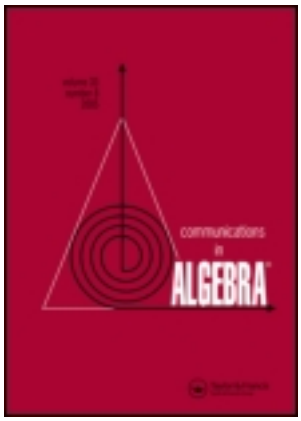

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NILPOTENT PRODUCTS OF CYCLIC GROUPS AND CLASSIFICATION OF *p*-GROUPS

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In 1940, Hall introduced the notion of isoclinism of groups for classifying all p-groups of order at most $p^5 (p > 3)$. According to this classification, all these groups are partitioned into ten families. This article intends to characterize all the families which have the nilpotent products of cyclic groups in themselves, and then determine the exact structures of these products.

Key Words: Classification of *p*-groups; Nilpotent product of cyclic groups; Isoclinism.

2010 Mathematics Subject Classification: 20D15; 20E99.

1. INTRODUCTION AND MOTIVATION

In 1940, Hall [5] defined an equivalence relation, called isoclinism, and applied it for classifying all p-groups of order at most p^5 ($p > 3$) into ten families $\Phi_1, \Phi_2, \ldots$ and Φ_{10}.

In 1950, Golovin [1] defined nilpotent products of groups and proved that every nilpotent group is a factor group of a nilpotent product of cyclic groups. Nilpotent products of cyclic groups are defined in a simple way and a unique decomposition theorem holds for them (see Golovin [2]). So these groups have suitable structures to study. On the other hand, we know that isoclinic groups have similar structures. Therefore, it would be desirable, if every isoclinism family contains a nilpotent product of cyclic group.

In 1960, Struik [8] conjectured that "every finite nilpotent group is isoclinic to a nilpotent product of cyclic groups," but unfortunately it was not true in general. She (Struik [8]) proved that there exists a group of order p^4 ($p \geq 3$), which is not isoclinic to any nilpotent product of cyclic groups. But it remained to find nilpotent groups which are isoclinic to nilpotent products of cyclic groups. In this article, we determine all p-groups of order at most p^5 ($p > 3$) with this property. To this end, we characterize all isoclinism families which contain nilpotent products of cyclic

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groups, among $\Phi_1, \Phi_2, \ldots$ and Φ_{10}. We also give the exact structures of all nilpotent products of cyclic groups in these families.

2. PRELIMINARIES

The following important equivalence relation was defined by Hall [5] for classifying finite p-groups.

Definition 2.1. Two groups G and H are isoclinic (or skew isomorphic), if there exist isomorphisms

$$\alpha : \frac{G}{Z(G)} \to \frac{H}{Z(H)} \quad \text{and} \quad \beta : G' \to H'$$

such that for all $g_1, g_2 \in G$, $\beta[g_1, g_2] = [h_1, h_2]$, where $h_i Z(H) = \alpha(g_i Z(G))$, for $i = 1, 2$. In this case, we write $G \sim H$ and say the pair (α, β) is an isoclinism between G and H.

The following lemma which is concluded from Definition 2.1, shows some similarities among isoclinic groups.

Lemma 2.2. *Let G and H be isoclinic. Then for any positive integer k, we have as follows*:

(i) $\gamma_{k+1}(G) \cong \gamma_{k+1}(H)$;
(ii) $G^{(k)} \cong H^{(k)}$;
(iii) $cl(G) = cl(H)$;
(iv) $Z(G) \cap G' \cong Z(H) \cap H'$;

where $\gamma_k(G)$ is the kth term of lower central series of G, $G^{(k)}$ is the kth term of derived series of G and $cl(G)$ denotes the nilpotency class of G.

Hall [3] defined the concept of basic commutator and showed that these commutators play an important role in the category of free groups. For example he proved that if F is a free group on free generators $x_1, x_2, \ldots, x_d$, then the basic commutators of weight n provide a basis for the free abelian group $\frac{\gamma_n(F)}{\gamma_{n+1}(F)}$. The number of basic commutators of weight n on d generators is obtained by the following theorem.

Theorem 2.3 (Witt Formula, Hall [3]). *The number of basic commutators of weight n on d generators is given by the formula*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m},$$

where $\mu(m)$ is the Möbius function defined by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & \text{if } m = p_1 \ldots p_s, \end{cases}$$

in which the p_i, $1 \le i \le k$, are the distinct primes dividing m.

Although Golovin [1] defined the nilpotent product of groups in general, in this article we need the following special case of the definition.

