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Estimating the robust domain of attraction and directional enlargement of attraction domain via Markov models

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SUMMARY

This paper proposes a new approach for estimating the robust domain of attraction (RDA) and directional enlargement of the DA for dynamical systems. The proposed method analyzes stability of dynamical systems by Markov modeling and employs invariant measure as the stability indicator. Markov chains analysis focuses on asymptotic behaviors of systems and ignores the transient ones. The proposed method expresses the problem of estimating RDA and directional enlargement of DA as an infinite dimensional linear problem. The resulting linear problem is converted to a finite dimensional optimization problem using approximated Markov transition function. As a novel application, the directional enlargement of DA is used in order to increase the critical clearing time of power systems. The efficiency of proposed methods is shown via simulations. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, we study the dynamic behavior of systems using Markov modeling. Two advantages of using Markov models for extracting dynamical behaviors are the following:

(i) Often, the statistical properties of the Markov model have closed forms and are easily numerically computable.
(ii) Using Markov chains, one can remove transient effects and only compute the asymptotic behavior of systems. So it takes less time than direct analysis of system orbits.

To transform the domain of attraction (DA) estimation to a finite dimensional optimization problem, the state space is divided into a number of subspaces, and the average of the probability of the state transition matrixes of these subspaces is calculated. A measure is required to indicate the aforementioned calculated average. The most popular measure used is the probability invariant measure. This measure is obtained using the distribution of the typical long trajectories of the system. In recent years, invariant measure has played an important role in the characterization of dynamical systems. It is an approximate tool to determine behaviors of dynamical systems. It is effectively used for detecting invariant sets or cyclic behavior of nonlinear systems [1].

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Our work contains three main results.

(i) We propose a new algorithm to approximate the invariant measure from Markov transition matrix.

(ii) We focus on computing and analyzing robust DA (RDA) of uncertain nonlinear systems. As most physical systems have uncertain parameters, finding RDA that guarantees the stability for different values of uncertainty is important and of most interest. Calculating actual RDA remains an unsolved problem; however, the following solutions are suggested for it in recent literatures.

1. Estimating RDA via parameter dependant Lyapunov function (LF) [2, 3].
2. Finding a common LF to prove robust local stability [2].
3. Robust domain of attraction estimation through generalized Zubove's method.

All the aforementioned methods have limitations. Parameter dependant LF is applicable only for time-invariant uncertainties. In addition, there is no general LF structure. In most literature, quadratic LF is used leading to a conservative estimation of DA. Although RDA of systems with, probably time varying, uncertainty can be estimated through common LF, finding such a common LF in general is impossible. In the third method, the viscosity solution of straightforward generalization of classical Zubove's equation is used to characterize RDA of a nonlinear system with time varying perturbations [4]. To solve the Zubove's equation, method of characteristic is used. This method requires the solution of the nonlinear system, and in fact the knowledge of DA, which is in general impossible [5].

Our work overcomes these limitations using invariant measure as an approximating tool. Although the model with finite number of subspaces employed in this work has less information than the original system, this simplification allows computing some dynamical properties such as finding invariant sets, enlarging DA, and estimating RDA for a large class of nonlinear systems effectively.

(iii) We propose a method for directional DA enlargement. In systems with large DA, disturbance is not a serious problem and its affect is usually removed with typical controllers. But in power systems with bounded DA, increasing critical fault clearing time, the maximum allowable time to clear the fault and keep the system stable, is very important. Methods of increasing critical clearing time of power systems can be classified into two main groups: (i) decreasing the length of fault trajectories by using fast voltage regulators, breaking resistors [6], and/or line reclosing[7]; and (ii) designing controllers to enlarge DA of post-fault system [6, 8]. Different methods have been proposed to enlarge DA. Some of these methods expand DA but not along specific directions [9, 10] and others enlarge DA directionally for a special class of nonlinear systems [11].

This work contains four sections. In Section 2, needed definitions are summarized. Introducing the main idea of this work which is describing stability analysis according to Markov model of a system, DA directional extension, and RDA estimation by means of Markov chains and invariant measure are the subjects of Section 3. And finally, the results are simulated in Section 4.

2. PRELIMINARIES

Let Ω be an n-dimensional open rectangular set in R^n, equipped with Lebesgue measure μ on σ-algebra of Borel sets $\mathrm{B}(\Omega)$ and T be a measurable nonsingular transition operator [1] on the measurable space $(\Omega, \mathrm{B}, \mu)$ such that

$$\begin{gathered} X(k+1) = T(\,X(k)\,) \qquad T : \Omega \to \Omega,\ \ \Omega \subset R^n \\ X(k) = [x_1(k), \ldots, x_n(k)]^T, x_i(k+1) = T_i(X(k)) \end{gathered} \tag{1}$$

It is assumed that system (1) has isolated equilibrium points. The following definitions relate to system (1).

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Definition 1 (State-space partitioning)
$\mathcal{A}$ is a state-space partitioning for Ω if it divides Ω into sets (cells) $A_i \;\; i = 1, \ldots, N$ such that they satisfy the following two conditions.

$$\bigcup_{i=1}^{N} A_i = \Omega$$
$$\mathring{A}_i \cap \mathring{A}_j = \phi \;\; \forall i \neq j$$

where $\mathring{A}_i$ is the interior of A_i and ϕ is the empty set.

Definition 2 (Center of a partition)
Let $\mathcal{A}$ be a state-space partitioning for $\Omega \subset R^n$. For simplicity, we suppose rectangular partitions as $A_i = [l_{1i}, h_{1i}] \times \ldots \times [l_{ni}, h_{ni}] \;\; i = 1,.., N$. The center of each partition A_i is a point like $C_i = [c_{1i}, \ldots, c_{ni}]^T$, where $c_{ji} = \frac{h_{ji} - l_{ji}}{2}$.

Definition 3 (Long-term orbit)
For the discrete nonlinear system (1), the set $\{X(k) \in \Omega | \;\; X(k) = T(X(k-1)) = \ldots = T^k(X(0)) \;\; k = 0, 1, \ldots\}$ is called the long-term orbit of (1) with initial condition $X(0)$.

Definition 4 (Asymptotic stable equilibrium point [12]*)*
$X_e \in \Omega$ is an asymptotic stable equilibrium point of the discrete nonlinear system (1) if

(i) X_e is a fixed point with respect to transition operator T such that $T^k(X_e) = X_e$ for all $k \in \mathbf{N}$.
(ii) All orbits $\{X(k)\}$ starting sufficiently near X_e stay near X_e and converge to X_e, as $k \to \infty$.

Definition 5 (Domain of attraction)
Consider the nonlinear system (1). The DA of an asymptotic stable equilibrium point X_e is $DA_{X_e} = \{X(k) \in \Omega | \lim_{h \to \infty} T^h(X(k)) = X_e\}$.

Definition 6 (Robust domain of attraction)
Consider an uncertain nonlinear system, with an isolated equilibrium state X_e, of the following form

$$\begin{aligned} &X(k+1) = T_U(X(k), \beta) \qquad T_U : \Omega \times B \to \Omega, \;\; \Omega \subset R^n, \;\; B \subset R^p \\ &T_U(X_e, \beta) = X_e \;\; \forall \beta \in B \end{aligned} \tag{2}$$

where $\beta \in B$ is an uncertainty vector, B is a measurable compact set in R^p, and T_U is a nonsingular uncertain transition operator. The RDA of system (2) is defined as $RDA_{X_e} = \{X(k) \in \Omega | \lim_{h \to \infty} T_U^h(X(k), \beta) = X_e; \forall \beta \in B\}$.

Obviously, $T_U(X_e, \beta) = X_e \;\; \forall \beta \in B$ implies that in this paper, a class of nonlinear systems is considered which has at least one isolated equilibrium point that is not sensitive to the variation of parameters.

Definition 7 (Fault running vector)
Consider a nonlinear system with its state at its stable equilibrium point X_e. Call it pre-disturbance system. Consider further occurrence of a disturbance so large that, if not removed, moves the state of the system out of the DA of X_e, thus resulting in instability. The trajectory of the state of the system due to this disturbance, provided it is not removed, passes the closure of the DA of X_e at a point X_c. We are considering only the specific class of disturbances that if removed in time leaves the structure of the system unchanged. Therefore, the system after the in time removal of disturbance, post-disturbance system, is stable and will converge to the same pre-disturbance stable equilibrium point X_e. For such systems, we define the fault running vector as the vector in the direction of the straight line connecting X_e to X_c (See Figure 1 that shows the fault running vector for the Van der Pol oscillator).

