

Frequency response calculation of non-linear torsional vibration in gear systems

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Abstract: The current paper focuses on the non-linear torsional vibration of a one-stage transmission gear system. Four different methods have been applied for solution of the equation of motion; the discretization method, the perturbation method, the Ritz method, and the step-wise time integration of the equation of motion. The time and frequency results from these analyses have been compared with each other, as well as those reported in literatures. Although all of these methods are accurate and computationally effective for finding the main spectral contribution, however, only the discretization method and the step-wise time-integration model are able to identify the other frequency components.

Keywords: non-linear vibration, frequency response, discretization method, perturbation method, Ritz method, step-wise time-integration method

1 INTRODUCTION

Rattling in change over gears of automobiles is an unwanted comfort problem [1]. Recently, there has been an increase of interest in the scrutiny of gear rattle. A large number of researchers had paid close attention to the gear rattle problem. Non-linear dynamic analysis of a single-degree-of-freedom (SDOF) system with clearances type non-linearity has been the subject of several recent investigations. Gear box dynamics is based on a periodically changing meshing stiffness complemented by a non-linear effect of backlash between the teeth. The regular and chaotic vibrations of such systems have been predicted theoretically and examined experimentally [2, 3]. The theoretical description of this phenomenon has been based mainly on single degree of freedom models or multi-degree models neglecting backlash [4]. Litak and Friswell [5] had examined the possibility of taming chaotic vibrations and reducing the amplitude by

non-feedback control methods. First, they had introduced a weak resonant excitation through an additional small external excitation term (torque) with a different phase. Second, they had examined the effect of an additional degree of freedom to account for shaft flexibility on one side of the gearbox. They had used a numerical method for approaching the problem. Modern research has shown that chaotic vibration can occur on impulsive systems with plays, which are confident of a non-linear element in mechanics. Therefore, the chaotic vibration on a rattling system has received attention. Feng and Pfeiffer [1] established a discrete stochastic model described by a mean map using the non-Gaussian closure technique. Souza *et al.* [6] present a study on the dynamics of the rattling problem in gearboxes under non-ideal excitation. They consider the motion to be a function of the system response and a limited energy source is adopted. So an extra degree of freedom was introduced in their problem and a numerical integration was used to detect some interesting geometrical aspects of regular and irregular motion of the system response. Kim *et al.* [7] analysed the effect of smoothening functions on the frequency response of an oscillator

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with clearance non-linearity. However, little research has been undertaken to study the problem of gear rattle due to backlash and clearance non-linearity, using innovative methods introduced in mathematics.

A pair of meshing gears with rigid, perfect, uniformly spaced involute teeth would transmit exactly uniform angular motion. However, the tooth faces of real gears are designed to deviate slightly from perfect involute surfaces, and they also contain machining errors that may vary tooth to tooth. Furthermore, when a pair of meshing rotating gears is transmitting torque, the teeth elastically deform, giving rise to an unsteady component in the relative angular motion of the two gears. This unsteady component is caused by the periodic variation in the stiffness of the gear mesh that is attributable to the periodic variation in the numbers of teeth in contact and the variation in the stiffness of the individual tooth pairs as the location of their mutual line of contact changes during rotation [8]. Gear profile errors are likely to affect the time domain vibration of the system, but have minimal influence on modal properties. Teeth profile errors also result in an excitation source, but time varying gear mesh stiffness effect dominates [9]. Many methods for calculating gear mesh stiffness are reported in the literature. These include the closed-form solutions of Cornell [10], the elastic energy methods of Daniewicz *et al.* [11] and the complex potential methods of Cardou and Tordion [12], as well as many finite-element studies. In the current study, a displacement-type vibratory excitation was used for calculation of restoring force which would be discussed in the following sections.

Although considerable amount of research has been devoted to gear rattle, few attempts have been made to investigate exact solution of the equation of motion of a single stage two transmission gear. Even the powerful and famous methods, such as perturbation and Ritz method had been on less interest in the analysis of gear rattle. The only analytical methods used for this purpose are the harmonic balance method (HBM) and the method of describing functions. Kim *et al.* [13] present a new multi-term HBM for non-linear frequency response calculations of a torsional subsystem containing a clearance type non-linearity. Their proposed HBM includes adaptive arc-length continuation and stability calculation capabilities to find periodic solutions in multi-valued non-linear frequency response regimes as well as to improve convergence. They have introduced essential steps of the proposed HBM calculations validated it by comparing time and frequency domain predictions with those yielded by numerical solutions and experimental studies. They focused on super- and sub-harmonic non-linear frequency response characteristics of an

oscillator with clearance non-linearity. In addition, they have investigated the number of harmonic terms that must be included in the HBM response calculations, with sinusoidal excitation. Their work was a direct extension of the previous describing function analysis that has been introduced by Comparin and Singh [14]. Sun and Hu [15] investigate non-linear dynamics of a planetary gear system with multiple clearances taken into account. They establish a lateral torsional coupled model with multiple backlashes, time-varying mesh stiffness, and error excitation. Their solutions were determined by using HBM from the equations in matrix form.

As explained above it would be thus of interest to study how other methods such as discretization method, perturbation method, and the method of Ritz treat with differential equations which have clearance type non-linearity. These methods proved to be efficient for solving a number of various problems. In this paper, their application to the equations of motion which have some kind of dead zone in their restoring force term is demonstrated. The main task of this study is to provide new regions and new solutions in the study of the gear rattle phenomena.

The current article, presents a model for the relative vibration of two transmission gear. Introduce the restoring force as the main source of non-linearity in the system, and use some different methods such as discretization method, perturbation, and Ritz method to solve the non-linear equation of motion.

For comparison purpose a numerical solution of the equation of motion using MATLAB Simulink model has been presented. The results of these models are compared and verified by the results of other researchers.

2 PROBLEM FORMULATION

This analysis is started with modelling the relative vibrations of a single stage transmission gear. Figure 1 shows a schematic picture of the physical system. The gear wheels are shown with moments of inertia I_1 and I_2 and are coupled by the linear torsional stiffness and viscous damping, represented by K and C , respectively. The equations of motion of the system may be written in terms of the two degrees of freedom, which represent the rotational angles of the gear wheels. These angles are those that remain after the steady rotation of the system is removed [16]. Thus, if the backlash is initially neglected, the governing equations would be

$$\begin{aligned} I_1 \ddot{\theta}_1 + K(\theta_1 - \theta_2) + C(\dot{\theta}_1 - \dot{\theta}_2) &= T_1 \\ I_2 \ddot{\theta}_2 - K(\theta_1 - \theta_2) - C(\dot{\theta}_1 - \dot{\theta}_2) &= -T_2 \end{aligned} \quad (1)$$

