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Isogeometric shape optimization of three dimensional problems

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1. Abstract

Shape optimization of 3-D structures is a challenging task. The newly developed isogeometrical analysis is a more suitable approach, in comparison with other numerical methods, to address the problem. Apart from the ability of defining the boundaries of the problem by more precision which cannot be easily achieved by other techniques, such as finite element and mesh-free methods, this method takes burden of several remeshings that are needed in the conventional approaches to structural shape optimization.

In this research work, the domain of interest is defined by using the Non-Uniform Rational B-Splines (NURBS). Also, a numerical analysis technique is developed which uses the NURBS basis functions in place of the conventional shape functions for approximation. The control points defining the design boundary surface are considered as the design variables of the optimization problem. A few examples are presented to demonstrate the performance of the method.

2. Key words: Structural optimization, Shape optimization, Isogeometric analysis, NURBS

3. Introduction

Shape optimization of three dimensional and shell structures has drawn the attention of many disciplines in industry for its role in improving the structural performance. The aim of structural shape optimization is to find the optimal geometry of a structure while keeping the behavioral and geometrical constraints fulfilled. Most frequently, weight, strain energy (mean compliance) or the value of natural frequencies of the structure constitute the objective function. Maximum stresses and displacements or their value at certain locations are usually considered as constraints of the optimization problem.

A considerable amount of literature has been published on structural shape optimization which goes back to more than 30 years ago. One of the first papers on the subject is [1, 2]. For a survey of the subject references [3-5] can be consulted. However, the geometrical optimization of three dimensional structures is addressed only in a very few of the published literature.

It is well known in the optimization community that to circumvent the problem of obtaining results with wiggly and irregular shapes and in order to obtain structures with smooth boundaries it is useful to employ some kind of geometrical parameterization, e.g. splines, for geometrical description and defining the boundaries of the problem [6-10]. However, due to the variations of the boundaries within the optimization process, there is a need for several adjustments between the evolving geometry and the analysis model. This, in practice, requires several finite element remeshings which is quite costly and is a serious obstacle in real life problems.

To overcome the problems mentioned above, and inspired by the developments in geometrical modeling and CAGD (Computer Aided Geometry Design) for description of complicated shapes, the idea of isogeometric analysis has recently been proposed by Hughes and his co-workers [11-15]. The proposed method has many features in common with the finite element method and some features in common with the meshfree methods [11] and attempts to take advantage of some interesting properties of splines and NURBS [16]. The main idea is to use the same basis functions which are employed for geometry description, for approximation and interpolation of the unknown field variables as well. By this means, when applied to structural shape optimization, besides having smooth boundaries in the whole optimization process, the problem of several finite element remeshings is also obviated.

The isogeometrical method has recently been employed by Wall and Cho *et. al.* [17, 18] for the purpose of shape optimization of plane problems. In this research work, in addition to a couple of plane examples, application of the method is extended to three dimensional problems.

In Section 2 the mathematical model for the shape optimization problem is concisely explained. In Section 3 a brief explanation of the isogeometric analysis method is presented. Section 4 is devoted to the derivation of the formulation for the problem at hand. Presentation of a few examples of two and three dimensional problems is the subject of Section 5. Finally, quality of the results is discussed briefly in Section 6.

4. Definition of the shape optimization problem

For a linear elastic body the aim of the shape optimization problem is to find, by means of boundary variation, the shape of the domain $\Omega \subseteq D$ that minimizes (or maximizes) the objective function with respect to the state (equilibrium) as well as the behaviour and geometric constraints. The optimization problem may be expressed as

$$\begin{aligned} &\min_{\Omega \subseteq D} && F(\mathbf{s}) \\ &\text{subject to} && h_j(\mathbf{s}) = 0, \quad j = 1,2,\ldots,n_h \\ & && g_k(\mathbf{s}) \leq 0, \quad k = 1,2,\ldots,n_k \end{aligned} \tag{1}$$

where $D \subseteq \mathbb{R}^3$ denotes the set of admissible shapes defined through local geometric constraints (e.g. $s_i^{\ell} \leq s_i \leq s_i^{u}$). The design variables defining the shape are denoted by s_i ($i = 1,\ldots,n$). h_i and g_k are the equality and inequality constraints, respectively. The objective function $F(\mathbf{s})$ can be the volume, weight or the strain energy.