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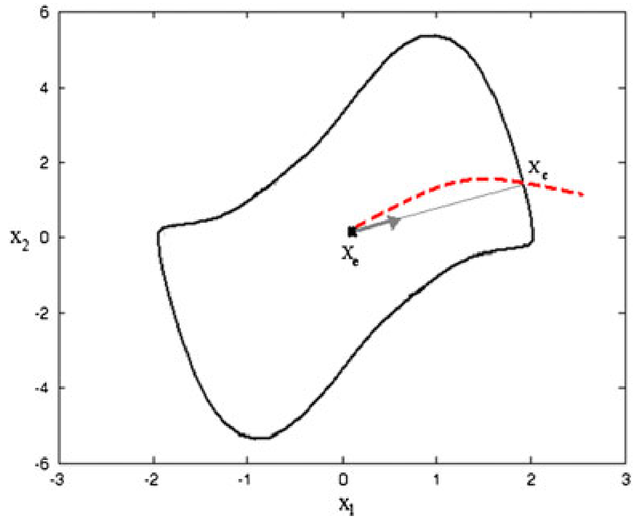

Figure 1. Fault running vector is in the direction of the line that contacts X_e to X_c. Dashed curve: state trajectory of distributed system; black curve: domain of attraction of pre-disturbance Van der Pol system.

An example of such a nonlinear system is the power system with a self-healing fault (disturbance) where the pre-fault and post-fault systems have the same structure and equilibrium state and where the trajectory of the fault-on system, if the fault is not removed in time, passes the DA of the equilibrium state.

3. MAIN RESULTS

We focus on the discrete dynamical system of equation (1). Results for the continuous-time systems can be deduced from the discrete ones. As we are concerned with estimating DA of systems (certain or uncertain), we should analyze their long-term orbits. Doing so is not practically possible in many systems because it takes a long time and may lead to computer round-off error. Therefore, in this paper, we use the method of Markov modeling of dynamical systems to remove the transient effects and calculate only the asymptotic behavior.

In the next part, we review some properties and theorems that are necessary for stability analysis of nonlinear systems via Markov models.

3.1. Stability analysis of Markov chains

Considering (1), $X(k+1)$ can be exactly obtained from $X(k)$ so we can construct a Markov chain for this system as follows [13].

$$\phi_{X_0} = \left\{ \phi_k | \quad \phi_k = X(k) = T^k(X(0)), \ \ 0 \leqslant k < n+1 \right\} \tag{3}$$

Definition 8
Let $X \in \Omega$ and $A \subset \Omega$. The n-step transition function, denoted by $p^n(X, A)$, is the probability that a Markov chain ϕ_{X_0} starting from an arbitrary point like $X_0 = X$ remains in the set A after n steps [12].

Proposition 1
For Markov chain (3), Markov transition function is proposed as $P(X, A) = \lim_{n\to\infty} p^n(X, A)$.

Proof
See [12, chapter 1, page 3]. □

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In the sequel (proposition 2), we show that the uniformly distributed $P(X, A)$ depends only on A. For analysis of the stability of the proposed Markov chain, we consider Theorem 1.

Theorem 1
The existence of a fixed point like X_e, which is asymptotically stable in the set $A \subset \Omega$, is exactly equal to the existence of a nonzero unique solution for the following invariant equation.

$$m(A) = \int_{\Omega} P(X, A)\ \mathrm{d}m(A)$$

Proof
See [12, chapter 1, page 20, asymptotic stability definition]. □

In the aforementioned theorem, $m \in M$ and M is the set of all probability Lebesgue measures on the topological space Ω.

Lemma 1
Closure of the DA of the equilibrium point X_e of the nonlinear system (1), $\overline{DA_{X_e}} \subset \Omega$, is the union of the members of support of probably measure m and obtained from following equation

$$\overline{DA_{X_e}} = SUPP\{m\}$$

where $SUPP\{m\} = \cup\{A|\ m(A) = \int_{\Omega} P(X, A)\ \mathrm{d}m(A) \neq 0\}$.

Proof
According to Theorem 1, every member of $SUPP\{m\}$ is asymptotically stable so it is contained in DA_{X_e} so it yields $\overline{DA_{X_e}} = SUPP\{m\}$. As A is a close set *SUPP*, $\{m\}$ is also close. □

3.2. Estimating domain of attraction

3.2.1. Proposed method for estimating domain of attraction. It is not practically possible to estimate DA of system (1) using Lemma 1 because it leads to an infinite dimensional problem in space M. In other words, because $DA \subset \Omega$, we should calculate $P(X, A) = \lim_{n \to \infty} p^n(X, A)$ for every $X \in \Omega$ which leads to an infinite dimensional problem. So, we use the idea of [10] and partition the state space Ω(according to Definition 1). Assuming that $P(X, A)$ has a uniform distribution, we calculate probability of transition of subspaces instead of calculating the probability of transition for every point $X \in \Omega$. So in the sequel, we convert the infinite dimensional problem of estimating DA, proposed in Lemma |1, to a finite dimensional one. To investigate the stability of state partitions, we use the discrete-time Markov chain that is a Markov process ϕ_n having a countable number of states A_n [13].

Definition 9 (Markov transition matrix)
Consider nonlinear system (1). For state-space partitioning $\mathcal{A}$ of Ω, the $N \times N$ Markov transition matrix P is defined as

$$P^{(n_1,n_2)} = \left[p_{ij}^{(n_1,n_2)}\right] = \left[prob\ \left[X(n_2) \in A_j \middle| X(n_1) \in A_i\right]\right];\ \sum_j p_{ij}^{(n_1,n_2)} = 1 \qquad (4)$$

Definition 10 (State probability vector)
The state probability vector for state-space partitioning $\mathcal{A}$ in nth transition is defined as $\vartheta(n) = (\vartheta_1(n), \ldots, \vartheta_N(n))$, where $\vartheta_i(n)$ is the probability of Markov chain to exist in A_i state in the nth transition [12]. In other words, $\vartheta_i(n) = prob\,[X(n) \in A_i]$ and $\sum_{i=1}^{N} \vartheta_i(n) = 1\ \ \forall n$ [13].

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In this paper, we assume the Markov chain related to system (1) to be

(i) A time-homogeneous Markov chain (or stationary Markov chains). This is a Markov chain for which $p_{ij}^{(n_1,n_2)}$ only depends on $m = n_2 - n_1$ [13], so we have

$$P^{(n_1,n_2)} = \left[p_{ij}^{(n_1,n_2)}\right] = \left[p_{ij}^{(m)}\right] = prob\,[X(n+m) \in A_j \,|\, X(n) \in A_i]; \quad \forall n \in \mathrm{N}$$

(ii) An irreducible chain which means that it is possible to get to any state (A_j) from any state (A_i).

(iii) Aperiodic or have at least one aperiodic state. The state i is said to be aperiodic if it returns to itself at irregular times or in the other words

$$\gcd\{n|\, prob[X(n+1) \in A_i \,|\, X(0) \in A_i] > 0\} = 1$$

where gcd is the greatest common divisor.

The aforementioned assumptions contain a large class of nonlinear systems. For these systems, the following definitions are considered

Definition 11 (n-step Markov transition matrix)
n-Step Markov transition matrix for a homogenous Markov process is defined as

$$P^{(n)} = \left[p_{ij}^{(n)}\right] = prob\,(X(k+n) \in A_j / X(k) \in A_i) \tag{5}$$

Proposition 2
For uniformly distributed $P(X, A)$, $p_{ij}^{(1)}$ can also be presented as

$$p_{ij}^{(1)} = \frac{m(T^{-1}(A_j) \cap A_i)}{m(A_i)} \quad i, j = 1, \ldots, N \tag{6}$$

Proof
See [10]. □

Proposition 3
For a homogeneous process, we have

(i)

$$P^{(n)} = P^n \tag{7}$$

(ii)

$$\vartheta(n) = \vartheta(0)P^{(n)} = \vartheta(0)P^n = \vartheta(n-1)P \tag{8}$$

Proof

(i) For a time-homogeneous Markov chain, we have $n_1 < n_2 < n_3 : \; p_{ij}^{(n_3-n_1)} = \sum_r p_{ir}^{(n_2-n_1)} p_{rj}^{(n_3-n_2)}$ (See Equations (16–110) of [13]).