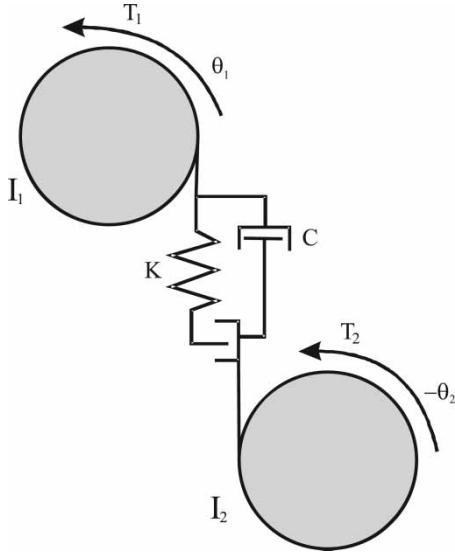


Fig. 1 Schematic representation of a one stage two transmission gear

where θ_1 and θ_2 are the absolute angular displacements of the gear wheels and the overdot represents differentiation with respect to time t . In fact, it is possible to reduce the above two equations to one, using the new relative angular displacement coordinate $\delta = \theta_1 - \theta_2$ [17]. This coordinate represents the relative angular displacement of the gear wheels. The equation of motion for δ is obtained by subtracting $1/I_2$ times of the second equation from $1/I_1$ times the first one. Thus, the equation of motion can be rewritten as

$$\ddot{\delta} + C\left(\frac{1}{I_1} + \frac{1}{I_2}\right)\dot{\delta} + K\left(\frac{1}{I_1} + \frac{1}{I_2}\right)\delta = \frac{T_1}{I_1} + \frac{T_2}{I_2} \quad (2)$$

The excitation torque $T_1(t)$, from an internal combustion engine, fluctuates significantly between low (around the compression stage) and high (around the ignition stage) values. Therefore, the $T_1(t)$ can be decomposed into mean T_m and the perturbation $T_p(t)$ parts. The fundamental frequency ω_p of $T_p(t)$ depends on type of the engine, number of cylinders, and crankshaft configuration. For example, the four-stroke engine with V-6, 120° crankshaft configuration fires three times within a crank revolution. Therefore, the firing frequency ω_p is $3\omega_e$, where ω_e represent the crankshaft rotational speed. Similarly, for an I-4 engine, $\omega_p = 2\omega_e$ [13]. Express $T_1(t)$ via Fourier series as

$$T_1(t) = T_m + \sum_{j=1}^{\infty} T_{pj} \sin(j\omega_p t + \phi_j) \quad (3)$$

Here, j represents the torque harmonic index. Under no power condition, the mean term T_m , should be equal to the drag torque $T_2(\omega_e, \Lambda)$ generated within

the transmission where the Λ is the gearbox temperature [18]. The temperature consideration in the gear transmission systems is out the scope of this study. Therefore, it can be written $T_m = T_2$.

The equation (2) can be simplified by introducing new parameter $I = I_1 I_2 / (I_2 + I_1)$ as follows

$$\begin{aligned} \ddot{\delta} + \frac{C}{I}\dot{\delta} + \frac{K}{I}\delta &= F(t) \\ &= F_m + \sum_{j=1}^{\infty} F_{pj} \sin(j\omega_p t + \phi_j) \end{aligned} \quad (4)$$

where I is the effective torsional inertia, $F_m = T_m/I$ is the effective mean torque, and $F_{pj} = T_{pj}/I_1$ is the effective amplitude for j th harmonic of pulsating torque [13]. In this article, the excitation is assumed to be a single harmonic with zero phase, then the final form of the equation of motion (4) becomes

$$\ddot{\delta} + \frac{C}{I}\dot{\delta} + \frac{K}{I}\delta = F_m + F_p \sin \omega t \quad (5)$$

In the above equation, the effect of backlash and gear transmission error has not been considered. Cornell [10] suggested the following formulation for restoring force in gears including backlash and transmission error effects

$$F_{\text{res}} = K_M(f_s(\delta) - f_e(\delta)) \quad (6)$$

where K_M is the gear mesh stiffness, $f_e(\delta)$ is the non-linear periodic transmission error function and $f_s(\delta)$ is a stepwise linear function which is the main source of non-linearity in the system. Figure 2 shows the $f_s(\delta)$ in practice. The mesh stiffness can be explained in terms of Fourier series as

$$K_M = K + \sum_{n=1}^{\infty} K_{Mn} \cos(n\omega_M t + \phi_{Mn}) \quad (7)$$

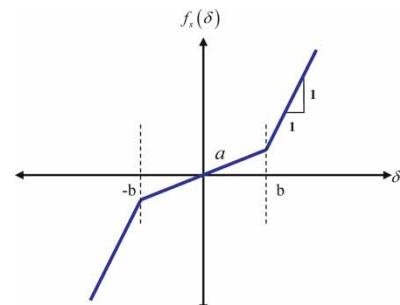


Fig. 2 Schematic picture of non-linear function $f_s(\delta)$ [13]

where K_{Mn} is the amplitude of harmonic mesh stiffness of contacting gears with frequency of ω_M and ϕ_{Mn} is the initial phase of periodic mesh stiffness. Ozguvan and Houser [19] showed that the mesh stiffness can be substituted with its mean value if the transmission error is corrected according to the number of contacting pairs. So the restoring force can be calculated from the following expression

$$F_{\text{res}} = K f_s(\delta) \quad (8)$$

In Fig. 2, α and 1 represent the linear stiffness of the first and second stages of transmission, respectively. The transmission is assumed to take place at $\pm b$ in a symmetric manner, where b is half the amount of backlash. Note that α is stiffness ratio of stiffnesses, can be viewed as a measure of severity of piecewise linear system. When $\alpha = 1$, $f_s(\delta)$ become a linear function. When α is very close to unity (but less than 1), the component may be viewed as weakly non-linear. It would exhibit stiffening type non-linearity from the first stage stiffness. In contrast, when $\alpha > 1$, the stiffness saturation type non-linearity is demonstrated. When $0 < \alpha < 0.5$ such a multi-staged torsional clutch damper, it could be classified as strongly non-linear systems [13]. These systems are the major interest of this paper.

Mathematically $f_s(\delta)$ is as follows [13]

$$f_s(\delta) = \begin{cases} \delta - (1 - \alpha)b & b < \delta \\ \alpha\delta & -b \leq \delta \leq b \\ \delta + (1 - \alpha)b & \delta < b \end{cases} = \delta + (1 - \alpha) \frac{|\delta - b| - |\delta + b|}{2} \quad (9)$$

Using this function, the equations of motion can be updated as follows

$$\ddot{\delta} + \frac{C}{I}\dot{\delta} + \frac{K}{I}f_s(\delta) = F_m + F_p \sin \omega t \quad (10)$$

This equation considers the effect of backlash and so it is the equation which would be studied in the current paper. In the next steps different methods have been applied for solution of this equation.