Various mathematical programming techniques may be used to solve the optimization problem such as sequential quadratic programming (SQP) [19, 20] or the method of moving asymptotes (MMA) [21]. The SQP method is used in this research.

In structural shape optimization, the parameters defining the geometry of the problem are taken as design variables. These parameters could be the coordinates of the boundary nodes of the finite element model or coefficients of polynomial representations of the boundaries which both suffer from serious drawbacks [22], or the control points of splines together with an automatic, unstructured and frontal finite element mesh generator. It is important to notice that in the conventional shape optimization by using finite elements it is advantageous to separate the design model from the analysis model [23] to have flexibility for geometry description. However, in the isogeometrical method, subject of this research, since these models are merged, this problem doesn't exist. Therefore, some of the control points which define the boundaries of the domain of the structure are considered as the design variables of the optimization problem.

5. Isogeometrical analysis

Inspiration of the so named isogeometrical analysis [11] is mainly due to the immense developments in the CAD (Computer Aided Design) technology, especially, part of it which is related to surface generation and definition of objects with complicated geometries (CAGD). For this purpose, splines and some new versions of them, i.e. non uniform rational B-splines (NURBS) and t-splines, are usually employed. Accepting the point that by using proper versions of splines, one is able to define complex geometries, any component of a field variable which satisfies a governing partial differential equation might be imagined as a surface comes the idea of discretizing the domain of interest by the defining points of these splines instead of, for instance, using the finite element meshes, finite difference grids, or the collection of points in the meshfree methods. Applicability and usefulness of this idea is verified in the case of ordinary differential equations by this research team [22]. Also, very promising results are reported by Hughes and his coworkers [11-15] where the idea is employed in several advanced problems in the area of computational mechanics.

The main advantages of the isogeometrical analysis method, compared to the other numerical methods, can be summarized as below:

- A considerable reduction in the size of system of equations, in comparison with the other numerical methods, when expecting an almost the same level of accuracy.
- Flexibility and accuracy in the definition of the geometry and its boundaries, so that in most cases a near exact boundary surface can be defined with a considerable ease.
- The possibility of keeping the original model in the whole process, without several remeshings, in problems with a varying domain of interest, e.g. problems which are solved in a Lagrangian frame or the structural shape optimization problems, due to the flexibility of the approach.
- Considerable ease in implementing adaptivity and mesh refinement.

- Accuracy in satisfaction of the essential boundary conditions. Note that, in other numerical methods, such as finite elements, finite difference and meshfree methods, this can be achieved at only some certain descretization points.
- Applicability of the method in problems that the coefficients of the differentials are functions themselves, which is the case, for example, in the functionally graded materials (FGM).

One of the main drawbacks of the method is that the defining points are not located on the solution surface. In other words, subsequent to finding the position of the control points, some extra effort is needed to obtain the solution surface. Furthermore, finding the correspondence between the physical points and parameters of the basis functions of splines needs either searching or solving a kind of inverse problem.

5.1 Surface and volume definition by NURBS

A NURBS volume is parametrically constructed as follows

$$V(r,s,t)=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3}\frac{N_{i,p_1}(r)\,N_{j,p_2}(s)\,N_{k,p_3}(t)\,\omega_{i,j,k}}{\sum_{e=0}^{n_1}\sum_{f=0}^{n_2}\sum_{g=0}^{n_3}N_{e,p_1}(r)N_{f,p_2}(s)N_{g,p_3}(t)\,\omega_{e,f,g}}P_{i,j,k} \qquad (2)$$