Substituting $n_2 - n_1 = k$ and $n_3 - n_2 = n$ yields $p_{ij}^{(n+k)} = \sum_r p_{ir}^{(k)} p_{rj}^{(n)}$ which is equal to $P^{(n+k)} = P^{(k)} P^{(n)}$. This consequently yields $P^{(n)} = P^{(n-1)} P = P^{(n-2)} P^2 = P^n$.

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(ii) From Equations (7–48) or (16–109) of [13], we have $\sum_i \vartheta_i(k) p_{ij}^{(k,n)} = \vartheta_j(n)$. For a time-homogeneous stationary Markov chain, this yields $\vartheta(n) = \vartheta(n-k)P^{(k)}$. For $k = 1$, this is equal to $\vartheta(n) = \vartheta(n-1)P$. □

Proposition 4
For a stationary Markov process, the state probability vector ϑ does not depend on n and is called a **stationary distribution** (or **invariant measure**) vector.

Theorem 2 (Perron–Frobenius theorem)
For irreducible and aperiodic Markov chains, there exist a unique invariant measure vector ϑ. In addition, $P^{(n)}$ converges to a rank-one matrix in which each row is the stationary distribution ϑ that is

$$\lim_{n\to\infty} P^{(n)} = \mathbf{1}\vartheta$$

where **1** is the column vector with all entries equal to 1.

Proof
See[13]. □

Theorem 3
The (closure of) DA of nonlinear system (1) with N state partitioning $\mathcal{A}$ can be estimated from the support of invariant measure vector ϑ. Where ϑ is calculated from the following equations.

$$\begin{aligned} &\vartheta = P\vartheta \quad ; \vartheta = (\vartheta_1, \ldots, \vartheta_N) \\ &\sum_{i=1}^{N} \vartheta_i = 1 \end{aligned} \tag{9}$$

Proof
Propositions 3 and 4 imply that $\vartheta = P\vartheta$, where $\sum_{i=1}^{N} \vartheta_i = 1$ and $\vartheta = (\vartheta_1, \ldots, \vartheta_N)$ is unique (see Perron–Frobenius theorem). In addition, as ϑ_i is the probability that the Markov chain exists in state A_i, we can conclude that ϑ_i is a stability weight. In other words, $\vartheta_i = 0$ shows that orbits do not exist in A_i and leave this state so DA includes states with nonzero invariant measure or equally $\overline{DA}$ is the support of ϑ. □

3.2.2. Proposed analytic form of Markov matrix. Considering Theorem 3, to estimate DA, we should calculate Markov matrix. There are different numerical algorithms to calculate P matrix from equation (6) [see chapter 6 of reference 10]. In the sequel, we provide a new analytic formula to determine P which is more accurate; moreover, we use this analytic form to estimate RDA.

Proposition 5
Some useful properties of the (probability) Lebesgue measure m and characteristic function χ are as follows.

(a) $m(A \cap B) = \int_A \chi_B(X)\,\mathrm{d}X = \int_B \chi_A(X)\,\mathrm{d}X$.

Proof
From [14], we have $\chi_{(A\cap B)} = \chi_{(A)} \cdot \chi_{(B)}$, which yields

$$m(A \cap B) = \int_\Omega \chi_{A\cap B}(X)\,\mathrm{d}X = \int_\Omega \chi_A(X).\ \chi_B(X)\,\mathrm{d}X = \int_A \chi_B(X)\,\mathrm{d}X = \int_B \chi_A(X)\,\mathrm{d}X$$

(b) $\chi_{T^{-1}(A)}(X) = \chi_A(T(X))$.

□

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Proof
Because T is nonsingular, we have

$$\chi_{T^{-1}(A)}(X) = 1 \Leftrightarrow X \in T^{-1}(A) \Leftrightarrow T(X) \in A$$
$$\Leftrightarrow [T_1(X), \ldots, T_n(X)]^T \in A \Leftrightarrow \chi_{(A)}(T(X)) = 1$$

□

Lemma 2
For nonlinear system (1), with state-space partitioning $\mathcal{A}$, the Markov matrix can be represented by the following analytic form

$$p_{ij} = \frac{\int\limits_{\Omega} \prod\limits_{q=1}^{n} H\ [(T_q - l_{qj}).(h_{qj} - T_q)]\ \prod\limits_{k=1}^{n} H\ [(x_k - l_{ki}).(h_{ki} - x_k)]\,\mathrm{d}X}{S_{D_j}} \tag{10}$$

where

$$S_{D_j} = \prod_{k=1}^{n} (h_{kj} - l_{kj})\ \ , dX = dx_1 \ldots \mathrm{d}x_n$$
$$H(x) = \begin{cases} 1 & x > 0 \\ 0.5 & x = 0 \\ 0 & x < 0 \end{cases}$$

And the state space portioning $\mathcal{A}$ is chosen as $A_i = [l_{1i}, h_{1i}] \times \ldots \times [l_{ni}, h_{ni}]\ i = 1,.., N$.

Proof
From proposition 5a, the P matrix as defined in Proposition 2 can be expressed as

$$p_{ij} = \frac{m(T^{-1}(A_j) \cap A_i)}{m(A_i)} = \frac{\int\limits_{\Omega} \chi_{T^{-1}(A_j)}(X).\ \chi_{A_i}(X)\,\mathrm{d}X}{\int\limits_{\Omega} \chi_{A_i}(X)\,\mathrm{d}X}$$

Therefore, from Proposition 5b, we have

$$p_{ij} = \frac{\int\limits_{\Omega} \chi_{A_j}(T(X))\ .\ \chi_{A_i}(X)\,\mathrm{d}X}{\int\limits_{\Omega} \chi_{A_i}(X)\,\mathrm{d}X} \tag{11}$$

According to the characteristic function definition [13], an acceptable $\chi_{A_i}(X)$ for A_i set is

$$H\left(\prod_{k=1}^{n} [(x_k - l_{ki}).(h_{ki} - x_k)]\right) \tag{12}$$

Substituting (12) in (11) completes the proof. □

3.3. *Robust domain of attraction estimation*

In this section, we generalize the stability Theorem 1, defined in the previous section, for RDA estimation.

Although finding the exact RDA is a difficult problem, different ways are proposed in literatures to estimate it. Some of these methods choose arbitrary values for uncertainty and estimate DA for these fixed values and estimate RDA from the intersection of these DA sets. These methods are not reliable because they just study DA variations for special values in uncertainty bound. On the other

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hand, as these methods are based on intersecting DAs, they usually use simple LFs for estimating DA [2]. The Lyapunov-based algorithms, which use quadratic structures obtain a conservative estimate of RDA and the other algorithms such as those using generalized Zubove's method [4] are only applicable for a special class of nonlinear systems.

According to Theorem 3, we propose a new method for RDA approximation, which is convenient for a large class of nonlinear systems (with time-homogeneous aperiodic chains).

Theorem 4
Consider nonlinear system (2) with uncertain parameter β, then the support of ϑ_β provides the estimated closure of RDA, where ϑ_β is obtained through the following optimization formulation.

$$\begin{aligned} \vartheta_\beta &= \underset{\beta \in B}{Inf}\ \vartheta(\beta) \\ s.t.\ \vartheta(\beta) &= P(\beta)\vartheta(\beta) \\ \sum_{i=1}^{N} \vartheta_i(\beta) &= 1 \end{aligned} \tag{13}$$

and $P(\beta)$ is calculated from Lemma 2 substituting T by T_U.

