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A posterior error estimation in the isogeometrical analysis method

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ABSTRACT: A new approach for error estimation and stress recovery in the isogeometrical analysis method is presented. In this approach, by making use of the superconvergent points, each of the components of the improved stress tensor is considered as an imaginary surface. This surface is generated by using the same NURBS' basis functions which are employed for approximation of the primary variables in the isogeometric analysis method. To demonstrate the performance of the method a couple of examples with available exact energy norm errors are solved and the obtained results are presented.

1 INTRODUCTION

Amongst the several numerical methods the currently suggested Isogeometric Analysis Method [1–6], which still is in its early stages of development, has a few interesting features that presents itself as a potential substitute to the other numerical techniques. In most of the currently practiced methods, such as finite elements and finite difference, as well as the meshfree methods, precise definition of the boundaries of the domain of problem is not easy to achieve. Also, in problems that need to be considered in a Lagrangian frame, that require several times of renewing the descretization of the domain of the problem, or for instance, in problems that the coefficients of the partial differentials are not constants, e.g. in analysis of the so called functionally graded materials (FGM), or in three dimensional structural shape optimization problems, the performance of the currently used methods is not always satisfactory [7].

In an attempt to overcome some of the above problems and inspired by the developments in geometrical modeling and CAGD (Computer Aided Geometry Design) for description of complicated shapes, the idea of has recently been proposed by Hughes and his co-workers [1–6]. The isogeometric analysis method has many features in common with the Finite Element Method (FEM) and some features in common with the meshfree methods [1] and attempts to take advantage of some interesting properties of splines and NURBS [8].

Since all numerical methods are approximate, the obtained solutions are inevitably erroneous and hence the question of the degree of reliability of the obtained results is a main concern of scientists and engineers. For this reason error estimation in numerical computations is as old as the numerical computations themselves [9]. Today the procedures available for error estimation are essentially reduced to two kinds [9]: the residual error estimators as the continuation of the original work of Babuska and in the second approach, suggested by Zienkiewicz and Zhu [9], a recovery process is followed to obtain more accurate representation of the unknowns.

In this research work, a new approach, for improvement of stresses and estimation of solution errors based on the isogeometrical analysis, is represented. Section 2 is devoted to a brief explanation of the isogeometric analysis method. In Section 3 derivation of a new formulation for the stress recovery and error estimation for the isogeometric analysis method is presented. Two examples of two dimensional problems is the subject of Section 4. Finally, conclusions are presented in Section 5.

2 ISOGEOMETRICAL ANALYSIS

Accepting the point that by using proper versions of splines, one is able to define complex geometries, any component of a field variable which satisfies a governing partial differential equation might be imagined as a surface comes the idea of discretizing the domain of interest by the defining points of these splines instead of, for instance, using the finite element meshes, finite difference grids, or the collection of points in the meshfree methods. Applicability and usefulness of this idea is verified in the case of ordinary differential equations by this research team [10]. Also, very promising results are reported by Hughes and his coworkers [1-6] where the idea is employed in several advanced problems in the area of computational mechanics.

The main advantages of the isogeometrical analysis method are flexibility and accuracy in the definition of the geometry and its boundaries, a considerable reduction in the size of system of equations, accuracy in satisfaction of the essential boundary conditions and considerable ease in implementing adaptivity and mesh refinement.

3 STRESS RECOVERY IN ISOGEOMETRICAL ANALYSIS

The main idea of the proposed method is inspired by the core idea of the isogeometrical analysis. That is, the components of the recovered stress field are considered as an imaginary (hyper-) surface. This surface is constructed by the same spline basis functions and NURBS shape functions that were employed for approximation of the components of the unknown displacement field vector. In two dimensional elasticity problems, similar to the isogeometrical analysis method itself, the *x* and *y* coordinates of the control points which are used for the analysis are considered as before and their *z* coordinates are calculated based on a criteria which is constructed by using the well known notion of superconvergent points in finite elements [11]. The so called superconvergent points are points that at them the obtained approximate solution for the gradient of the primary variable has a better accuracy in comparison with the other points of the domain of interest. This expression was originally employed by Barlow [11] and it can be shown that at these special points the convergence rate is one order higher than that was used for the approximation of the unknown function. In finite elements these superconvergent points are the same as the Gauss integration points. One should note that, in this respect, the number of Gauss points is the minimum that can render an exact integration of the employed polynomial approximation. Here, a similar approach is followed and the knot spans are considered as the elements in the finite element method. To find the unknown coordinates of the defining control points of the stress recovery components, the least square method is employed and the distance between the surface obtained from the isogeometrical analysis and the stress recovery surface at the superconvergent points is minimized.