3 THE DISCRETIZATION METHOD

It has been illustrated that the non-linearity of the system is due to the non-linear restoring force which behaves in a stepwise linear manner. It means there are intervals which within them, the proposed differential equation becomes a linear one. Those intervals are $[-\infty, -b]$, $[-b, b]$, and $[b, \infty]$. The motion starts at one of these intervals with a

linear governing equation, but when the amplitude of motion goes out of this interval the equation of motion changes but it would be remain linear. The initial conditions of this part of motion, which are about δ and $\dot{\delta}$ are those that it exits the last interval. The motion continues in this manner and it goes on and on. These were the main concepts of the discretization method. It is assumed the motion to be started at the initial conditions $\delta_0 = 0$ and $\dot{\delta}_0 = 0$. It means that the relative vibration of the transmission gears starts at the middle stage stiffness with a zero velocity.

With these backgrounds the procedure can be started. At the first the equation of motion is discretized in to three linear differential equations. This can easily be done as follows

$$\left\{ \begin{array}{l} \ddot{\delta} + \frac{C}{I}\dot{\delta} + \frac{K}{I}\delta = \left(F_m + \frac{K}{I}(1 - \alpha)b \right) + F_p \sin \omega t \\ \delta > b \\ \ddot{\delta} + \frac{C}{I}\dot{\delta} + \frac{K}{I}\alpha\delta = F_m + F_p \sin \omega t \\ -b \leq \delta \leq b \\ \ddot{\delta} + \frac{C}{I}\dot{\delta} + \frac{K}{I}\delta = \left(F_m - \frac{K}{I}(1 - \alpha)b \right) + F_p \sin \omega t \\ -b > \delta \end{array} \right. \quad (11)$$

These are linear ordinary differential equations with some constant coefficients and can be solved using the method of reducing order. The solution of equation (11) is expressed in the form of equation (12)

$$\delta(t) = \begin{cases} \delta_1(t) & \delta > b \\ \delta_2(t) & -b < \delta < b \\ \delta_3(t) & -b > \delta \end{cases} \quad (12)$$

where

$$\delta_1(t) = e^{Re(\lambda_{1,2})t} [C_1 \cos(\text{Im}(\lambda_{1,2})t) + C_2 \sin(\text{Im}(\lambda_{1,2})t)] + A + B_1 \sin \omega t + B_2 \cos \omega t \quad (13)$$

$$A = \frac{I \cdot F_m}{K} + (1 - \alpha)b \quad (14)$$

$$B_1 = \frac{(I \cdot \omega^2 - K) \cdot I \cdot F_p}{-I(I\omega^2 - K)\omega^2 - C^2\omega^2 + K(I\omega^2 - K)} \quad (15)$$

$$B_2 = \frac{C \cdot I \cdot \omega \cdot F_p}{-I(I\omega^2 - K)\omega^2 - C^2\omega^2 + K(I\omega^2 - K)} \quad (16)$$

$$\lambda_{1,2} = \frac{-(C/I) \pm \sqrt{(C/I)^2 - 4(K/I)}}{2} \quad (17)$$

and

$$\delta_2(t) = e^{Re(\lambda'_{1,2})t} [C'_1 \cos(\text{Im}(\lambda'_{1,2})t) + C'_2 \sin(\text{Im}(\lambda'_{1,2})t)] + A' + (B'_1 \sin \omega t + B'_2 \cos \omega t) \quad (18)$$

$$A' = \frac{I \cdot F_m}{K \alpha} \quad (19)$$

$$B'_1 = \frac{(I \cdot \omega^2 - K \varepsilon) \cdot I \cdot F_p}{-I(I\omega^2 - K\alpha)\omega^2 - C^2\omega^2 + K\alpha(I\omega^2 - K\alpha)} \quad (20)$$

$$B'_2 = \frac{C \cdot I \cdot \omega \cdot F_p}{-I(I\omega^2 - K\alpha)\omega^2 - C^2\omega^2 + K\alpha(I\omega^2 - K\alpha)} \quad (21)$$

$$\lambda'_{1,2} = \frac{-(C/I) \pm \sqrt{(C/I)^2 - 4(K/I)\alpha}}{2} \quad (22)$$

and

$$\delta_3(t) = e^{Re(\lambda_{1,2})t} [C''_1 \cos(\text{Im}(\lambda_{1,2})t) + C''_2 \sin(\text{Im}(\lambda_{1,2})t)] + A'' + (B''_1 \sin \omega t + B''_2 \cos \omega t) \quad (23)$$

$$A'' = \frac{I \cdot F_m}{K} - (1 - \alpha)b \quad (24)$$

$$B''_1 = B_1 \quad (25)$$

$$B''_2 = B_2 \quad (26)$$

In the above equations C_1 , C_2 , C'_1 , C'_2 , C''_1 , and C''_2 are some constants and should be evaluated using initial conditions as explained in the previous paragraphs.

Thus, the exact solution of the equation (10) is obtained treating the problem as an initial value problem.

4 THE PERTURBATION METHOD

The perturbation method is applicable to problems where a small parameter ε is associated with the non-linear term of the differential equation. The solution is formed in terms of a series of the perturbation parameter ε , the results being a development in the neighbourhood of the solution of the linearized problem. For finding the steady-state solution of the equation of vibration, the perturbation expansion of some parameters of the system is developed. The coefficients of the secular terms set to zero and the main equation is reduced to some linear equations with constant coefficients which can be solved easily using the

theory of the differential equations. For simplicity, it is assumed that there is no damping term in the equation of motion.

In order to solve the equation of motion with perturbation method, the restoring force term $F_{\text{res}} = K/I f_s(\delta)$ ought to be approximated with a polynomial. Making the perturbation parameter as smaller as possible and also for the convenience, the polynomial is chosen a third order one

$$F_{\text{res}} \approx \omega_0^2 \delta + \varepsilon \delta^3 \quad (27)$$

where ε is chosen as the perturbation parameter.

Now, the equation of motion is updated as follows

$$\ddot{\delta} + (\omega_0^2 \delta + \varepsilon \delta^3) = F_m + F_p \cos \omega t \quad (28)$$

It should be noted that in equation (28), the excitation term $\sin \omega t$ has been represented by $\cos \omega t$. In fact the phase of the excitation has no effect on the steady-state response.