Proof
Definition 6 easily implies that

$$\overline{RDA_{X_e}} = \bigcap_{\beta \in B} \overline{DA_{X_e}}(\beta) \tag{14}$$

where $\overline{DA_{X_e}}(\beta)$ is the closure of the DA of system (2), if we suppose a fixed value for β. According to Theorem 3 and Lemma 1 for a fixed β, we have a solution as

$$\overline{DA_{X_e}}(\beta) = SUPP(\vartheta(\beta)),\ \ \vartheta(\beta) = P(\beta)\vartheta(\beta),\ \sum_{i=1}^{N} \vartheta_i(\beta) = 1 \tag{15}$$

where $\vartheta(\beta)$ is a vector of invariant measures. In addition, from 14 and 15, it is clear that

$$\begin{aligned} \overline{RDA_{X_e}} &= \bigcap_{\beta \in B} SUPP(\vartheta(\beta)) = \bigcap_{\beta \in B} \{A_i |\ \ \vartheta_i(\beta) \neq 0\} = \bigcap \left\{ A_i |\ \inf_{\beta \in B}\ \vartheta_i(\beta) \neq 0 \right\} \\ &= SUPP\left(\inf_{\beta \in B}\ \vartheta(\beta) \right) \end{aligned} \tag{16}$$

In other words, $\overline{RDA}_{X_e} = SUPP\ \ \vartheta_\beta$. □

According to Theorem 4, we propose an analytic formula to find RDA. According to (16), A_i is contained with RDA if $\vartheta_i(\beta)$ has a nonzero global minimum on B. This global minimum is numerically found using the proposed method of [15]. The advantage of our proposed method is that we express the problem of estimating RDA in the form of a simple optimization problem which is useful for a large class of nonlinear systems.

3.4. Directional extension of domain of attraction

Considering the following system with equilibrium state $X_e = 0$, the goal is determining an optimal value for controlling parameters $\alpha = [\alpha_1\ \dots\ \alpha_h]^T$ to extend the DA of the system along a desired direction like $e = [e_1\ \dots\ \ e_n]^T$.

$$\dot{X} = f(X, \alpha) \quad f : R^n \times R^{+^h} \to R^m$$

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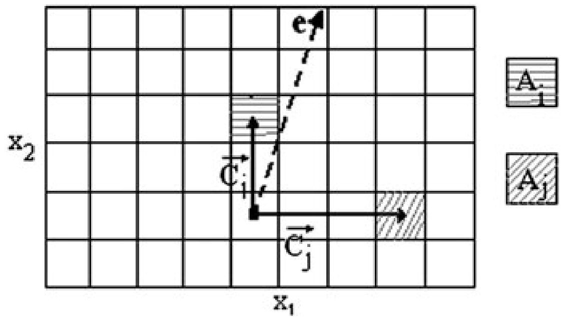

Figure 2. Extending domain of attraction along e ($\eta_i = C_i \cdot e$ is greater than $\eta_j = C_j \cdot e$).

Although directional extension of DA is of interest in many applications such as power systems, finding the controlling parameter(s) to achieve this feat is not very easy. In other words, extending DA along directions of interest and computing its sensitivity to controlling parameters along these directions needs an analytic estimate of DA which directly depends on α. We do this by Markov modeling of the following discrete-time system

$$x_i(k+1) = T_i(X(k), \alpha) \quad i = 1, \ldots, n \tag{17}$$

Then, we propose Theorem 5 for DA directional extension along the vector e.

Theorem 5
Consider nonlinear system (17) with the controlling parameters α and state-space partitioning $\mathcal{A}$. Let e be the direction of interest, the optimal α's that lead to DA extension along this desired vector, found through the following optimization problem

$$\begin{aligned} &\underset{\alpha}{\text{minimize}} \quad J(\alpha) = -|\eta_i| \quad \vartheta_i(\alpha) \\ &\text{s.t}: \begin{cases} \vartheta(\alpha) = P(\alpha)\vartheta(\alpha) \\ \sum_{i=1}^{N} \vartheta_i(\alpha) = 1 \end{cases} \end{aligned} \tag{18}$$

where, $\eta_i = C_i \cdot e = \sum_{j=1}^{n} c_{ji}\, e_j$, N is the number of state space partitions, and C_i is the center of A_i as described in Definition 2.

Proof
As shown in Figure 2, $\eta_i = C_i \cdot e$ leads to increasing η_i in A_i s along e. So, to minimize $J(\alpha)$, we should find optimal control parameters that maximize $\vartheta_i(\alpha)$ with greater η_i or in other words maximize $\vartheta_i(\alpha)$ of partitions along e. According to Theorem 3, this is equal to DA enlargement along e. □

In this paper, we effectively use directional extension of DA for increasing critical clearing time of nonlinear systems such as power systems. Literatures that are increasing critical clearing time are extending DA globally [9, 10]. But in this work, we find the fault running vector by simulating the faulted system and extend DA along this vector according to Theorem 5. Numerical results verify the efficiency of the proposed idea.

4. NUMERICAL EXAMPLES

This part contains three numerical examples. In Section 4.1, DA of the equilibrium point of a Van der Pol system is estimated through Theorem 3. We focus on estimating RDA according to Theorem 4, in Section 4.2. Section 4.3 is about directional extension of DA, using Theorem 5. To show the problem formulation more precisely, we propose a numerical algorithm in Appendix 1.

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4.1. Domain of attraction estimation

In this part, we apply our method for two different classes of nonlinear systems. We estimate the DA of a Van der Pol oscillator (which has a bounded DA) and compare the estimated region with the approximated area obtained from [1], and we apply the method on system (20) to prove the capability of estimating unbounded DAs.

Consider the following discrete Van der Pol system with bounded DA

$$\begin{aligned} x_1[k+1] &= (-x_2[k])T_s + x_1[k] \\ x_2[k+1] &= \{x_1[k] - (1-(x_1[k])^2)x_2[k]\}T_s + x_2[k] \end{aligned} \tag{19}$$

where T_s is the sampling time.

To estimate DA, the invariant measure vector $\vartheta = (\vartheta_1, \ldots, \vartheta_N)$ should be determined from (9). Before solving (9), it is necessary to find Markov transition matrix $P = [p_{ij}]_{N\times N}$. According to Appendix 1,

(i) We choose $\Omega = [-3,3]\times[-3,3]$ and $N = N_1.N_2 = 1024$ which yields $\Delta_1 = \Delta_2 = 0.187$, $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]$ $i = 1,.., 1024, l_{1i} = -3 + 0.187\, remaider\left(\frac{i}{32}\right), l_{2i} = -3 + 0.187 quotient\left(\frac{i}{32}\right), h_{1i} = l_{1i} + 0.187$ and $h_{2i} = l_{2i} + 0.187$.
(ii) Substituting (19) in (25) and choosing $M_1 = M_2 = 30$ and $T_S = 1$, P matrix is obtained as

$$P = [p_{ij}],\ \ p_{ij} = \frac{\sum_{m_1=0}^{30} \sum_{m_2=0}^{30} [h_T(m_1, m_2, i, j).h_X(m_1, m_2, i, j)]}{S_D/0.04}$$

where

$$\begin{aligned} h_T(m_1, m_2, i, j) &= H\ \left[\left(T_1\left(X^{m_1,m_2}\right) - l_{1j}\right).\left(h_{1j} - T_1\left(X^{m_1,m_2}\right)\right)\right] \\ &\quad \times H\ \left[T_2\left(X^{m_1,m_2}\right) - l_{2j}\right).\left(h_{2j} - \left(T_2\left(X^{m_1,m_2}\right)\right)\right] \\ T_1\left(X^{m_1,m_2}\right) &= -x_2^{m_2} + x_1^{m_1},\ T_2\left(X^{m_1,m_2}\right) = x_1^{m_1} - \left(1 - {x_1^{m_1}}^2\right)x_2^{m_2} + x_2^{m_2} \\ h_X(m_1, m_2, i, j) &= H\ \left[\left(x_1^{m_1} - l_{1i}\right).\left(h_{1i} - x_1^{m_1}\right)\right]\ H\ \left[\left(x_2^{m_2} - l_{1i}\right).\left(h_{1i} - x_2^{m_2}\right)\right] \\ x_1^{m_k} &= 0.2\,.m_k, x_2^{m_k} = 0.2\,.m_k \\ S_D &= 0.187^{\,2} \end{aligned}$$

To find invariant measure vector, we use 'Linprog' instruction of Matlab as $\vartheta = linprog\,(Zeros(N,1),\ A,\ B,\ A_{eq},\ B_{eq})$ with the following chosen values for the matrixes

$$A_{eq} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ p_{11}-1 & p_{12} & p_{13} & \cdots & p_{1N} \\ p_{21} & p_{22}-1 & p_{23} & \cdots & p_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{N1} & p_{N2} & p_{N3} & \cdots & p_{NN}-1 \end{bmatrix}_{N+1\times N} \qquad B_{eq} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{N+1\ \times 1}$$

Because the invariant measure vector ϑ is positive, A and B are defined as

$$A = \begin{bmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & -1 \end{bmatrix}_{N\times N} \qquad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{N\times 1}$$

Also, 'Cool Colormap' is applied and Figure 3 is plotted by 'Surfc' instruction.