where $P_{i,j,k}$ are $(n_1+1)\times(n_2+1)\times(n_3+1)$ control points, $\omega_{i,j,k}$ are the associated weights, and $N_{i,p_1}(r)$, $N_{j,p_2}(s)$ and $N_{k,p_3}(t)$ are the normalized B-splines basis functions of degree p_1, p_2 and p_3, respectively. The i-th B-spline basis function of degree p_1 (order p_1+1), denoted by $N_{i,p_1}(r)$, is defined recursively as:

$$N_{i,0}(r)=\begin{cases}1 & if \quad r_i \le r \le r_{i+1}\\ 0 & otherwise\end{cases}$$
$$N_{i,p_1}=\frac{r-r_i}{r_{i+p_1}-r_i}N_{i,p_1-1}(r)+\frac{r_{i+p_1+1}-r}{r_{i+p_1+1}-r_{i+1}}N_{i+1,p_1-1}(r) \qquad (3)$$

where $\mathbf{r}=\{r_0,r_1,...,r_{m_1}\}$ is the knot vector and r_i are a non-decreasing sequence of real numbers, which are called knots. The knot vectors $\mathbf{s}=\{s_0,s_1,...,s_{m_2}\}$ and $\mathbf{t}=\{t_0,t_1,...,t_{m_3}\}$ are employed to define the $N_{j,p_2}(s)$ and $N_{k,p_3}(t)$ basis functions for other directions in a 3D volume. A knot vector, for instance in *r* direction, is called *open* if the first and last knots have a multiplicity of p_1+1. In this case, the number of knots is equal to $m_1=n_1+p_1+1$. Also, the interval $[r_i,r_{i+1})$ is called a knot span where at most p_1+1 of the basis functions $N_{i,p_1}(r)$ are non-zero which are $N_{i-p_1,p_1}(r)$, … , $N_{i,p_1}(r)$.

As examples, for $p_1=0,1$ and 2 and by using a uniform knot vector $\mathbf{r}=\{0,1,2,3,4,5,...\}$ the basis functions are shown in the Figure 1. For more details Reference [16] can be consulted.

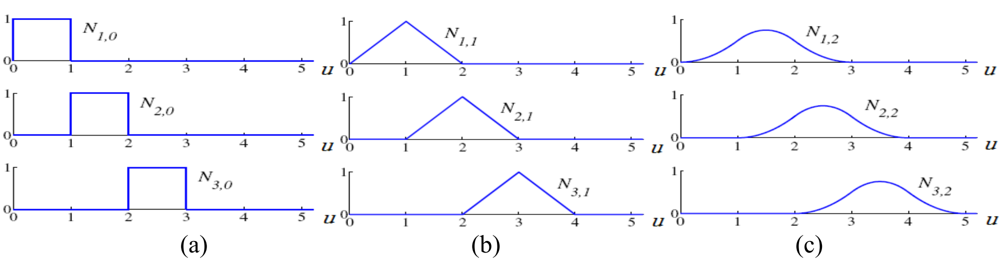


Figure 1: Basis Functions of (a) order 0, (b) order 1 and (c) order 2.

In the isogeometrical analysis, the domain of interest is generated by using NURBS. Furthermore, the basis functions are also employed for the purpose of approximation which plays a similar role as the interpolating shape functions in the finite element method. For construction of the solution surfaces for each of components of the primary variables of the governing differential equation, the position of these control points are chosen and an extra component with an unknown value is associated to them which is determined when solution is achieved. Note that the mentioned imaginary surfaces belong to a space with an extra dimension with respect to the physical space where the domain of the problem has been defined.

6. Numerical Formulation for 3D Elasticity Problems

In isogeometric analysis method, the domain of problem might be divided into subdomains or patches so that B-spline or NURBS parametric space is local to patches. A patch is like an element in the finite element method and the approximation of unknown function can be written over a patch. Therefore, the global coefficient matrix, which is similar to the stiffness matrix in elasticity problems, can be constructed by employing the conventional assembling which is used in finite element method.

