k})\}$$

for all $x \in X$. So $Q = T$. Now suppose φ satisfies the condition (2), then one can easily see that $\tilde{\psi}(0,0) = 0$. From (21),

$$\|\frac{f(2x)}{4} - f(x)\| \leq \max\{\frac{1}{|4|}\psi(x,x), \frac{M}{|4|}\psi(0,0),$$
$$\frac{1}{|4||k-1|}\psi(x,0), \frac{1}{|4||k-1|}\psi(2x,0), \frac{1}{|2||k(k-1)|}\psi(0,0)\}. \quad (23)$$

Replacing x by $2^n x$ in (23) and multiplying it by $\frac{1}{|4|^n}$, we have

$$\|\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}\| \leq \frac{1}{|4|}\max\{\frac{1}{|4|^n}\psi(2^n x, 2^n x), \frac{M}{|4^n|}\psi(0,0),$$
$$\frac{1}{|k-1||4^n|}\psi(2^n x,0), \frac{1}{|4^n||k-1|}\psi(2^{n+1}x,0), \frac{1}{|2||k(k-1)||4^n|}\psi(0,0)\}$$

which tends to zero, by (2), when $n \to \infty$. Thus $\{\frac{f(2^n x)}{4^n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete non-Archimedean normed space Y and so is convergent. Hence for any $x \in X$,

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

exists. Also for $m, n \in \mathbb{N}$, we get

$$\|\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}\| \leq \frac{1}{|4|} max\{\frac{1}{|4^i|}\psi(2^i x, 2^i x), \frac{M}{|4^i|}\psi(0,0),$$
$$\frac{1}{|k-1||4^i|}\psi(2^i x,0), \frac{1}{|4^i||k-1|}\psi(2^{i+1}x,0), \frac{1}{|2||k(k-1)||4^i|}\psi(0,0) : m \leq i < n\}.$$

Letting $m = 0$ and passing the limit $n \to \infty$ in this equation, we get (10). The other parts of proof is similar to the case that (8) is valid. □

Corollary 1. *Let $k \in \mathbb{N}$, $\epsilon, p \in \mathbb{R}$, $p \neq 2$,$\epsilon > 0$and $|2| < 1$. Suppose that the function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, x_1, ..., x_k)\| \leq \epsilon(\|x\|^p + \sum_{i=1}^{k} \|x_i\|^p). \quad (24)$$

Then there exists a unique quadratic function $Q : X \to Y$ such that satisfies in (4)

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|}\max\{\frac{1}{|k-1|}\ \epsilon\|x\|^p,\ 3\ \epsilon\|x\|^p\},\ p > 2$$
$$\|f(x) - Q(x)\| \leq \frac{1}{|4|}\max\{\frac{|4|}{|k-1||2|^p}\ \epsilon\|x\|^p,\ \frac{3|4|}{|2|^p}\ \epsilon\|x\|^p\},\ p < 2.$$

Proof. Define $\phi : X^{n+1} \to [0,\infty)$ by

$$\phi(x, x_1, ..., x_n) = \epsilon(\|x\|^p + \sum_{i=1}^{k} \|x_i\|^p).$$

One can see that for $p < 2$, the conditions (2) is valid and for $2 < p < \infty$ the conditions (8) is satisfied. Now using Theorem 2, we may complete the proof. □

References

1. L.M. Arriola, and W.A. Beyer, *Stability of the Cauchy functional equation over p-adic fields*, Real Anal. Exchange Vol. 31 (2005/2006),125–132.
2. J. Aczel, and J., Dhombers, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.
3. C. Borelli and G.L. Forti, *On a genereal Hyers-Ulam stability result*, Internat. J. Math. Math. Sci. Vol. 18(1995), 229–236.
4. P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. Vol. 27(1984), 76–86.
5. S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, Vol. 62(1992), 59–64.
6. M. Eshaghi Gordji, and M.B. Savadkouhi, *Stability of Cubic and Quartic Functional Equations in Non-Archimedean Spaces*, Acta Applicandae Mathematicae, Vol. 110, (2010) No. 3, 1321–1329.
7. M. Eshaghi Gordji, R. Khodabakhsh, S.M. Jung and H. Khodaei, *AQCQ–functional equation in non–Archimedean normed spaces*, Abs. Appl. Anal., Vol. 2010, Article ID 741942, (2010)22 pages.
8. M. Eshaghi Gordji and M.B. Savadkouhi, *Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces*, Appl. Math. Lett. Vol. 23(2010) No.10, 1198–1202.
9. M. Eshaghi Gordji and M.B. Savadkouhi, *Stability of cubic and quartic functional equations in non-Archimedean spaces*, Acta Appl. Math. Vol. 110 (2010), 1321–1329.
10. M. Eshaghi Gordji, *Nearly ring homomorphisms and nearly ring derivations on non-Archimedean Banach algebras*, Abs. Appl. Anal., Vol. 2010, Article ID 393247, (2010)12 pages.
11. M. Eshaghi Gordji and Z. Alizadeh, *Stability and superstability of ring homomorphisms on non–Archimedean Banach algebras*, Abs. and Appl. Anal., Vol. 2011, Article ID:123656, (2011), 10 pages.
12. Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. Vol. 14 (1991), 431–434.
13. P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. appl. Vol. 184 (1994), 431–436.
14. A. Grabiec, *The generalized Hyers-Ulam-Rassias stability of a class of functional equations*,Publ. Math. Debrecen, Vol. 48 (1996), 217–235.
15. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA Vol. 27 (1941), 222–224.
16. D. H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables* Birkhäuer, Base *l.* 1998.
17. M. Janfada, R. Shourvarzi, *On solution and HyeresUlamRasstas stability of a generalized quadratic equation*, Inter. J. of Nonlin. Sci., Vol. 10 (2010) No. 2, 231–237.
18. M. Janfada, R. Shourvarzi, *Solutions and the generalized Hyers-Ulam-Rassias stability of a generalized quadratic-additive functional equation*, Abs. and Appl. Anal. Vol. 2011, Article ID 326951, (2011)19 pages.