It is assumed that the solution of equation (28) can be represented by an expansion having the form

$$\delta = \delta_0 + \varepsilon \delta_1 + \varepsilon^2 \delta_2 + O(\varepsilon^3) \quad (29)$$

where δ_0 is the generating solution, ε is a small perturbation parameter, and δ_1 , etc., are the added corrective terms. Since the frequency is influenced by amplitude of oscillation the frequency ω is perturbed by letting

$$\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + O(\varepsilon^3) \quad (30)$$

where ω_1 , ω_2 , and so on are unknown functions of amplitude at this point as the corrective terms for the frequency. If only the third-order correction is desired, the equations (29) and (30) conclude in

$$\delta^3 = \delta_0^3 + 3\varepsilon \delta_0^2 \delta_1 + 3\varepsilon^2 \delta_0 \delta_1^2 + 3\varepsilon^2 \delta_0^2 \delta_2 \quad (31)$$

$$\omega_0^2 = \omega^2 - 2\varepsilon \omega_0 \omega_1 - \varepsilon^2 (\omega_1^2 + 2\omega_0 \omega_1) \quad (32)$$

The excitation is also expressed in terms of ε as

$$F_m = \varepsilon^2 F_{mo} \quad (33)$$

$$F_p = \varepsilon^2 F_{po} \quad (34)$$

Substituting equations (29), (31), (32), (33), and (34) into (28) gives

$$\begin{aligned} \ddot{\delta}_0 + \varepsilon \ddot{\delta}_1 + \varepsilon^2 \ddot{\delta}_2 + [\omega^2 - 2\varepsilon\omega_0\omega_1 - \varepsilon^2(\omega_1^2 + 2\omega_0\omega_1)] \\ \times (\delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2) + \varepsilon(\delta_0^3 + 3\varepsilon\delta_0^2\delta_1 + 3\varepsilon^2\delta_0\delta_1^2 \\ + 3\varepsilon^2\delta_0^2\delta_2) = \varepsilon^2(F_{mo} + F_{po} \cos \omega t) \end{aligned} \quad (35)$$

Since perturbation parameter ε could have been chosen arbitrary, the coefficients of various powers of ε must be equal to zero. This leads to a system of equations that can be solved successively

$$\ddot{\delta}_0 + \omega^2 \delta_0 = 0 \quad (36)$$

$$\ddot{\delta}_1 + \omega^2 \delta_1 = 2\omega_0\omega_1\delta_0 - \delta_0^3 \quad (37)$$

$$\begin{aligned} \ddot{\delta}_2 + \omega^2 \delta_2 = 2\omega_0\omega_1\delta_1 + (\omega_1^2 + 2\omega_0\omega_1)\delta_0 \\ - 3\delta_0^2\delta_1 + F_{mo} + F_{po} \cos(\omega t) \end{aligned} \quad (38)$$

The solution to the first equation, subject to the initial conditions, $\delta_0(0) = A$, $\dot{\delta}_0(0) = 0$ is

$$\delta_0 = A \cos(\omega t) \quad (39)$$

Which is called the generating solution. Substituting δ_0 into the right side of equation (37), it is obtained as

$$\begin{aligned} \ddot{\delta}_1 + \omega^2 \delta_1 &= 2\omega_0\omega_1 A \cos \omega t - A^3 \cos^3 \omega t \\ &= \left(2\omega_0\omega_1 - \frac{3}{4}A^2\right) A \cos \omega t - \frac{A^3}{4} \cos 3\omega t \end{aligned} \quad (40)$$

It is to be noted here that the forcing term $\cos \omega t$ would lead to a secular term $t \cos \omega t$ in the solution for δ_1 (i.e., it is a condition of resonance). Such terms violate the initial stipulation that the motion is to be periodic; hence, the condition $2\omega_0\omega_1 - (3/4)A^2 = 0$ is imposed. So

$$\omega_1 = \frac{3A^2}{8\omega_0} \quad (41)$$

The solution $\delta_1(t)$ is the sum of the complementary function and the particular integral, that is

$$\delta_1(t) = C_1 \sin \omega t + C_2 \cos \omega t + \frac{A^3}{32\omega^2} \cos 3\omega t \quad (42)$$

Imposing the initial conditions $\delta_1(0) = \dot{\delta}_1(0) = 0$, gives

$$\delta_1(t) = \frac{A^3}{32\omega^2} (\cos 3\omega t - \cos \omega t) \quad (43)$$

The second corrective term $\delta_2(t)$ is obtained from equation (38). The solution of this linear differential equation also contains a non-harmonic term. To eliminate the secular term from $\delta_2(t)$, it must be

$$\omega_2 = \frac{\omega_1 A}{32\omega^2} - 3 \frac{A^4}{128\omega_0\omega^2} - \frac{F_{po}}{2\omega_0 A} - \frac{\omega_1^2}{2\omega_0} \quad (44)$$

Assuming zero initial conditions and suppressing the secular term, the solution of equation (38) is written as

$$\begin{aligned} \delta_2(t) &= \frac{A^3}{128\omega^4} \left(3 \frac{A^2}{8} - \omega_0\omega_1 \right) (\cos 3\omega t - \cos \omega t) \\ &\quad + \frac{A^5}{1024\omega^4} (\cos 5\omega t - \cos \omega t) \\ &\quad + \frac{F_{mo}}{\omega^2} (1 - \cos \omega t) \end{aligned} \quad (45)$$

And the solution at this point from equation (29) becomes

$$\begin{aligned} \delta(t) &= \delta_0(t) + \varepsilon \delta_1(t) + \varepsilon^2 \delta_2(t) = A \cos \omega t + \frac{\varepsilon A^3}{32\omega^2} \\ &\quad \times (\cos 3\omega t - \cos \omega t) + \frac{\varepsilon^2 A^3}{128\omega^4} \left(\frac{3A^2}{8} - \omega_0\omega_1 \right) \\ &\quad \times (\cos 3\omega t - \cos \omega t) + \frac{\varepsilon^2 A^5}{1024\omega^4} \\ &\quad \times (\cos 5\omega t - \cos \omega t) + \frac{\varepsilon^2 F_{mo}}{\omega^2} (1 - \cos \omega t) \end{aligned} \quad (46)$$

And the frequency response of the system is

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 \quad (47)$$

where ω_1 and ω_2 are obtained from the equations (41) and (44), respectively.

5 THE RITZ METHOD

The Ritz method has been applied extensively in studies of non-linear differential equations. This method is based on such ideas as satisfying the equation at certain points of the motion or satisfying the equation in the average. In this method, an approximate solution is assumed in the form of a finite series of trial functions multiplied by undetermined coefficients. Then the square resulting error

in the differential equation is integrated over the domain of the system and minimization of the results permits the determination of the coefficients. Thus, if the equation of motion was expressed as $E(\ddot{\delta}, \dot{\delta}, \delta, t) = 0$ the response of the system is approximated with a harmonic expression such as

$$\delta = \sum_{j=0}^N (a_j \cos(j\omega t) + b_j \sin(j\omega t)) \quad (48)$$

Using this approximation it is tried to minimize the following equation with respect to a_j and b_j

$$J = \int_0^{T=2\pi/\omega} E^2(\ddot{\delta}, \dot{\delta}, \delta, t) dt \quad (49)$$

The restoring force in equation (9) is approximated with a polynomial based on the mean square error. This helps the integration in equation (49) to be evaluated analytically. To minimize the error integration the simplex search method has been applied. The results of this method would be discussed in the results section.

In order to find the error integration analytically, using the Ritz method, first the non-linear function $f_s(\delta)$ must be approximated with a polynomial. Figures 3(a) and (b) shows two different approximation based on the fifth- and third-order polynomials, respectively.