Definition 2.4. Let $A_1, A_2, \ldots, A_k$ be cyclic groups. Then n-nilpotent product of $A_1, A_2, \ldots, A_k$, denoted by $A_1 \overset{n}{*} A_2 \overset{n}{*} \cdots \overset{n}{*} A_k$, is defined to be the group $\frac{F}{\gamma_{n+1}(F)}$, where F is the free product of $A_1, A_2, \ldots, A_k$.

It is clear that the nilpotency class of a k-nilpotent product of cyclic groups is k. The following theorems, which are frequently used hereafter, give the structures of the center and the terms of lower central series of some k-nilpotent products of cyclic groups.

Theorem 2.5 Magidin [7]. *Let k be a positive integer and p a prime number such that $k \leq p$. Let $C_1, \ldots, C_r$ be cyclic p-groups generated by $x_1, \ldots, x_r$, respectively. Let p^{α_i} be the order of x_i, and assume that $1 \leq \alpha_1 \leq \cdots \leq \alpha_r$. If G is the k-nilpotent product of the groups C_i, $G = C_1 \overset{k}{*} C_2 \overset{k}{*} \cdots \overset{k}{*} C_r$, then $Z(G) = \langle x_r^{p^{\alpha_r - 1}}, \gamma_k(G) \rangle$.*

Theorem 2.6. *Let $G = \mathbb{Z}_{r_1} \overset{k}{*} \mathbb{Z}_{r_2} \overset{k}{*} \cdots \overset{k}{*} \mathbb{Z}_{r_t}$, such that $r_{i+1} \mid r_i$ for all i, $1 \leq i \leq t-1$. If $(p, r_1) = 1$ for any prime p less than or equal to k, then:*

(i) *If $k \leq c$, then $\gamma_{c+1}(G) = 1$;*
(ii) *If $c < k \leq 2c$, then $\gamma_{c+1}(G) = \mathbb{Z}_{r_2}^{(f_2 - f_1)} \oplus \cdots \oplus \mathbb{Z}_{r_t}^{(f_t - f_{t-1})}$, where $f_j = \sum_{i=1}^{k-c} \chi_{c+i}(j)$ for all j, $1 \leq j \leq t$, and $\mathbb{Z}_r^{(d)}$ denotes the direct sum of d copies of the cyclic group $\mathbb{Z}_r$.*

Proof. Since the nilpotency class of G is k, (i) is clear. The proof of (ii) is similar to Theorem 3.3 in Hokmabadi et al. [6]. □

3. MAIN RESULTS

Throughout this section, we use the notations $\Phi_1, \Phi_2, \ldots$ and Φ_{10} for isoclinism families, which are introduced by Hall [5] , in order to classifying all p-groups of order at most p^5 ($p > 3$). The following lemmas provide some required information about these families.

Lemma 3.1 Hall [4]. *Let G be a p-group of order p^n. If $G^{(m)}$ is not trivial, then we have $2^m + m \leq n$.*

Lemma 3.2. *Let G be a p-group of order p^5. Then G' is an abelian group of order at most p^3.*

Proof. By Lemma 3.1, G' is an abelian group. Let the order of G' be p^4. Then the derived subgroup of G coincides with its Frattini subgroup. Thus G is cyclic which is a contradiction. On the other hand, G is nilpotent. Hence $|G'|$ is not p^5, and then the result holds. □

Every group H with the property $Z(H) \subseteq H'$ is called a stem group. Hall [5] proved that every isoclinism family includes a stem group and the order of a

stem group divides the order of any member of its family. Therefore, the orders of stem groups in an isoclinism family are equal to a unique number. Moreover, by Definition 2.1, the structure of center factors in a family is also unique. In the following lemma, we try to accumulate some useful statement and information which were sporadically proved by Hall [5]. This lemma actually determines both structure of center factor and the order of a stem group in families $\Phi_1, \Phi_2, \ldots$ and Φ_{10}.

Lemma 3.3. *Let $v_i = (N_i, x_i)$ be a pair such that N_i and x_i denote the structure of center factor and the order of a stem group in family Φ_i, respectively. Then we have $v_1 = (1, 1)$, $v_2 = (\mathbb{Z}_p \oplus \mathbb{Z}_p, p^3)$, $v_3 = (E_1, p^4)$, $v_4 = (\sum_{i=1}^{3} \mathbb{Z}_p, p^5)$, $v_5 = (\sum_{i=1}^{4} \mathbb{Z}_p, p^5)$, $v_6 = (E_1, p^5)$, $v_7 = (E_1 \times \mathbb{Z}_p, p^5)$, $v_8 = (\Phi_2(2^2), p^5)$, $v_9 = (\Phi_3(1^4), p^5)$, and $v_{10} = (\Phi_3(1^4), p^5)$, in which E_1 is the extra special p-group of order p^3 and exponent of p, $\Phi_2(2^2)$ is the group of order p^4 and type (2^2) in family Φ_2, and $\Phi_3(1^4)$ is the group of order p^4 and type (1^4) in family Φ_3.*

Invoking Lemma 3.3, we can prove the following useful lemma.