19. M. Janfada, Gh. Sadeghi, *Generalized Hyeres–Ulam stability of a quadratic functional equation with involution in quasi-β-normed spaces*, J. Appl. Math. & Informatics Vol. 29 (2011) No. 5 - 6, 1421 –1433.
20. Y.J. Cho, C. Park, R. Saadati, *Functional inequalities in non-Archimedean Banach spaces*, Appl. Math. Lett. Vol. 23 (2010) No. 10, 1238–1242.
21. S.M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. Vol. 222 (1998), 126–137.
22. S.M. Jung, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl. Vol. 232 (1999), 384–393.
23. S.M. Jung, *Quadratic functional equations of pexider type*, Internat. J. Math. Math. Sci. Vol. 24(2000), 351–359.
24. S.M. Jung, *Quadratic functional equations of Pexider type*, Internat. J. Math. Math. Sci. Vol.24 (2000) , 351–359.
25. S.M. Jung, *Stability of the quadratic equation of Pexider type*, Abh. Math. Sem. Univ. Hamburg, Vol. 70(2000), 175–190.
26. S. M. Jung, *Hyers-Ulam-Rassias stability of Functional Equations in Mathematical Analysis*, Hardonic Press Inc. Palm Harbor, Florida. 2001.
27. Pl. Kannappan, *Quadratic functional equations and inner product spaces*, Result. Math. Vol. 27 (1995)No. 2, 368–372.
28. H.M. Kim and J.M. Rassias, *Generalization of the Ulam stability problem for Euler–Lagrange quadratic mapping*, J. Math. Anal. Appl. Vol. 336 (2007), 277–296.
29. M.S. Moslehian and Th.M. Rassias, *Stability of functional equations in non-Archimedian spaces*, *Appl. Anal. Disc. Math.* Vol. 1 (2007), 325–334.
30. Th.M. Rassias, *Functional Equations and Inequalities*, Kluwer Academic, Dordrecht, 2000.
31. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. Vol. 72 (1978), 297–300.
32. Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai. Vol. 43 (1998), 89–124.
33. Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. Vol. 62 (2000), 23–130.
34. R. Saadati, C. Park, *Non-Archimedean $\mathcal{L}$-fuzzy normed spaces and stability of functional equations* Comput. Math. Appl. Vol. 60 (2010) No. 8, 2488–2496.
35. Gh. Sadeghi, R. Saadati, M. Janfada, J.M. Rassias, *Stability of Euler-Lagrange quadratic functional equations in non-archimedean normed space*, Hacet. J. Math. Stat. Vol. 40(2011) No. 4, 571–579 .
36. F. Skof, *Proprietá localie approssimazione dioperatori*, Rend. Sem. Mat. Fis. Milano. Vol. 53 (1983), 113–129.
37. S.M. Ulam, *A Collection of Mathematical Problem* Interscience Tracts in Pure and Applied Mathematics, intercience Pubisher, New York, 1960.

Mohammad Janfada received M.Sc. from Ferdowsi University of Mashhad and Ph.D at Ferdowsi University of Mashhad. He is now an associate professor in Ferdowsi University of Mashhad. His research interests include theorey of semigroup of operators, operator theory, functionl equations and stability.

Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, P.O. Box 1159-91775, Iran.
e-mail: mjanfada@gmail.com

Rahele Shourvarzi received M.Sc. from Sabzevar Tarbiat Moallem University, Iran at 2011. Her research interests are functional equations, stability.

Department of Mathematics, Sabzevar Tarbiat Moallem University, Sabzevar, P.O. Box 397, Iran.
e-mail: rshurvarzy@gmail.com