In Figure 3, actual DA and estimated DA of the aforementioned system have been displayed. The invariant measure of each partition, varies between zero and one, is defined through Theorem 3 and specified with color toolbars. In Figure 3, we also compare the estimated DA obtained from our

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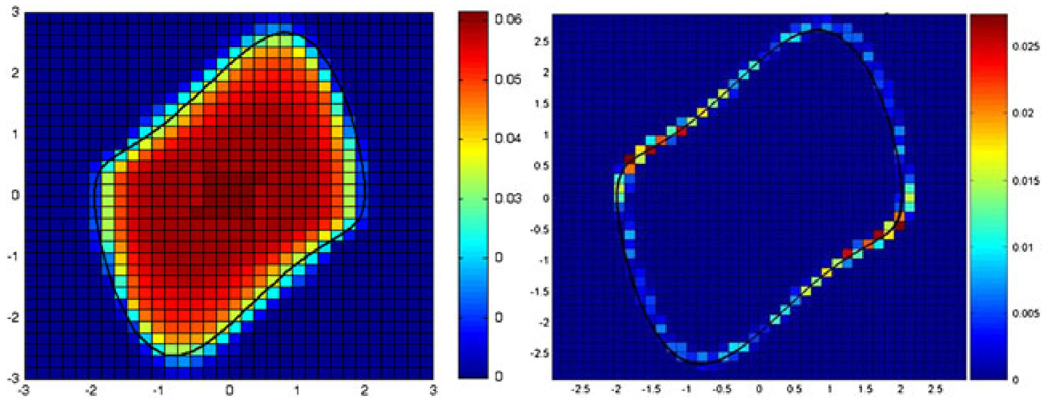
(a) (b)

Figure 3. (a) Our estimation: The estimated domain of attraction of Van der Pol system for $N = 32 \times 32$ partitions is illustrated through color bar and the actual domain of attraction is shown in black curve. (b) Estimated region of [1]: The estimated domain of attraction of Van der Pol system for $N = 64 \times 64$ partitions.

proposed method with another estimated region that is illustrated in [1]. The comparison shows that although [1] uses a finer partitioning, which leads to more complicated calculations, our estimated region is more accurate.

Most of the existing estimation methods, such as Lyapunov-based ones [2, 3], are not convenient for systems with unbounded DA. We show that the proposed method can effectively be used for such systems. The results of estimating the unbounded DA of system (20) are shown in Figure 4. From Figures 3 and 4, one concludes that it is better to choose Ω such that it contains DA (see Figure 3(a)). Note that, $DA_{X_e} \subseteq \Omega$ is not a necessary condition and does not affect the estimation accuracy (see Figure 4). So, it is sufficient only to consider $X_e \in \Omega$.

To estimate the system, described in equation (20), we set parameters as $\Omega = [-5, 5] \times [-5, 5]$, $N_1 = N_2 = 35$, $\Delta_1 = \Delta_2 = 0.28$, $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]$ $i = 1,.., 1225$, $l_{1i} = -5 + 0.28\, remaider\, \left(\frac{i}{35}\right), l_{2i} = -5 + 0.28\, quotient\, \left(\frac{i}{35}\right)$ and $h_{1i} = l_{1i} + 0.28$, $h_{2i} = l_{2i} + 0.28$ the estimation method is the same as that of the previous part.

$$\begin{aligned} x_1[k+1] &= (x_2[k])T_s + x_1[k] \\ x_2[k+1] &= \left(-x_2[k] - x_1[k] + \frac{1}{16}(x_1[k])^5\right) T_s + x_2[k] \end{aligned} \tag{20}$$

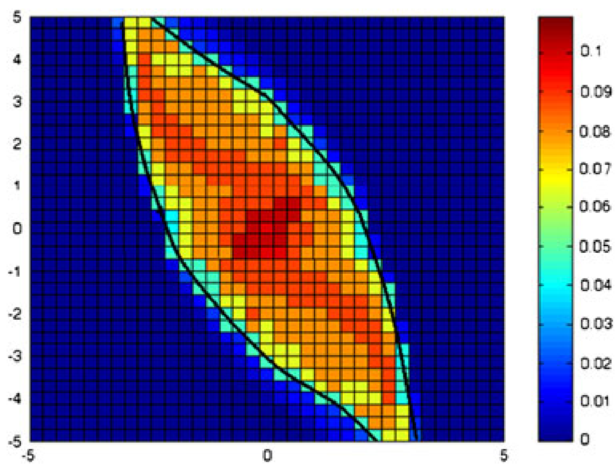

Figure 4. The estimated domain of attraction of the system with unbounded domain of attraction for $N = 35 \times 35$ partitions is illustrated through color bar and the actual domain of attraction is shown in black curve.

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4.2. Robust domain of attraction estimation

Consider the following system

$$\begin{aligned} x_1[k+1] &= \left(-x_1[k] + (x_2[k])^2 + \beta(-2(x_2[k]) - 2(x_2[k])^2)\right) T_s + x_1[k] \\ x_2[k+1] &= ((3x_1[k] - 2x_2[k] + x_1[k]x_2[k]) + \beta(-2x_1[k] + 2x_2[k]))\, T_s + x_2[k] \end{aligned} \tag{21}$$

with uncertain parameter $\beta \in [0,\ 1]$.

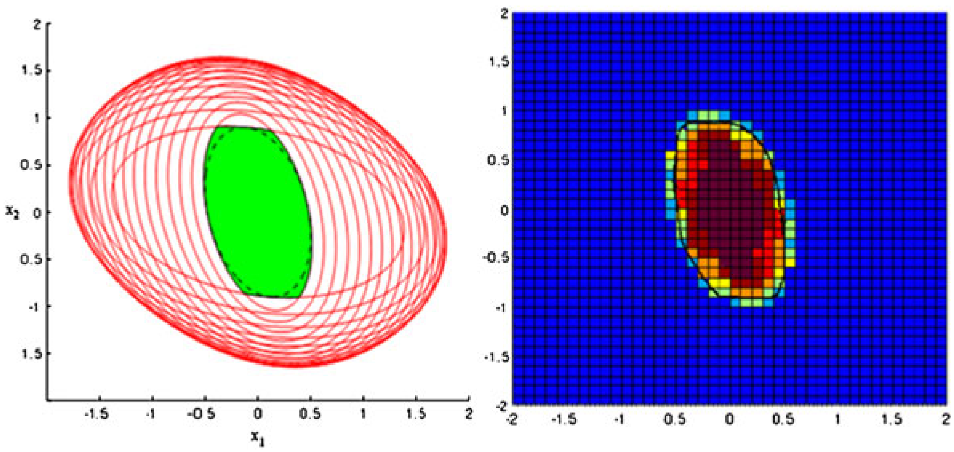
This system is an example of [2] which estimates the RDA of polynomial systems with parameter dependent LFs. In Figure 5(a), the green space is the actual RDA in which stability is guaranteed for different values of β, and the dashed line is the estimated RDA by [2]. Figure 5(b) is the estimated RDA obtained from Theorem 4. According to part (i) of Appendix 1, RDA estimation is the same as estimating DA, as described in part 4.1. We set $\Omega = [-2,2] \times [-2,2]$, $N_1 = N_2 = 40$, $\Delta_1 = \Delta_2 = .1$, $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]\ \ i = 1,..,1600, l_{1i} = -2 + 0.1\, remaider\ \left(\frac{i}{40}\right), l_{2i} = -2 + 0.1\, quotient\ \left(\frac{i}{40}\right), h_{1i} = l_{1i} + 0.1$ and $h_{2i} = l_{2i} + 0.1$.