By using the NURBS basis functions, t he approximated displacement functions $\mathbf{u}^p = [u, v, w]$ can be written as

$$\mathbf{u}^p(r,s,t) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} R_{i,j,k}(r,s,t)\mathbf{u}^p_{i,j,k} \tag{4}$$

where $R_{i,j,k}(r,s,t)$ is the rational term in eq. (2). It is noted that the geometry is also approximated by B-spline basis functions which is mentioned in eq. (2). In other words, every displacement component is considered as an extra unknown variable of a NURBS' solution surface.

By using the local support property of NURBS basis functions, the above relation can be summarized as it follows in any given $(r,s,t) \in [r_i, r_{i+1}) \times [s_j, s_{j+1}) \times [t_k, t_{k+1})$.

$$\mathbf{u}^p(r,s,t) = \sum_{e=i-p_1}^{i}\ \sum_{f=j-p_2}^{j}\ \sum_{g=k-p_3}^{k} R_{e,f,g}(r,s,t)\mathbf{u}^p_{e,f,g} = \mathbf{RU} \tag{5}$$

The strain-displacement matrix $\mathbf{B}$ can be constructed from the following fundamental equations

$$\boldsymbol{\varepsilon} = \mathbf{Du} \quad \rightarrow \quad \boldsymbol{\varepsilon} = \mathbf{BU} \tag{6}$$

where $\mathbf{D}$ is the differential operation matrix. Following a standard approach for the derivation of the finite elements formulation, the matrix of coefficient can easily be obtained. For example, by implementing the virtual displacement method with existence of body forces $\mathbf{b}$ and traction forces $\mathbf{f}$ we can write

$$\int_{V^p} \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV - \int_{V^p} \delta\mathbf{u}^T \mathbf{b}\, dV - \int_{\Gamma^p} \delta\mathbf{u}^T \mathbf{t}\, d\Gamma = 0, \tag{7}$$

Now, by substituting $\delta\boldsymbol{\varepsilon} = \mathbf{B}\delta\overline{\mathbf{P}}$ from eq. (6) and the constitutive equation $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, in eq. (7) and by dropping the coefficient of $\delta\mathbf{U}^T$, the matrix of coefficients can be obtained as

$$\mathbf{K}^p = \int_{V^p} \mathbf{B}^T \mathbf{CB}\, dV \tag{8}$$

As it is noted, in the equations above, all of the variables are written in terms of the parameters r, s and t which is similar to mapping in the standard finite element method where the base or unit elements are used. However, calculation of the partial differentials is somehow different and needs special care.

In this research the standard Gauss quadrature over each knot space is used for numerical integration. The proper number of gauss points depends on the order of the NURBS basis functions.

7. Numerical Examples

To demonstrate the performance of the method, in this section, four examples are presented. The objective, in all of the examples, is to minimize the weight of the structure where the stresses are limited to a target stress. The upper and lower bounds of the design variables in each example are chosen by regarding the practicality of the problem.

Example 1. A cantilever beam with a point load at its free end top corner is considered. The problem definition is illustrated in Figure 2. The magnitude of the force, the Poisson's ratio and the modulus of elasticity are 300, 0.15 and 1500, respectively. The objective is minimization of the volume where the von Misses stresses are limited to 4000. In this Figure the different kinds of control points which are used in the process of shape optimization, comprising linked, variable and fixed, are specified. Number of design variables is equal to 6. Open knot vectors for a bi-quadratic NURBS surface are assumed to be as follows in one patch that yields 18 control points.

$$\mathbf{r} = \{0,0,0,0.25,0.5,0.75,1,1,1\} \quad \text{and} \quad \mathbf{s} = \{0,0,0,1,1,1\}.$$

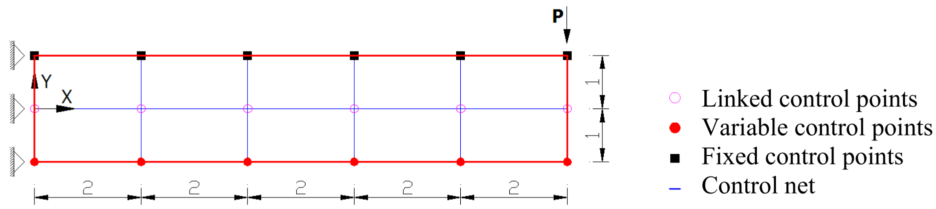


Figure 2: 2D cantilever beam

After carrying out the shape optimization process, location of the control points as well as the optimimum shape are shwon in Figure 3. The stress contours and the iteration history are depicted in Figure 4.