Lemma 3.4. *Let $u_i = (z_i, t_i)$ be a pair such that z_i, and t_i denote the nilpotency class and the order of derived subgroup in family Φ_i, respectively. Then we have $u_1 = (1, 1)$, $u_2 = (2, p)$, $u_3 = (3, p^2)$, $u_4 = (2, p^2)$, $u_5 = (2, p)$, $u_6 = (3, p^3)$, $u_7 = (3, p^2)$, $u_8 = (3, p^2)$, $u_9 = (4, p^3)$, and $u_{10} = (4, p^3)$.*

Proof. It is easy to obtain t_1 and z_i, for all i, $1 \le i \le 10$. Suppose that H_i denotes a stem group of Φ_i. If $i = 2, 4, 5$, then Lemma 3.3 implies that $(\frac{H_i}{Z(H_i)})' = 1$ and hence $|H_i'| = |Z(H_i)|$. On the other hand, $|Z(H_2)| = |Z(H_5)| = p$ and $|Z(H_4)| = p^2$. So the result follows for $i = 2, 4, 5$. If $i = 3, 6, 7$, then $(\frac{H_i}{Z(H_i)})' \cong \mathbb{Z}_p$ and hence $|H_i'| = p|Z(H_i)|$, by Lemma 3.3. This proves the lemma for $i = 3, 6, 7$. Since $\frac{H_8}{Z(H_8)} \cong \Phi_2(2^2)$, then $\frac{H_8}{Z(H_8)}$ belongs to Φ_2 and has order p^4. As the order of the derived subgroup in family Φ_2 is p, then $|H_8'| = p|Z(H_8)|$. Therefore, $|H_8'| = p^2$. Similarly for $i = 9, 10$ the result holds, since $\frac{H_9}{Z(H_9)}$ and $\frac{H_{10}}{Z(H_{10})}$ are in Φ_3 and have order p^4 . □

Lemma 2.2 shows that every two isoclinic groups have the same nilpotency class. Therefore, if $G \sim H$, in which $G = \mathbb{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbb{Z}_{p^{\alpha_2}} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{p^{\alpha_t}}$ and $H = \mathbb{Z}_{p^{\beta_1}} \overset{m}{*} \mathbb{Z}_{p^{\beta_2}} \overset{m}{*} \cdots \overset{m}{*} \mathbb{Z}_{p^{\beta_r}}$, then we must have $n = m$. In particular, let $1 < n < p$ and $\alpha_i = \beta_j = 1$ for all $i > 1$ and $j \ge 1$. Then using Theorem 2.6, it can be easily seen that if $G \sim H$, then we have $r = t$. Furthermore, the next theorem implies that the converse of the last statement is also true.

Theorem 3.5. *Let n and α be positive integers and p be a prime number such that $p > n$. Let $A_1, A_2, \ldots, A_t, B_1, B_2, \ldots, B_t$ be cyclic p-groups generated by $x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t$, respectively. Suppose that x_1 is of order p^α and other generators are of order p. If $G = A_1 \overset{n}{*} A_2 \overset{n}{*} \cdots \overset{n}{*} A_t$ and $H = B_1 \overset{n}{*} B_2 \overset{n}{*} \cdots \overset{n}{*} B_t$, then $G \sim H$.*

Proof. Since every element of an n-nilpotent product of cyclic groups has a unique form, the map $\theta : G \to H$, defined by $\theta(x_i) = y_i$, for all i, $1 \le i \le t$, is a homomorphism. Using Theorem 2.5, one can see that θ induces the isomorphism $\bar{\theta}$

from $\frac{G}{Z(G)}$ to $\frac{H}{Z(H)}$. Moreover, $\theta|_{G'}$ is an isomorphism from G' to H'. Now, it is clear that $(\bar{\theta}, \theta|_{G'})$ is an isoclinism between G and H. □

Now, we are going to determine isoclinism families including a nilpotent product of cyclic groups. Let p be a prime number greater than 3 and G be an n-nilpotent product of cyclic groups. If G lies in $\Phi_1, \Phi_2, \ldots$ and Φ_{10}, then $n \leq 4$, by Lemma 3.4. Whereas 1-nilpotent product and direct product of cyclic groups coincide, we have $G \in \Phi_1$ if and only if $n = 1$. In what follows, the results for $n = 2, 3, 4$ are given.