According to the Ritz method, two different approximate solutions is assumed in the form of a five and seven terms series of trial functions which is described in the equations (50) and (51), respectively

$$\begin{aligned} \delta(t) = & a_0 + a_1 \sin \omega t + b_1 \cos \omega t + a_3 \sin 3\omega t \\ & + b_3 \cos 3\omega t \end{aligned} \quad (50)$$

$$\begin{aligned} \delta(t) = & a_0 + a_1 \sin \omega t + b_1 \cos \omega t + a_3 \sin 3\omega t \\ & + b_3 \cos 3\omega t + a_5 \sin 5\omega t + b_5 \cos 5\omega t \end{aligned} \quad (51)$$

The response of equivalent linear system can be obtained by Ritz method based on approximating $f_s(\delta)$ with a linear function. This approximating function has been shown in Fig. 4. In order to increase accuracy, the following approximation has been considered for $\delta(t)$

$$\begin{aligned} \delta(t) = & a_0 + a_1 \sin \omega t + b_1 \cos \omega t + a_3 \sin 3\omega t \\ & + b_3 \cos 3\omega t + a_5 \sin 5\omega t + b_5 \cos 5\omega t \\ & + a_7 \sin 7\omega t + b_7 \cos 7\omega t + a_9 \sin 9\omega t \\ & + b_9 \cos 9\omega t \end{aligned} \quad (52)$$

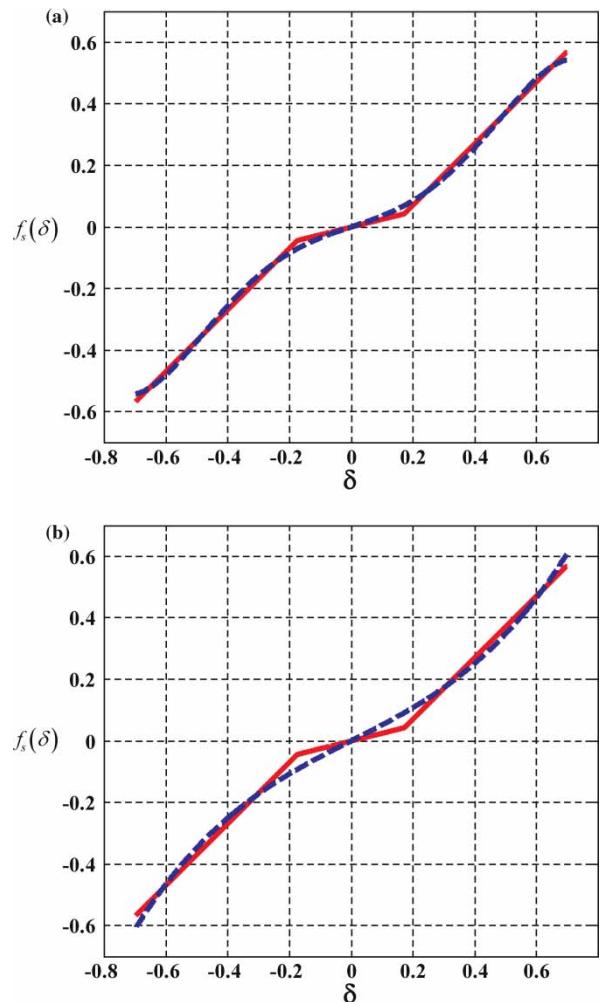


Fig. 3 The approximated restoring non-linear force $f_s(\delta)$ based on (a) a fifth-order and (b) a third-order polynomial

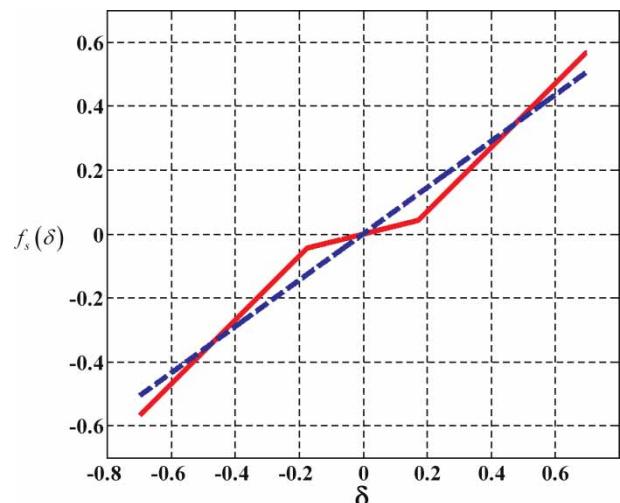


Fig. 4 The approximated restoring non-linear force $f_s(\delta)$ with a linear function

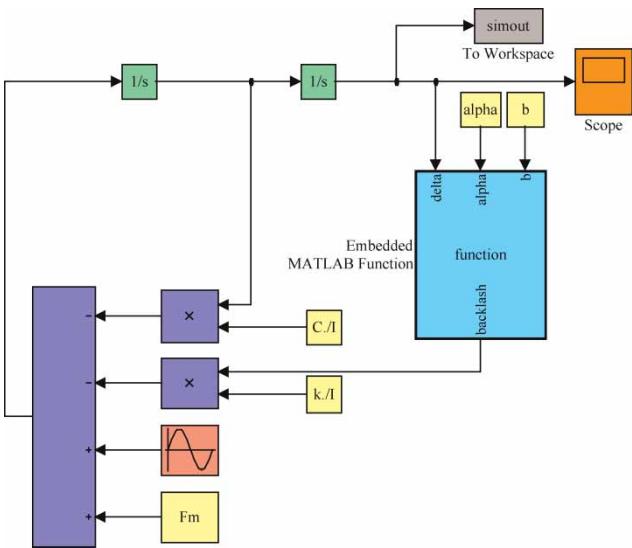


Fig. 5 The Simulink model used for solving the differential equation (10)

The results of these approximations would be discussed in results and discussion section.

6 THE STEPWISE TIME-INTEGRATION METHOD USING SIMULINK MODEL

From the transmission gear system model, shown in Fig. 1, together with the equations (9) and (10) a complete simulation model of the gear system can be developed, as shown in Fig. 5. This simulation model has been constructed MATLAB using the Simulink toolbox. The Simulink toolbox enables a

Table 1 Numerical values for system parameters [20]

F_m	F_p	B	C	α	K	I
0.005	0.03	10°	0.05	0.25	1	1

model to be constructed, using existing block functions held in its library. Each block contains elements of inertia, damping stiffness, friction, and backlash. The parameter values may be interactively selected and revised.

In Fig. 5, embedded MATLAB function block is predicted to compute the non-linear function $f_s(\delta)$ using equation (9). This Simulink model computes the response of the system at the specified excitation frequency. For finding the frequency response of the system this Simulink model is to be run for several excitation frequencies.