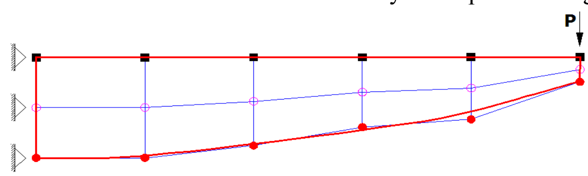
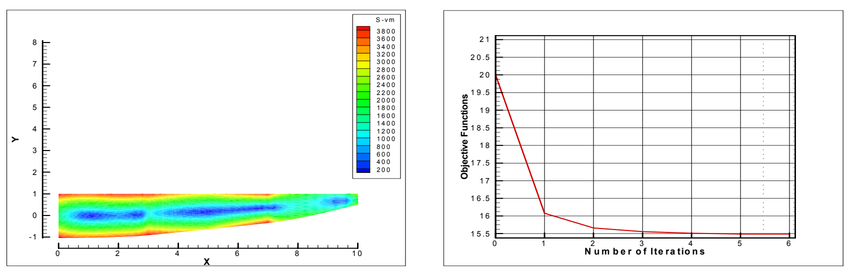


Figure 3: Optimum shape of Example 1

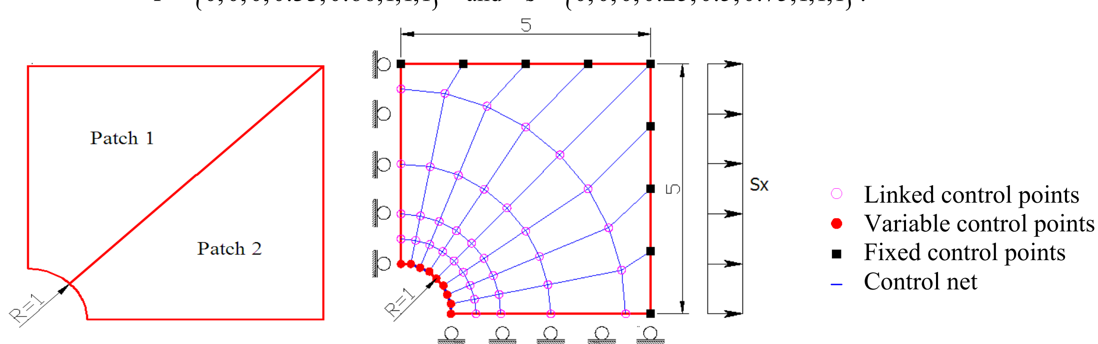


(a) (b)

Figure 4: (a): von Misses stress contours, (b): iteration history

Example 2. The problem definition of the second example is illustrated in Figure 5. The magnitude of the uniform tensile force, the Poisson's ratio and the modulus of elasticity are taken as 10, 0.15 and 1500, respectively. Weight minimization with von-Misses stress limited to 40 is our objective. In this example, because of symmetry in both directions, a quarter of a large plate with a circular hole under an axial stress is considered. The control points which describe the hole boundary as well as the linked control points are varied during the optimization process. The von-Misses stress is checked to be smaller than the target stress. The domain of intrest is discretized into two patches and the corresponding open knot vectors are cosidered as