Theorem 3.6. *If $i \neq 2$, then there is not any 2-nilpotent product of cyclic p-groups in Φ_i.*

Proof. Let $G = \mathbb{Z}_{p^{\alpha_1}} \overset{2}{*} \mathbb{Z}_{p^{\alpha_2}} \overset{2}{*} \cdots \overset{2}{*} \mathbb{Z}_{p^{\alpha_t}}$, such that $\alpha_t \leq \alpha_{t-1} \leq \cdots \leq \alpha_1$. Since the nilpotency class of G is two, it may lie in Φ_2, Φ_4 or Φ_5, by Lemma 3.4. Let H be a stem group in family Φ_4 or Φ_5. Since the center factor of H is an elementary abelian $p =$ group, H^p is contained in the center of H. Therefore, $H^{'}p = 1$. Hereupon the exponent of derived subgroup in families Φ_4 and Φ_5 is p and its order is at most p^2. Hence if G lies in Φ_4 or Φ_5, then we must have $t = 2$ and $\alpha_2 = 1$, by Theorem 2.6. On the other hand, one can use Theorem 2.5 to see that the structure of the center factor of the group $\mathbb{Z}_{p^{\alpha_1}} \overset{2}{*} \mathbb{Z}_p$ is $\mathbb{Z}_p \oplus \mathbb{Z}_p$, which is a contradiction, by Lemma 3.3. Therefore, G does not lie in Φ_4 or Φ_5. □

The above theorem shows that any 2-nilpotent product of cyclic p-groups of order at most p^5 belongs to Φ_2.

By an argument similar to the proof of Theorem 3.6, one can easily deduce the next theorem.

Theorem 3.7. *If p is an odd prime number and $i \neq 6$, then there is not any 3-nilpotent product of cyclic p-groups in Φ_i.*

Now let G be a 4-nilpotent product of cyclic p-groups. Theorem 2.6 implies that the order of the third term of lower central series of G is greater than p^4 and so is the order of its derived subgroup. Then G does not lie in Φ_i for all i, $1 \leq i \leq 10$, by Lemma 3.2.

The above argument shows that if $n > 1$, then every n-nilpotent product of cyclic groups of order at most p^5 belongs to ϕ_2 or ϕ_6. In the sequel we would like to introduce the structure of all nilpotent products of cyclic groups which lie in families Φ_2 and Φ_6. As Hall [5] noted, the members of the family Φ_2 are characterized by either of the following two properties:

I. Their center is of index p^2;
II. They have more than one abelian subgroup of index p, without themselves being abelian.

It is easy to see that every two groups with either of these properties must be isoclinic. This fact enables us to prove the following theorem.

Theorem 3.8. *Let G be a 2-nilpotent product of cyclic p-groups. Then G belongs to Φ_2 if and only if $G \cong \mathbb{Z}_{p^\alpha} \overset{2}{*} \mathbb{Z}_p$, for any positive integer α.*

Proof. The necessity can be deduced by Theorem 2.6 and Lemma 3.3. For sufficiency, let $G = \mathbb{Z}_{p^{\alpha_1}} \overset{2}{*} \mathbb{Z}_p$. Then Theorem 2.5 implies that $\frac{G}{Z(G)}$ is an abelian group of order p^2, and hence the result holds. □

Let G and H be two stem groups of order p^5. Hall [5] showed that if $\frac{G}{Z(G)} \cong E_1 \cong \frac{H}{Z(H)}$, then $G \sim H$ and he nominated this family by Φ_6. In other words, he proved that there exists just one stem group, up to isoclinism, of order p^5 such that its center factor is isomorphic to E_1. This fact helps us to state the following lemma.

Theorem 3.9. *Let G be a 3-nilpotent product of cyclic p-groups. Then G belongs to Φ_6 if and only if $G \cong \mathbb{Z}_{p^\alpha} \overset{3}{*} \mathbb{Z}_p$, for any positive integer α.*

Proof. Necessity follows from Theorem 2.6 and Lemma 3.3. Now let $G = \mathbb{Z}_p \overset{3}{*} \mathbb{Z}_p$. By Theorem 2.5, we have $Z(G) = \gamma_3(G)$, and so G is a stem group, where $\frac{G}{Z(G)} \cong E_1$. Also Struik [8] showed that the order of this group is p^5. Thus $G \in \Phi_6$. Thereupon, the result follows by Theorem 3.5. □

Our main results of this article are accumulated in the following proposition.

Proposition 3.10. *Let p be a prime number greater than 3 and G be an n-nilpotent product of cyclic p-groups. Then G does not lie in families Φ_i, for $3 \leq i \leq 10$ and $i \neq 6$. Moreover:*

(i) *$G \in \Phi_1$ if and only if $n = 1$;*
(ii) *$G \in \Phi_2$ if and only if $n = 2$ and $G \cong \mathbb{Z}_{p^\alpha} \overset{2}{*} \mathbb{Z}_p$;*
(iii) *$G \in \Phi_6$ if and only if $n = 3$ and $G \cong \mathbb{Z}_{p^\alpha} \overset{3}{*} \mathbb{Z}_p$.*

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