7 RESULTS AND DISCUSSION

The main objective of this analysis was to study the non-linear torsional vibration of the gear system. Four different methods of analysis were undertaken. In order to demonstrate the validity of these methods, a system as described by Kim *et al.* [13] is selected. Table 1 shows the values of the parameters used for simulating the system. Figure 6 illustrates the corresponding variations in relative displacement at different values of excitation frequency using the discretization method. In all these figures the motion reaches a steady-state after some time and continues its motion in a periodic manner.

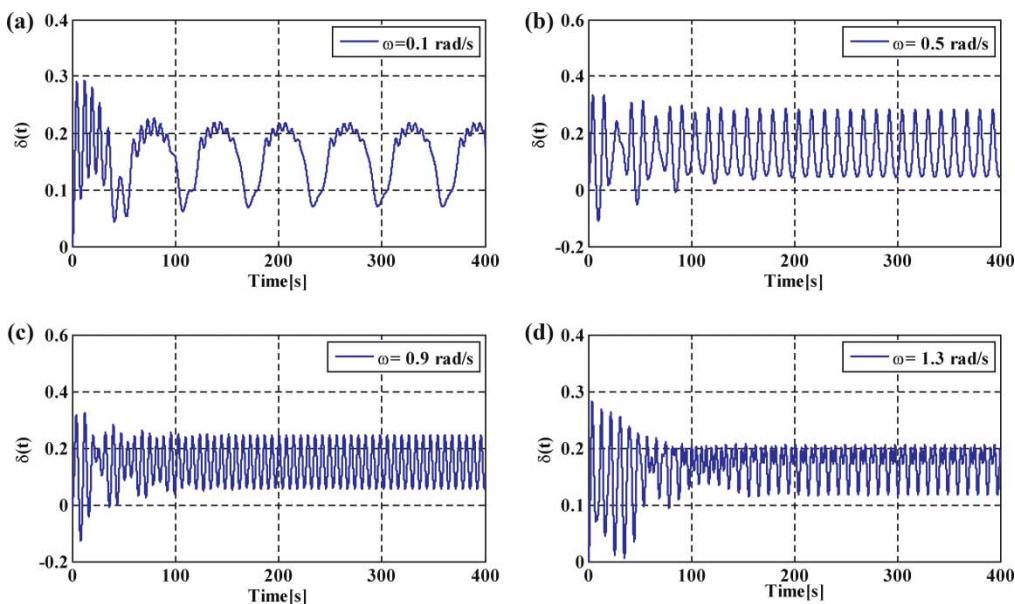


Fig. 6 Time response of the system with different frequency using the discretization method

A discrete Fourier transform analysis of Fig. 6 reveals frequency spectra having strong components at the frequency around 0.8 and 1.4 Rad/s, shown in Fig. 7.

Figure 8 shows the numerically predicted spectrum of vibration for the gear system using the perturbation method. The main spectral contribution occurs at $\omega = 0.9$ Rad/s. This frequency is in close proximity to that frequency of the power spectrum in Fig. 7 of the discretization method results. However, the second frequency has not been found using the perturbation method. The main reason of this limitation can be summarized as follows.

1. The damping effect has been ignored in the differential equation of motion.
2. The value of perturbation parameter, δ is not small enough (it is around 0.5).
3. The restoring non-linear force $f_s(\delta)$ has been approximated for convenience by only a third-order polynomial.
4. Only three terms of perturbation expansion of the system (the second-order approximation) has been considered.
5. The perturbation method makes restriction on finding all the frequency components simultaneously and for each frequency component the equation of motion should be solved separately.

Figures 9 and 10 show the frequency response of the system using the Ritz method based on the fifth- and the third-order polynomial approximation for $f_s(\delta)$, respectively. Also the frequency response of the equivalent linear system using the Ritz method is shown in Fig. 11.

The time response of the system using the Simulink model can be seen in Fig. 12 for different frequencies. In comparing Figs 6 and 12 it can be seen that there is

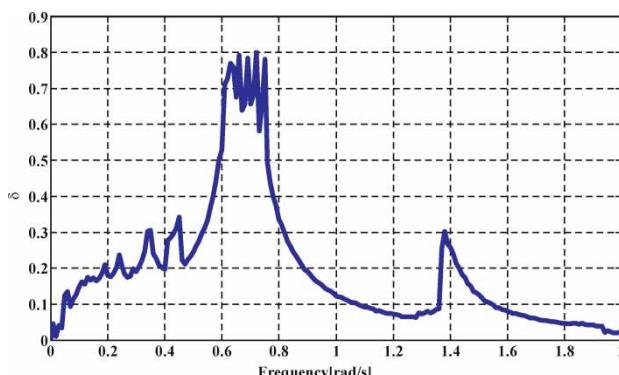


Fig. 7 Frequency response of the system obtained by the discretization method

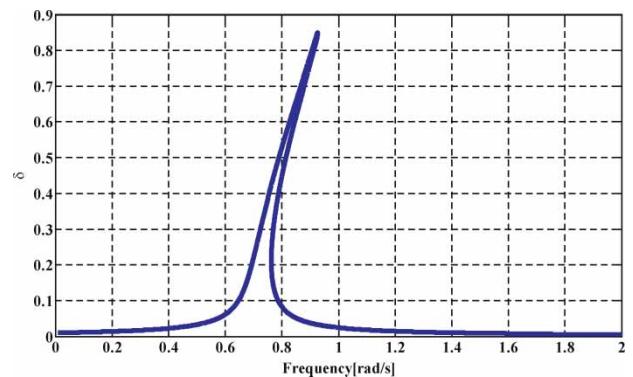


Fig. 8 The spectrum of vibration of the system obtained by the perturbation method

no noticeable difference in the dynamic responses. Consequently, for a non-linear system of this type, it would be reasonable to accept the Simulink model to be as accurate as the more complex discretization model.

A frequency analysis of the response of Fig. 12 is shown in Fig. 13. This figure shows that the Simulink model has been able to identify more frequency components whereas Figs 8 and 9 from the perturbation and Ritz models, respectively, is unable to identify these components. Consequently, using the perturbation and Ritz method to model non-linear systems, it is not possible to capture the total dynamic response of the system, thereby limiting the practical use of such models. It is shown that the Simulink model predictions agree well with the results obtained by Kim and Singh [20]. They presented a new multi-term HBM for non-linear frequency response of the same system, where the spectrum of the vibration is shown in Fig. 14. These results were found to agree with the Simulink model. Therefore, this model is valid and