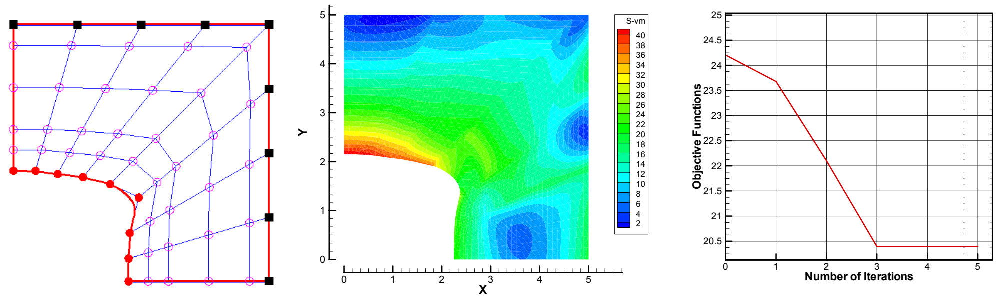
$$\mathbf{r} = \{0,0,0,0.33,0.66,1,1,1\} \quad \text{and} \quad \mathbf{s} = \{0,0,0,0.25,0.5,0.75,1,1,1\}.$$



(a) (b)

Figure 5: Problem definition of Example 2 (a): discretization and (b): Initial design of problem

The resulted optimum positions of the control points, the stress contours and the optimization iteration history are illustrated in figure 6.



(a) (b) (c)

Figure 6: (a) Optimum shape, (b) von Misses stress contours and (c) iteration history

Example 3. A three dimensional cantilever beam, illustrated in Figure 7, is considered. Dimensions of the initial design are shown in the Figure and. The magnitudes of the concentrated loads are 150 each. The modulus of elasticity and the Poisson's ratio are taken as 1500 and 0.3, respectively. Our goal is to minimize its weight and the stresses in longitudinal direction, S_{xx} , are considered as constraints of the optimization problem and assumed to be limitted to 2500. The domain of interest is described by 99 control points and the assumed knot vectors are

$$\mathbf{r} = \{0,0,0,0,0.1,0.2,0.4,0.5,0.7,0.8,0.9,1,1,1,1\}$$

$$\mathbf{s} = \{0,0,0,1,1,1\}, \qquad \mathbf{t} = \{0,0,0,1,1,1\}.$$

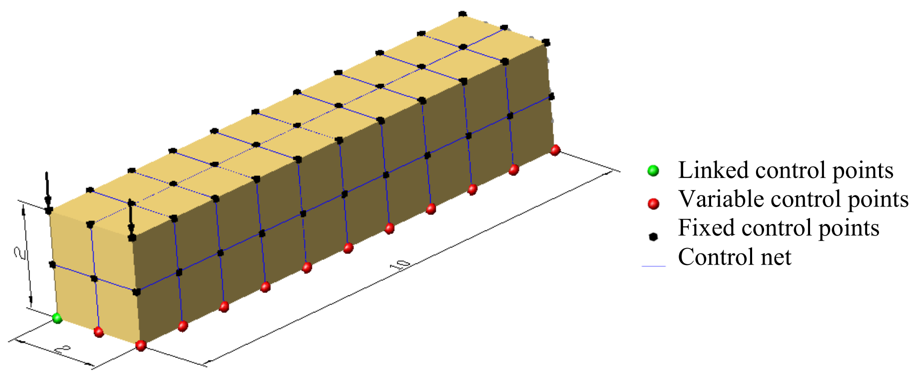


Figure 7: Problem definition of 3D cantilever Beam

In this example, the bottom edge variable control points are free to move in both Y and Z directions and the bottom mid-width control points can move vertically. The optimum layout, the control net and the stress contours are shown in Figure 8.

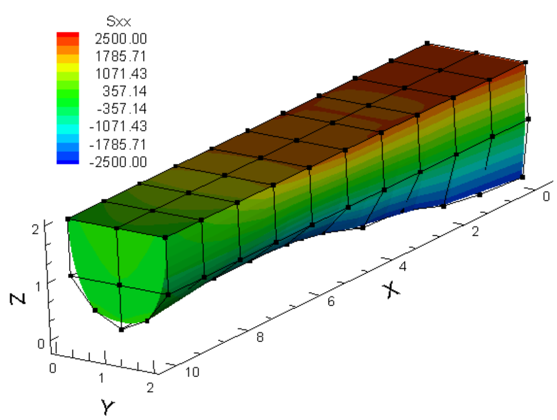


Figure 8: Optimum shape with the S_{xx} stress contours

Example 4. A beam with clamped supports at its ends, as is illustrated in Figure 9, subjected to a distributed uniform loading with its intensity equal to 100 is assumed. The modulus of elasticity and the Poisson's ratio are taken as 1500 and 0.3, respectively. Weight of the beam is considered as the objective function and the initial domain and control points are also shown in the Figure. The knot vectors $\mathbf{r}$ and $\mathbf{s}$ are the same as before, but $\mathbf{t}$ is considered as