Markov matrix in RDA estimation problem is a function of parameter β and is computed from equation (27). To decrease complexity of cost function $\vartheta(\beta)$ of equation (13), we choose $M_1 = M_2 = 10$ and easily construct $P(\beta)$ matrix

$$P = [p_{ij}],\ \ p_{ij} = \frac{\sum_{m_1=0}^{10} \sum_{m_2=0}^{10} [h_T(m_1, m_2, i, j, \beta).h_X(m_1, m_2, i, j)]}{S_D/.16}$$

where

$$\begin{aligned} h_T &= H\left[\left(T_1\left(X^{m_1,m_2}, \beta\right) - l_{1j}\right).\left(h_{1j} - T_1\left(X^{m_1,m_2}, \beta\right)\right)\right] \\ &\quad \times H\left[T_2\left(X^{m_1,m_2}, \beta\right) - l_{2j}\right).\left(h_{2j} - \left(T_2\left(X^{m_1,m_2}, \beta\right)\right)\right] \\ T_1\left(X^{m_1,m_2}, \beta\right) &= \left(x_2^{m_2}\right)^2 - x_1^{m_1} + \beta\left(-2x_2^{m_2}\left(1 + x_2^{m_2}\right)\right) + x_1^{m_1} \\ T_2\left(X^{m_1,m_2}, \beta\right) &= 3x_1^{m_1} - 2x_2^{m_2} + x_1^{m_1}x_2^{m_2} + \beta\left(-2x_1^{m_1} + 2x_2^{m_2}\right) + x_2^{m_2} \\ h_X(m_1, m_2, i, j) &= H\left[\left(x_1^{m_1} - l_{1i}\right).\left(h_{1i} - x_1^{m_1}\right)\right]\ H\left[\left(x_2^{m_2} - l_{2i}\right).\left(h_{2i} - x_2^{m_2}\right)\right] \\ x_1^{m_k} &= 0.4\,.m_k, x_2^{m_k} = 0.4\,.m_k \\ S_D &= 0.01 \end{aligned}$$



(a) (b)

Figure 5. (a) Actual robust domain of attraction of the system for $\beta \in [0,2]$ is shown in green space and estimated region with Lyapunov function is shown in dashed curve. (b) The estimated robust domain of attraction for $N = 40 \times 40$ partitions in comparison with actual robust domain of attraction is shown in black curve.

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$\vartheta(\beta)$ is obtained from solving optimization problem (13). To solve this problem, we use 'fmincon' instruction. Invariant vector, a function of β, is minimized as follows

$$\mathrm{fmincon}(\vartheta(\beta), \mathrm{A}, \mathrm{b}, \mathrm{Aeq}, \mathrm{Beq}\vartheta(\beta) = P(\beta)\vartheta(\beta))$$

where $A_{eq} = [1\ 1 \ldots .\ 1\ ]_{N+1\times 1}$ $B_{eq} = [1]_{1\times 1}$ demonstrates $\sum_{i=1}^{N} \vartheta_i(\beta) = 1$. A and B are defined as follows to indicate $\vartheta_i(\beta) \geqslant 0\ \ i = 1, \ldots, N$.

$$A = \begin{bmatrix} -1 & \ldots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & -1 \end{bmatrix}_{N\times N} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{N\times 1}$$

According to Figure 6, for such system, both methods define an acceptable estimate of DA, but as in Lyapunov-based methods, estimated DA is usually limited to quadratic structures of LF, this method is not applicable for systems with non-quadratic DAs and the result is very conservative. To show it more precisely, we introduce another test system defined in (22). Values of parameters (Ω, N, ...) have been stated in the figure description and estimating steps are as in system (21). Figure 6 shows that the Lyapunov-based answer is not as appropriate as the previous one. In comparison with the results of Lyapunov-based methods, our proposed method does not depend on system structure and if we use a finer partitioning, we will obtain a more accurate answer.

The more popular methods of estimating RDA use common LFs to prove robust local stability [2], estimating RDA via parameter dependent LFs [2, 3] and RDA estimation through generalized Zubove's method [4]. These methods have limitations that we overcome through our proposed method. For example, in Lyapunov-based algorithms, RDA of systems with probably time varying uncertainty can be estimated through common LF but finding such a common LF in general is impossible. Parameter dependent LF is applicable only for time-invariant uncertainties. In addition, there is not a general LF structure and most literatures use quadratic LF, which leads to a conservative estimation of DA as we mentioned in Figures 5 and 6. On the other hand, to find RDA through generalized Zubove's method, the viscosity solution of straightforward generalization of classical Zubove's equation is used. This method is concerned with exact determination of DA [5]. To solve Zubove's equation, method of characteristic is used, but this method requires solution of nonlinear system and in fact the knowledge of DA which is mostly impossible.

Another disadvantage of parameter dependent Lyapunov-based methods is that the stability is not exactly guaranteed in estimated region. For example, in Figure 5(a), one may choose a β which has

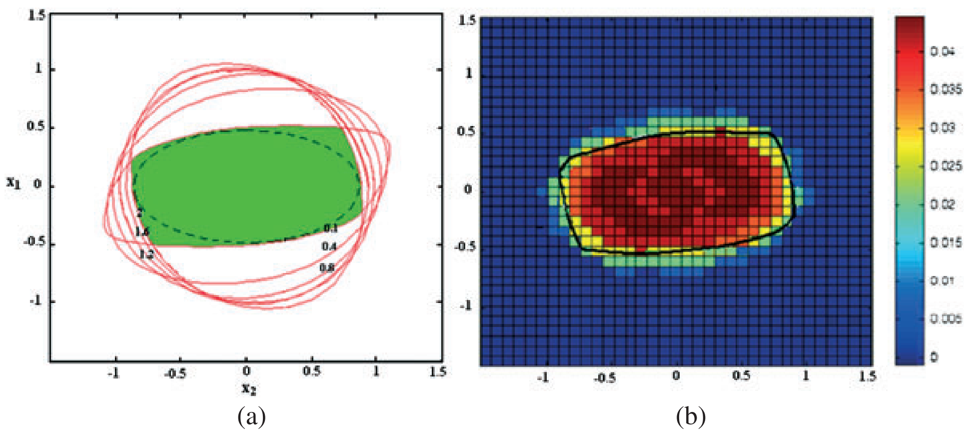

Figure 6. (a) Actual robust domain of attraction of the system for $\beta \in [0.1, 2]$ is shown in green space and estimated region with Lyapunov function is shown in dashed curve. (b) The estimated robust domain of attraction for $N = 35 \times 35$ partitions in comparison with actual robust domain of attraction is shown in black curve.

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not been previously considered in the intersection but using it leads to a different estimated RDA. Our proposed method overcomes this limitation by considering all values of parameter β.

In comparison with estimation algorithms proposed in [2–5], one disadvantage of Markov modeling is that the estimated RDA (as it can be seen in Figure 5(b)) includes real RDA so in boundary partitions stability is not guaranteed. To overcome this limitation and have a more accurate estimate of RDA, we suggest refining any partition sets which has measure greater than $\frac{1}{N}$, where N is the number of initial partitions.

$$\begin{aligned} x_1[k+1] &= \left(-x_2[k] + x_1[k]((x_1[k])^2 + (x_2[k])^2 - 1)\right) T_s + x_1[k] \\ x_2[k+1] &= \left(\beta\,(x_1[k] + x_2[k]((x_1[k])^2 + (x_2[k])^2 - 1))\right) T_s + x_2[k] \end{aligned} \tag{22}$$

4.3. *Directional extension*

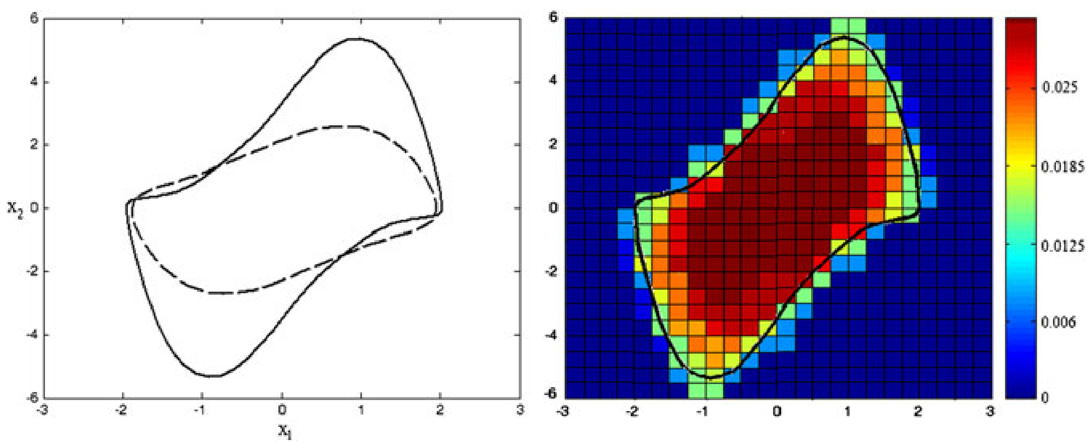
Consider the following system with the controlling parameter α.

$$\begin{aligned} x_1[k+1] &= (-x_2[k])T_s + x_1[k] \\ x_2[k+1] &= \left\{x_1[k] - \alpha(1-(x_1[k])^2)x_2[k]\right\} T_s + x_2[k] \end{aligned} \tag{23}$$

The proposed idea of Theorem 5 is applied to determine the appropriate α that extends the DA of the system along a desired direction $\vec{e}$. Using Theorem 5, $\alpha = 3.231$ is obtained as the optimal controlling parameter. In Figure 7(a), the extended DA along $\vec{e}_1 = [1\,\vec{i} \quad 4\,\vec{j}]^T$ is compared with the initial DA and Figure 7(b) illustrates the invariant measure of state-space partitioning which is obtained as follows.