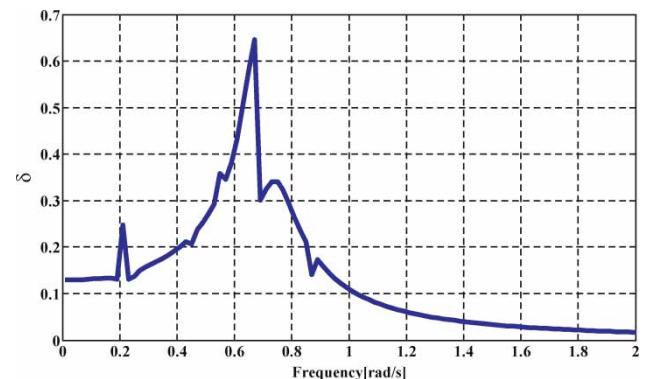


Fig. 9 Frequency response of the system using Ritz method, approximating $f(\delta)$ with a fifth-order polynomial and using equation (50) for $\delta(t)$

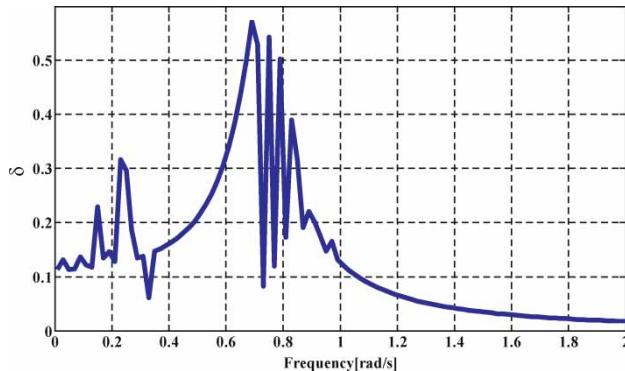


Fig. 10 Frequency response of the system using Ritz method, approximating $f_s(\delta)$ with a third-order polynomial and using equation (51) for $\delta(t)$

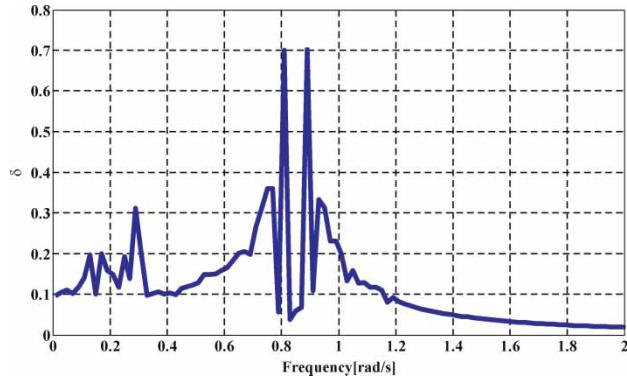


Fig. 11 Frequency response of the equivalent linear system using Ritz method and equation (52) for $\delta(t)$

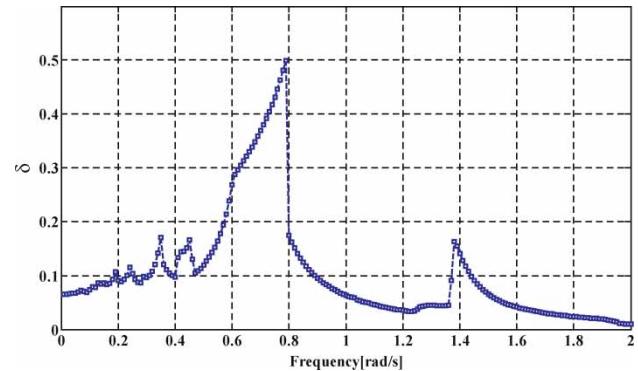


Fig. 13 Frequency response of the system using MATLAB Simulink model

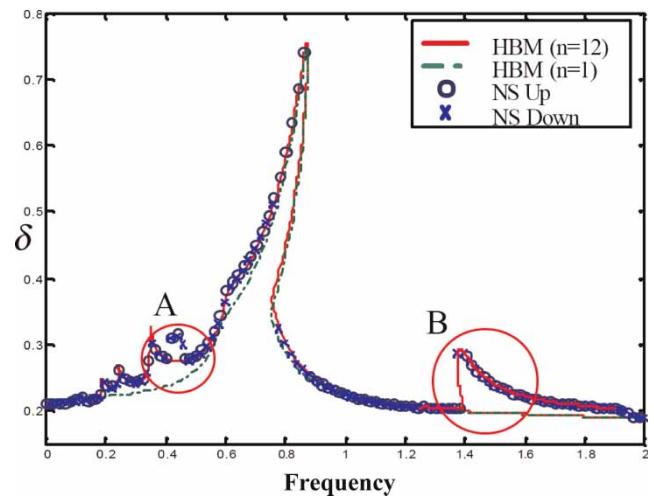


Fig. 14 Frequency response of the system obtained by Singh and Kim using HBM method [20]

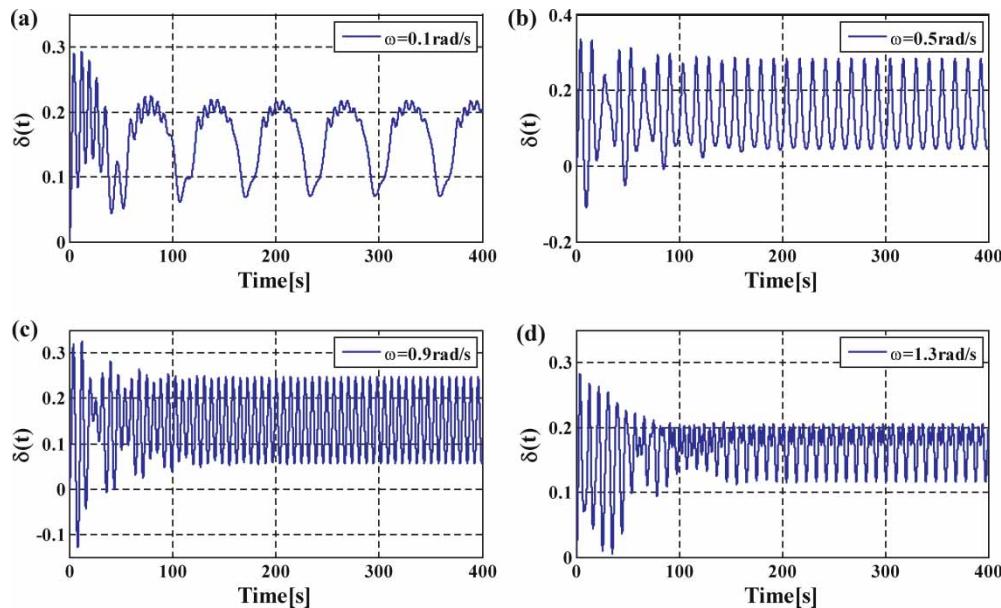


Fig. 12 The time response of the system at different frequency using the Simulink model

capable of modification and extension to investigate more complex systems with different loading conditions and characteristics.