Denoting the components of the stress recovery surface by σ^*, they can be constructed by using the NURBS' basis functions as

$$\sigma^* = \sum_{i=1}^{m}\sum_{j=1}^{n} R_{i,j}(u,v)P_{i,j}\ , \tag{1}$$

where *m* and *n* are the number of control points of the domain in the *x* and *y* dimensions, respectively. Also, $R_{i,j}$ are the NURBS shape functions and $P_{i,j}$ are the coordinates of control points where

$$\mathbf{R} = \left[R_{1,1}, R_{1,2}, \ldots, R_{1,n}, R_{2,1}, R_{2,2}, \ldots, R_{2,n}, \ldots, R_{m,n}\right]^T \tag{2}$$

$$\mathbf{P} = \left[P_{1,1}, P_{1,2}, \ldots, P_{1,n}, P_{2,1}, P_{2,2}, \ldots, P_{2,n}, \ldots, P_{m,n}\right]^T . \tag{3}$$

From which it follows that

$$\sigma^* = \mathbf{R}^T\mathbf{P} \tag{4}$$

As it is observed, for two dimensional problems only the *z* component of the control points (i.e. vector **p**) is the unknown to be determined. To do this, the sum of least squares between the obtained stresses by the isogeometrical analysis and the recovered stresses at the Gauss points are minimized. For this purpose, the function $F(\mathbf{P})$ is defined as

$$F(\mathbf{P}) = \sum_{j=1}^{k_y}\sum_{i=1}^{k_x}(\sigma^*_{i,j} - \bar{\sigma}_{i,j})^2 \tag{5}$$

where $\bar{\sigma}$ is the stress obtained from isogeometrical analysis and k_x and k_y the number of the Gauss points in the *x* and *y* directions, respectively. By substituting from (4) into (5) it follows

$$F(\mathbf{P}) = \sum_{l=1}^{K}(\mathbf{R}_l^T\mathbf{P}_l - \bar{\sigma}_l)^2 \tag{6}$$

where *k* is the number of Gauss points in each patch. From the stationary conditions of $F(\mathbf{P})$, as it follows, the *z* coordinates of the surface of improved stresses are obtained.

$$\frac{\partial F(\mathbf{P})}{\partial P_{i,j}} = 0 \quad \Rightarrow \ \mathbf{AP} = \mathbf{B} \ \Rightarrow \ \mathbf{P} = \mathbf{A}^{-1}\mathbf{B} \tag{7}$$

where **A** and **B** are defined as

$$\mathbf{A} = \sum_{i=1}^{K}\mathbf{R}_i\mathbf{R}_i^T \qquad ; \qquad \mathbf{B} = \sum_{i=1}^{K}\mathbf{R}_i\bar{\sigma}_i \quad .$$

Now, having the coordinates of the control points of the improved stresses, the related surface can be constructed by NURBS. As it will be demonstrated in the following, this surface has a higher level of accuracy and it can be employed as a potential error estimator for the isogeometric analysis method. In other words, the difference between the stresses obtained from isogeometrical analysis and this improved stress surface over each knot element is taken as an error index.

4 NUMERICAL EXPERIMENTS

To demonstrate the performance of the method, in this section, two plane stress linear elasticity problems are presented. The first is a cantilever beam subjected to a load at the free end. The second is an infinite plate with a circular hole subjected to a unit tensile traction in the horizontal direction. The problems have analytical solutions and therefore are useful for verification.

4.1 *Cantilever beam*

A cantilever beam with its length and width equal to $L = 10$ and $D = 2$, respectively, subject to a vertical shear distributed load equal to $P = 300$ at its free end is considered. The problem definition is depicted in Figure 1. The material properties are taken as Young's modulus $E = 1500$ and Poisson's ratio $\nu = 0.15$. The analytical stresses for this problem are available in References [12,13].

The isogeometrical analysis is carried out by using a net of 105 control points and two similar adjacent patches as illustrated in Figure 3. The employed knot vectors, in the horizontal (ξ) and vertical (η) directions are: $\xi = \{0, 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1\}$ and $\eta = \{0, 0, 0.3, 0.5, 0.7, 1, 1\}$. Here, in accordance with the results reported in [14] second order

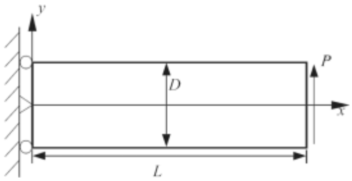


Figure 1. Cantilever beam subjected to a load at the free end.

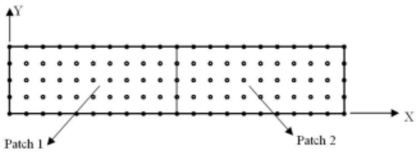


Figure 2. Net of control points for isogeometrical analysis of the cantilever beam.

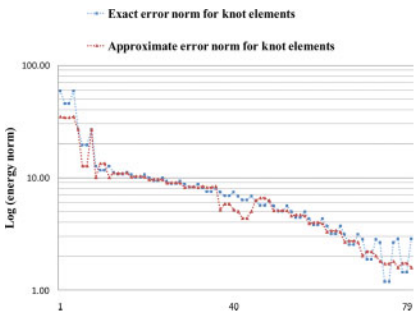


Figure 3. Comparison between the exact and approximate energy norm errors for knot elements.