(i) We choose $\Omega = [-3,3] \times [-6,6]$, $N_1 = N_2 = 24$ which yields $\Delta_1 = 0.25, \Delta_2 = 0.5$, $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]$ $i = 1,..,24^2$, $l_{1i} = -3 + 0.25 remaider \left(\frac{i}{24}\right), l_{2i} = -6 + 0.5 quotient \left(\frac{i}{24}\right), h_{1i} = l_{1i} + 0.25$ and $h_{2i} = l_{2i} + 0.5$.
(ii) Substituting (23) in (27) and choosing $M_1 = 12,\; M_2 = 24$, P matrix is obtained as

$$P = [p_{ij}],\;\; p_{ij} = \frac{\sum\limits_{m_1=0}^{30} \sum\limits_{m_2=0}^{30} [h_T(m_1, m_2, i, j, \alpha).h_X(m_1, m_2, i, j)]}{S_D/0.25}$$



(a) (b)

Figure 7. (a) Comparison of the initial domain of attraction (dashed curve) and the extended one along $\vec{e}_1 = [1\,\vec{i} \quad 4\,\vec{j}]^T$ for $\alpha = 3.231$. (b) The invariant measure of partitions for $\alpha = 3.231$.

where

$$h_T = H\ [(T_1(X^{m_1,m_2},\alpha) - l_{1j}).(h_{1j} - T_1(X^{m_1,m_2},\alpha))]$$
$$\times H\ [T_2(X^{m_1,m_2},\alpha) - l_{2j}).(h_{2j} - (T_2(X^{m_1,m_2},\alpha))]$$
$$T_1(X^{m_1,m_{2_1}}) = -x_2^{m_2} + x_1^{m_1},\ T_2(X^{m_1,m_2}) = x_1^{m_1} - \alpha\left(1 - {x_1^{m_1}}^2\right)x_2^{m_2} + x_2^{m_2}$$
$$h_X(m_1, m_2, i, j) = H\ \left[\left(x_1^{m_1} - l_{1i}\right).\left(h_{1i} - x_1^{m_1}\right)\right]\ H\ \left[\left(x_2^{m_2} - l_{2i}\right).\left(h_{2i} - x_2^{m_2}\right)\right]$$
$$x_1^{m_k} = 0.5\,.m_k, x_2^{m_k} = 0.5\,.m_k$$
$$S_D = 0.125$$

(iii) We solve optimization problem (18) to find α. The optimal controlling parameter and its related invariant vector can be easily computed from the following instruction.

$$\text{fmincon}(J(\alpha) = -|\eta_i|\ \ \vartheta_i(\alpha), \text{A, b, Aeq, Beq}\vartheta(\alpha) = P(\alpha)\vartheta(\alpha))$$

where $A_{eq} = [1\ 1 \ldots .\ 1\ \ 0]_{N+1\times 1}$ $\quad B_{eq} = [1]_{1\times 1}$. Because invariant measure vector ϑ and α are positive, A and B are defined as

$$A = \begin{bmatrix} -1 & \ldots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & -1 \end{bmatrix}_{N+1\times N+1} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{N+1\ \times 1}$$

where the output of ‘fmincon’ instruction is $[\vartheta;\alpha]^T$ vector.

4.4. *Increasing critical clearing time of power system*

In this part, we show that directional enlargement of DA can effectively be used for increasing critical clearing time of power systems.

Consider the following two-area four-machine power system. Specifications of this system are defined in Appendix 2. We suppose the exciter gains as controlling parameters of system with nominal value $k_A = [200\ \ \ 200\ \ \ 200\ \ \ 200\ ]$. Using power system toolbox (PST), written by Graham Rogers, we plot the faulted system flows (see Figure 8). The original paper about PST was written by J. H. Chow [16]. The fault running vector of system after three different faults is defined in Table I.

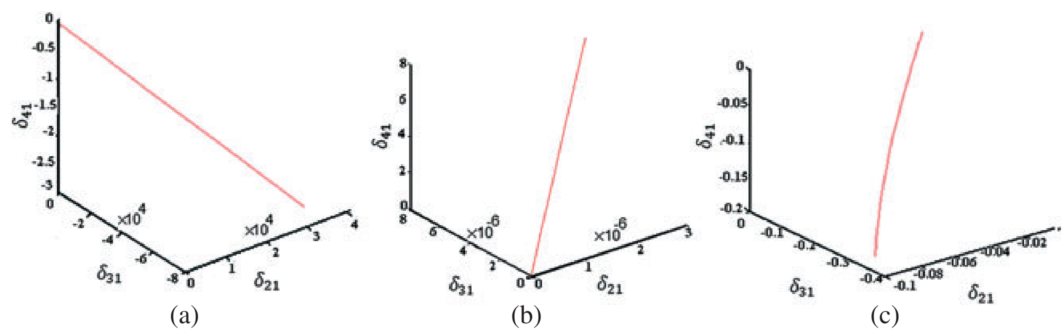


Figure 8. Faulted power system flows (a) short circuit of the fifth bus; (b) short circuit of the sixth bus; and (c) Open circuit of lines 5–6.

Table I. Fault running vector of system.

Fault type	Fault running vector
Short circuit of the fifth bus	$e_1 = [\ 0.39\ \ \ -0.85\ \ \ -0.35\ ]$
Short circuit of the sixth bus	$e_2 = [\ 0.3\ \ 0.6\ \ 0.74\ ]$
Open circuit of lines 5–6	$e_3 = [\ -0.24\ \ \ -0.85\ \ \ -0.46\ ]$

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Table II. Optimal controlling parameter for extending domain of attraction along fault running vectors.

Fault running vector	Optimal value of exciter gain
e_1	$k_{A-1} = [91 \quad 90 \quad 91 \quad 90]$
e_2	$k_{A-2} = [88 \quad 86.5 \quad 88 \quad 86.5]$
e_3	$k_{A-3} = [102 \quad 122.8 \quad 102 \quad 122.8]$

Table III. Improvement of critical clearing time.

	Critical clearing time (s)	
	For Nominal exciter gain	For optimal exciter gain
Short circuit of the fifth bus	0.0012	0.0033
Short circuit of the sixth bus	0.0012	0.0036
Open circuit of lines 5–6	0.21	0.342

For each type of faults, we use Theorem 5 to extend DA along fault running vector related to that fault. The numerical steps are the same as stated in part 4.3. The optimal gain for each fault is defined in Table II. Finally, in Table III, the critical clearing time which is obtained from nominal value of exciter gain is compared with the increased clearing time obtained from optimal controlling parameters.

5. CONCLUSIONS

In this work, we propose a new method for estimating the RDA. Although this work is based on the average quantities of state space and therefore does not use the exact information of real system, it is able to effectively find estimated RDA by solving just a simple optimization problem. Another advantage of this work is its capability of estimating RDA for a large class of nonlinear systems (systems with time-homogeneous aperiodic chains).

In addition, we apply the method for directional enlargement of the DA and find the optimal controlling parameters by solving an optimization problem. The method is applied for increasing critical clearing time of power systems by enlarging DA of power system along fault running vector.

One disadvantage of using Markov modeling for DA estimation is that the estimated DA includes real DA; so in boundary partitioning, the stability is not guaranteed. To overcome this limitation, we suggest refining any partition sets which has great invariant measure. Proposing a new algorithm for addressing such boundary partitions will be considered in our future work.

APPENDIX 1

In the sequel, we propose a numerical algorithm to find the answers to problems which were demonstrated in Section 4. For simplicity, the algorithm is applied for two-dimensional systems but it can similarly be used for n dimensional systems.