As another point of view, it is important to have a qualitative comparison of computational time between utilized methods in the current study. In fact, the computational time is depended on several parameters such as time step, desired precision, etc., which varies method by method. Consequently, it is not possible to compare the computational time exactly. However, generally speaking, it can be said that the perturbation method is the fastest among these methods, because it leads to a closed form solution and can be computed quickly. After that, the stepwise time-integration method using MATLAB Simulink toolbox needs less computational time to find the problem solution. The Ritz method and discretization method require more computational time than other methods. It may be explained by considering the details of each method. The stepwise time integration uses primitive calculations which take less computation time than simplex minimizing procedure in Ritz method and evaluation of complicated relations of discretization method.

8 CONCLUSION

In the current paper, some mathematical models of a one-stage transmission gear system are presented. Four different techniques based upon the discretization method, the perturbation method, Ritz method, and the stepwise time-integration method using Simulink model predicts time and frequency response of the system in adequate manner. These results were found to agree with the HBM results. Although all of these techniques predict the main spectral contribution, however, only the discretization method and the Simulink model are able to identify the other frequencies components. The Simulink model in compare with the other model is relatively easy to apply even for more complex arrangements.

REFERENCES

- 1** Feng, Q. and Pfeiffer, F. Stochastic model on a rattling system. *J. Sound Vibr.*, 1998, **215**(3), 439–453.
- 2** Kahraman, A. and Singh, R. Nonlinear dynamics of a spur gear pair. *J. Sound Vibr.*, 1990, **142**, 49–75.
- 3** Kahraman, A. and Blankenship, G. W. Experiments on nonlinear dynamic behavior of an oscillator with clearance and periodically time-varying parameters. *Trans. ASME, J. Appl. Mech.*, 1997, **64**, 217–226.
- 4** Warminski, J., Litak, G., and Szabelski, K. Synchronization and chaos in a parametrically and self-excited system with two degrees of freedom. *Nonlinear Dyn.*, 2000, **22**, 135–153.
- 5** Litak, G. and Friswell, M. Nonlinear vibration in gear systems. *Chaos Solitons Fractals*, 2003, **16**, 795–800.
- 6** de Souza, S. L. T., Caldas, I. L., Balthazar, J. M., and Brasil, R. M. L. R. F. Analysis of regular and irregular dynamics of a non ideal gear rattling problem. *J. Braz. Soc. Mech. Sci.*, 2002, **24**(2), 111–114.
- 7** Kim, T. C., Rook, T. E., and Singh, R. Effect of smoothing functions on the frequency response of an oscillator with clearance nonlinearity. *J. Sound Vibr.*, 2003, **263**, 665–678.
- 8** Mark, W. D. Analysis of the vibratory excitation of gear systems: basic theory. *J. Acoust. Soc. Am.*, 1978, **63**(5), 1409–1430.
- 9** Sargeant, M. A., Drew, S. J., and Stone, B. J. Coupled torsional and transverse vibration of a back-to-back gearbox rig. *Proc. IMechE, Part K: J. Multi-body Dynamics*, 2005, **219**, 259–273.
- 10** Cornell, R. W. Compliance and stress sensitivity of spur gear teeth. *Trans. ASME, J. Mech. Des.*, 1981, **103**, 447–459.
- 11** Daniewicz, S. R., Collins, J. A., and Houser, D. R. The stress intensity factor and stiffness for a cracked spur gear tooth. *Trans. ASME, J. Mech. Des.*, 1994, **116**, 697–700.
- 12** Cardou, A. and Tordion, G. V. Calculation of spur gear tooth flexibility by the complex potential method. *Trans. ASME, J. Mech. Transm. Autom. Des.*, 1985, **107**, 38–42.
- 13** Kim, T. C., Rook, T. E., and Singh, R. Super- and sub-harmonic response calculations for a torsional system with clearance nonlinearity using the harmonic balance method. *J. Sound Vibr.*, 2005, **281**, 965–993.
- 14** Comparin, R. J. and Singh, R. An analytical study of automotive neutral gear rattle. *Trans. ASME, J. Mech. Des.*, 1990, **112**, 237–245.
- 15** Sun, T. and Hu, H. Y. Nonlinear dynamics of a planetary gear system with multiple clearances. *Mech. Mach. Theory*, 2003, **38**, 1371–1390.
- 16** Warmiński, J., Litak, G., and Szabelski, K. Dynamic phenomena in gear boxes. In *Applied nonlinear dynamics and chaos of mechanical systems with discontinuities*, World Scientific Series of Nonlinear Science A (Eds M. Wiercigroch and B. de Kraker), vol. 28, 2000, pp. 177–206 (World Scientific, Singapore).
- 17** Kim, T. C., Rook, T. E., and Singh, R. Effect of nonlinear impact damping on the frequency response of a torsional system with clearance. *J. Sound Vibr.*, 2005, **281**, 995–1021.
- 18** Trochon, E. and Singh, R. Effects of automotive gearbox temperature on transmission rattle noise. In Proceedings of Noise-Con 98, Dearborn, MI, USA, 1998, pp. 151–156.
- 19** Ozguvan, H. N. and Houser, D. R. Mathematical models use in gear dynamics. *J. Sound Vibr.*, 1988, **121**, 383–411.
- 20** Kim, T. C. and Singh, R. Frequency domain analysis of rattle in gear pairs and clutches. In Proceedings of Inter-Noise, August 2002, paper N613 (available on CD-ROM).

APPENDIX**Notation**

A	vibration amplitude (m)	K_{Mn}	amplitude of harmonic mesh stiffness of contacting gears (Nm/rad)
b	backlash or stage transmission point (rad)	t	time (s)
C	viscous damping coefficient (Nms)	T_1, T_2	excitation torques (Nm)
$f_s()$	non-linear function (rad)	T_m	mean torque (Nm)
f_{Mn}	initial phase of periodic mesh stiffness of contacting gears (rad)	T_p	perturbed torque (Nm)
F_m	effective mean torque (N/kg m ³)	δ	stiffness ratio
F_p	effective amplitude of single harmonic pulsating torque (N/kg m ³)	$\dot{\delta}, \ddot{\delta}$	relative angular displacement, velocity, and acceleration (rad, rad/s, rad/s ²)
F_{pj}	effective amplitude for j th harmonic of pulsating torque (N/kg m ³)	ε	perturbation parameter
F_{res}	restoring force (N)	θ_1, θ_2	absolute angular displacement of transmission gears (rad)
I	effective torsional moment of inertia (kg m ⁴)	$\dot{\theta}_1, \dot{\theta}_2$	absolute angular velocity of transmission gears (rad/s)
I_1, I_2	gears moment of inertia (kg m ⁴)	$\ddot{\theta}_1, \ddot{\theta}_2$	absolute angular acceleration of transmission gears (rad/s ²)
j	the torque harmonic index	ω	angular velocity (rad/s)
J	square error integration	ω_e	crankshaft rotational speed (rad/s)
K	torsional stiffness (Nm/rad)	ω_M	frequency of periodic mesh stiffness of contacting gears (rad/s)
		ω_p	fundamental frequency of perturbed torque (rad/s)