$$\mathbf{t} = \{0,0,0,0.33,0.66,1,1,1\} .$$

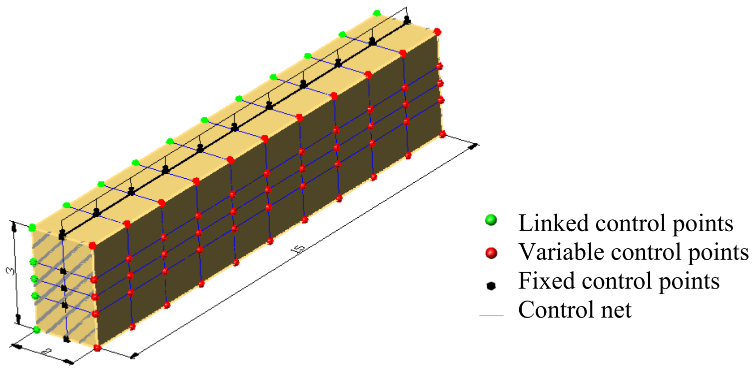


Figure 9: Problem definition for example 4

Similar to the previous example, the normal stresses in the longitudinal direction of the beam are considered as constraints and are limmited to 600. The diffrenrt types of control points, i.e. fixed, variable and linked, are also shown in Figure 9. All the control points in the front and back faces of the beam are allowed to move in Y direction and the rest of the control points are assumed to be fixed. The final shape with its control net as well as the contours are depicted in Figure 10. In this example, 65 percent of the initial beam volume is saved during 15 optimizing iteration. It is interesting to note that only 165 control points are used for discretization of the domain which results in a system of linear equations which is less than 500 for each analysis in this problem. Also, despite having such a few number of control points, the smoothness of the shape is not badly violated.

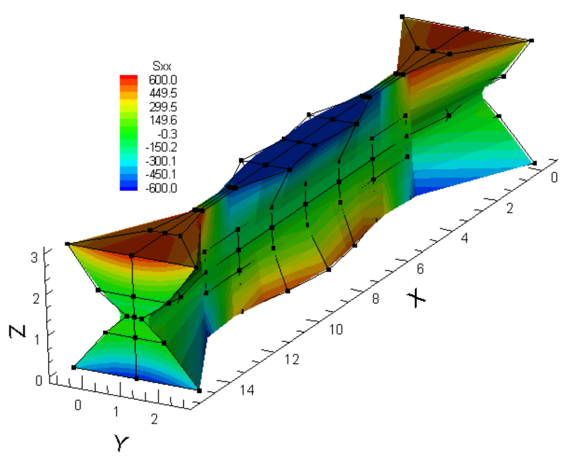


Figure 10: The resulted optimum shape for Example 4

8. Conclusion

The idea of using the isogeometrical analysis method in three dimensional structural shape optimization problems is experienced in this contribution. As it was expected, this approach allows us to define the geometry of the problem with more precision. Also, in this method variations of the boundary surfaces does not harm the smoothness of the obtained optimal designs and it usually results in very pleasant shapes. Furthermore, an important point about this approach is that the adaptation of the analysis model to the evolving geometry is done without any extra effort. In addition, due to the desirable characteristics of NURBS, it does not have any damaging effect on the quality of descretization and hence the accuracy of the analysis, which is not the case when other analysis methods are employed. Another interesting point about the method, unlike any other numerical method, apart from resulting in a much smaller system of equations, is that the essential boundary conditions can be satisfied all along the boundary surface and not just at the descretization points. In general, the experienced method possesses many interesting features and it seems that it has the potential of making the other methods obsolete in the near future.

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