For all examples of Section 4, step (i) is similar. The other steps are a bit different so we divide these steps to three individual parts from (a) to (c), which are, respectively, used for estimating DA, estimating RDA, and directional extension of DA.

Let $\Omega \subset R^2$ be a rectangular set, containing real DA. One can consider Ω as $\Omega = [l_1, h_1] \times [l_2, h_2]$. Ω should be chosen such that $X_e \in \Omega$, where X_e is an isolated stable equilibrium point of the system and estimating its DA is considered.

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(i) Divide Ω to rectangular cells $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]$ $i = 1,.., N$.

Divide $[l_1, h_1]$ to N_1 subintervals denoted by $[l_{1k}, h_{1k}]$ $k = 1, \ldots, N_1$ and $[l_2, h_2]$ to N_2 subintervals defined by $[l_{2l}, h_{2l}]$ $l = 1, \ldots, N_2$. Construct A_i partitions such that

$$l_{1i} = l_1 + \Delta_1.remainder\left(\frac{i}{N_1}\right),\ l_{2i} = l_2 + \Delta_2.quotient\left(\frac{i}{N_1}\right)$$
$$h_{1i} = l_{1i} + \Delta_1,\ \ h_{2i} = l_{2i} + \Delta_2$$

where $\Delta_j = \frac{h_j - l_j}{N_j}$ $j = 1, 2$.

(ii) Calculate the Markov transition matrix

Let $A_i = [l_{1i}, h_{1i}] \times [l_{2i}, h_{2i}]$ and $A_j = [l_{1j}, h_{1j}] \times [l_{2j}, h_{2j}]$ be two partitions in Ω. The Markov transition matrix, which is illustrated in Definition 9, is calculated from Lemma 2 as follows:

1. Calculate the Markov transition matrix for estimating DA

$$p_{ij} = \frac{\int\limits_{\Omega} \prod\limits_{q=1}^{2} H\ [(T_q(X) - l_{qj}).(h_{qj} - T_q(X))]\ \prod\limits_{k=1}^{2} H\ [(x_k - l_{ki}).(h_{ki} - x_k)]\,\mathrm{d}X}{S_D} \tag{24}$$

where

$$S_D = \Delta_1.\Delta_2\ \ , dX = dx_1.dx_2$$

To calculate p_{ij} easily, we convert the aforementioned integral into the following summation

$$p_{ij} = \frac{\sum\limits_{m_1=0}^{M_1} \sum\limits_{m_2=0}^{M_2} \left(\prod\limits_{q=1}^{2} H\,[(T_q(X^{m_1,m_2}) - l_{qj}).(h_{qj} - T_q(X^{m_1,m_2}))]\ \prod\limits_{k=1}^{2} H\ [(x_k^{m_k} - l_{ki}).(h_{ki} - x_k^{m_k})] \right)}{S_D \cdot \frac{M_1}{h_1 - l_1} \cdot \frac{M_2}{h_2 - l_2}} \tag{25}$$

where $X^{m_1,m_2} = [x_1^{m_1}\ \ x_2^{m_2}]^T$, $x_k^{m_k} = \frac{h_k - l_k}{M_k}.m_k$

2. Calculate the Markov transition matrix for estimating RDA

$$p_{ij}(\beta) = \frac{\int\limits_{\Omega} \prod\limits_{q=1}^{2} H\ [(T_q(X,\beta) - l_{qj}).(h_{qj} - T_q(X,\beta))]\ \prod\limits_{k=1}^{2} H\ [(x_k - l_{ki}).(h_{ki} - x_k)]\,\mathrm{d}X}{S_D} \tag{26}$$

This easily implies that

$$p_{ij}(\beta) = \frac{\sum\limits_{m_1=0}^{M_1} \sum\limits_{m_2=0}^{M_2} \left(\prod\limits_{q=1}^{2} H[(T_q(X^{m_1,m_2},\beta) - l_{qj}).(h_{qj} - T_q(X^{m_1,m_2},\beta))]\ \prod\limits_{k=1}^{2} H[(x_k^{m_k} - l_{ki}).(h_{ki} - x_k^{m_k})] \right)}{S_D \cdot \frac{M_1}{h_1 - l_1} \cdot \frac{M_2}{h_2 - l_2}} \tag{27}$$

3. Calculate the Markov transition matrix for extending DA

Replace β by α and follow part b.

(iii) Find invariant measure vector

1. Find invariant measure vector for DA estimation problem

To find invariant measure vector, equation (9) from Theorem 3 should be solved. There are different MATLAB instructions to solve such problems. In this paper, we rewrite equation (9) in the form of the following optimization problem and use 'Linprog' instruction to solve it (the problem formulation is described in details in part 4.1).

$$\begin{aligned} &\min\ C^T \vartheta \\ &s.t.\ \vartheta = P\vartheta\ \ ; \vartheta = (\vartheta_1, \ldots, \vartheta_N) \\ &\sum_{i=1}^{N} \vartheta_i = 1, C = [0 \ldots 0]_{1\times N}^T \end{aligned} \tag{28}$$

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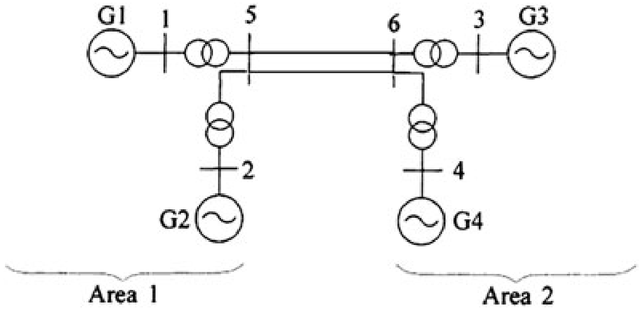

Figure 9. Power system with four machines.

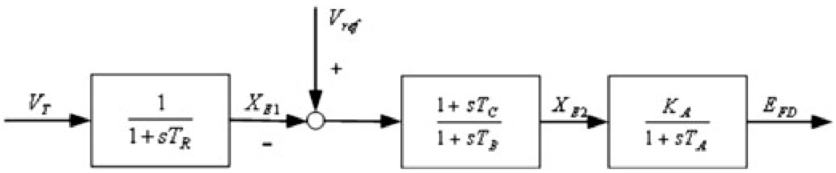

Figure 10. Block diagram of the exciter.

2. Find invariant measure vector for RDA estimation problem
   To calculate invariant measure vector, we solve equation (13) from Theorem 4 with 'Fmincon' instruction (see problem formulation in part 4.2 for more details).
3. Find optimal control parameter
   Solution of optimal control problem which is defined in Theorem 5 is similar to part b.

(iv) Estimating DA or RDA

1. Estimating DA
   According to Lemma 1, we have $\overline{DA_{X_e}} = \bigcup \{A_i : \vartheta_i \neq 0\}$. As shown in the figures, we use 'Surfc' MATLAB instruction to demonstrate invariant measure of each state partition.
2. Estimating RDA
   Theorem 4 yields $\overline{RDA_{X_e}} = \bigcap \left\{ A_i | \inf_{\beta \in B} \vartheta_i(\beta) \neq 0 \right\}$ which is easily shown by Surfc.

APPENDIX 2

As a case study, we use the two-area four-machine system introduced in [17] (see Figure 9). Excepting damping coefficients, all generators' parameters are equal and indicated in 100 MVA and 20 KV bases.

$$R_{ai} = .0025\, pu, X_{di} = 1.8\ pu, X_{qi} = 1.7\ pu, X'_{di} = 0.3\ pu, X'_{qi} = 0.55\ pu,$$
$$T'_{doi} = 8\ \text{sec},\ T'_{qoi} = .4\ \text{sec},\ H_{gi} = 6.5\ \text{sec},\ D_g = [9, 10, 12, 11]$$

We choose exciters as follows. Figure 10 is the block diagram of the exciter (IEEE Type AC4A) and its parameters values are as follows.

$$T_{Ri} = .01\ \text{sec},\ T_{Ai} = .01\ \text{sec}, T_{Bi} = 10\ \text{sec}, T_{Ci} = 1\ \text{sec}, K_{Ai-no\min al} = 200\ \ i = 1, 2, 3, 4$$
$$V_{r\max} = 5\ pu, V_{r\min} = -5\ pu$$

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