NURBS' basis functions are employed. Also, a three by three Gauss quadrature is used which is in agreement with the results of Reference [15].

The exact and approximate energy norm of errors for the knot elements of this problem are shown in Figure 4. As it is observed, there is a relatively good correspondence between the two which indicates the effectiveness of the proposed error estimator.

The calculated effectivity index for the whole domain was 0.86. As it was pointed out before, this index is obtained by dividing the sum of approximate error norm of all the knot elements of the domain to the sum of exact error norms. This measure of the quality of error estimation, tends to one for a more effective error estimator [11]. Distribution of the exact and approximated errors within the domain of the cantilever beam problem is illustrated in Figure 4.

4.2 *Infinite plate with a hole*

As the second example, the problem of infinite plate with a hole subject to horizontal tensile tractions is considered. A square portion of the infinite plate is depicted in Figure 5. Due to symmetry only a quarter of the problem is modeled for numerical analysis as a plane stress problem. The modulus of elasticity and the Poisson's ratio are taken as $E = 1000$ and $\nu = 0.3$, respectively and the intensity of the traction force is

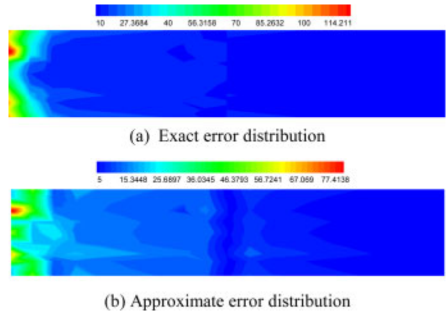


Figure 4. Distribution of errors in cantilever beam.

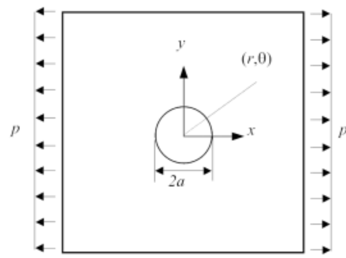


Figure 5. A square portion of an infinite plate with a circular hole under horizontal tensile tractions.

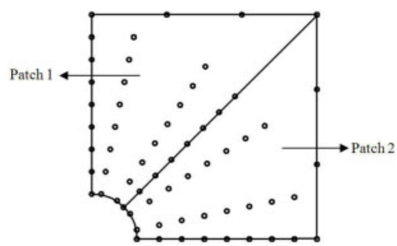


Figure 6. Control points for isogeometrical analysis of plate with a circular hole.

assumed $p = 1$. The analytical stresses for this problem can be found in References [12,13].

For isogeometrical analysis of this problem, a control net of 63 points and two similar adjacent patches, as shown in Figure 6, are used. Similar to the previous example, NURBS' basis functions of order two with the following knot vectors are employed in the radial and circumferential directions:

$\xi = \{0,0,0.2,0.3,0.4,0.5,0.6,0.7,0.8,1,1\}$ and $\eta = \{0,0,0.3,0.7,1,1\}$.

The exact and approximate energy norm of errors for the knot elements are compared in Figure 7. The calculated value for the effectivity index for this problem was 0.81. Distribution of the exact and approximated energy norm errors within the domain of this problem is illustrated in Figure 8.

5 CONCLUSION

The proposed method in this article for error estimation in isogeometrical analysis of elasticity problems

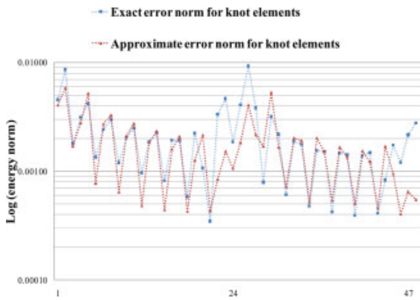
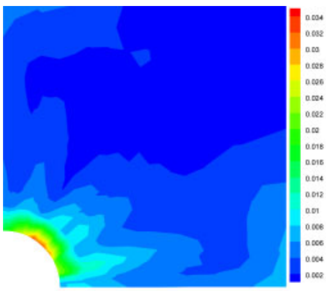
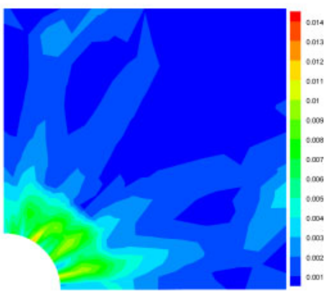


Figure 7. Comparison between the exact and approximate errors of knot elements for infinite plate with a hole.



(a) Exact error norm



(b) Approximate error norm

Figure 8. Distribution of errors for the problem of plate with a hole.

makes use of the abilities of NURBS' basis functions in creating surfaces of any shape and seems a natural extension of the isogeometric analysis method. Indeed, the same basis functions that are used for defining of the domain of the problem are employed as the approximation and interpolation functions as well as the construction of the recovered stress components. From the results of the examples that were taken into consideration it is concluded that the obtained effectivity index is quite reasonable. This technique can be employed as a simple and engineering approach for error estimation and adaptive solution in the isogeometric analysis method.

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