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DETERMINATION OF STRESS INTENSITY FACTORS FOR JOINTED BRITTLE ROCK MEDIUM USING ELEMENT FREE GALERKIN METHOD

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Abstract

Various numerical techniques such as finite element and boundary element methods are commonly used to analysis engineering problems. These methods encounter mesh-related difficulties in dealing with fracture mechanics problems. To overcome these difficulties, a number of meshless methods have been developed in recent years. In this paper the Element Free Galerkin method based on the linear elastic fracture mechanics is used to model the jointed rock medium under axial loads. The stress intensity factors are calculated on the tip of the joints by using J-integrals. The visibility criterion and a cubic spline weight function are applied to model rock fractures. In addition, the Lagrange multipliers method is employed to enforce the boundary conditions. To verify the computational capability and accuracy of the method, a couple of examples of jointed samples in mode I as well as mixed mode are considered and the stress intensity factors are determined. The obtained results by this technique, in comparison with analytical methods, show a good accuracy and denote that the Element Free Galerkin method can be used as a proper tool in rock fracture mechanics.

Keywords: Stress Intensity Factors; Rock Fracture Mechanics; Element Free Galerkin Method; J-integrals.

1. Introduction

Rock mass commonly contains fractures in the forms of joints and microcracks, and their failure strongly depends on the propagation of these pre-existing flaws. Propagation of rock mass discontinuities is studied in rock fracture mechanics. Stress intensity factors (SIF) in linear elastic fracture mechanics are the main parameters capable to characterize the stress field in the vicinity of the crack tip. These factors depend on the geometry of the fracture, applied stresses and the initial fracture length. Based on the loading type that a material is subjected to, there are three basic crack propagation modes in a fracture process (Fig. 1), namely: Mode I (extension, opening), Mode II (in-plane shear), and Mode III (out-of-plane shear). Any combination of these modes may occur as a mixed mode. When the stress intensity factors reaches a critical values at some point in a structure, a fracture will initiate and propagate. Therefore determination of stress intensity factors is an essential task: it can be obtained from the stress field, the displacement field or from energy quantities [1]. In practice, because of the mechanical and geometrical complexity of most of the problems, commonly a numerical method such as the finite element or boundary element methods is employed to calculate stress intensity factors [2, 3].

Finite element and boundary element methods encounter mesh-related difficulties in dealing with fracture mechanics problems. To alleviate these difficulties, various mesh free methods such as element-free Galerkin method (EFGM) was developed [4].

In numerical studies, the stress intensity factors is calculated by methods such as displacement extrapolation method [5], stress extrapolation method [6], J-integral [2], Griffith's energy calculations [2], and the stiffness derivative technique [2]. In this paper, the element-free Galerkin method (EFGM) based on linear elastic fracture mechanics was applied to determine the tension mode (mode I) and mixed mode (mode I and II) Stress intensity factors with the use of J-integrals in fractured rock medium under axial loads. To evaluate the performance of the EFGM, two examples were considered and the results are compared with analytical and finite element solutions.

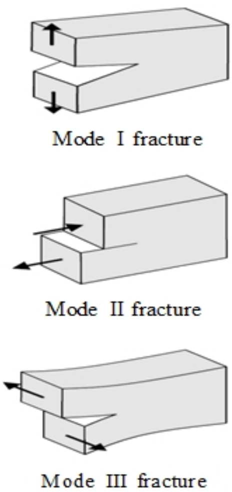


Fig. 1. Three basic modes of fracture.

2. The EFG method

The EFG method is one of the most promising meshless methods. It essentially consists of two aspects; construction of meshless approximation using the moving least-square (MLS) technique and formulation of Galerkin weak form to govern the numerical approximation.

2.1. *The MLS approximation*

According to Lancaster et al. [7], the local approximation u^h of a field variable u(x) defined in the solution domain, Ω, is expressed as the inner product of a vector of the polynomial basis, $p(x)$, and a vector of the coefficients, $a(x)$

$$u^h = P^T(x).a(x) = \sum_{i=1}^{m} p_j(x)a_j(x) \quad (1)$$

where m is the number of monomials in the polynomial basis. In the present study on 2-D problems, a linear basis, i.e., $p^T = (1, x, y)$ corresponding to m=3, is used. If the values at a set of nodes $x_i, i = 1, 2, ..., n,$ are known, the vector $a(x)$ can be determined by minimizing a weighted, discrete L_2 error norm defined as

$$J = \sum_{i=1}^{n} w(x, x_i)[u^h(x_i) - u_i]^2 \quad (2)$$

where $w(x, x_i)$ is a weight function defined over a compact support (also called the domain of influence of node i), u_i the nodal value at x_i, and n the number of nodes whose domain of influence contains the evaluation point, x. The weight function rules the nodal influence and plays an essential role in the MLS approximation. The present study employs the cubic spline as the weight function,

$$w(s) = \begin{cases} \frac{2}{3} - 4s^2 + 4s^3 & \text{for} \quad s \le \frac{1}{2} \\ \frac{4}{3} - 4s + 4s^2 - \frac{4}{3}s^3 & \text{for} \quad \frac{1}{2} < s < 1 \\ 0 & \text{for} \quad s \ge 1 \end{cases} \quad (3)$$

where the weight parameter s is a normalized distance, i.e., $s = |x - x_i|/r_0$ (r_0 is the radius of influence domain).

The stationarity of J with respect to $a(x)$ leads to the solution of $a(x)$

$$a(x) = A^{-1}(x)B(x)u \quad (4)$$

where

$$[A(x)]_{IJ} = \sum_{i=1}^{n} w(x, x_i)p_I(x_i)p_J(x_i), \quad I, J = 1,2,..., m \quad (5)$$

$$[B(x)]_{IJ} = w(x, x_J)p_I(x_J), \quad J = 1,2,..., n, I = 1,2,..., m. \quad (6)$$

$$u^T = (u_1, u_2, ..., u_n) \quad (7)$$

A is usually called the ‘‘moment matrix’’. Substitution of $a(x)$ into Eq. (1) gives

$$u^h(x) = \sum_{i=1}^{n} \phi_i(x)u_i \quad (8)$$

with $\phi_i(x)$ being the shape function in the following form:

$$\phi_i(x) = \sum_{j=1}^{n} p_j(x)(A^{-1}(x)B(x))_{ji} \qquad (9)$$

Note however, that the shape function obtained does not possess the Kronecker delta properties, i.e., $\phi_i(x_i) \neq 1$ and $\phi_i(x_j) \neq 0$.

2.2. *The Galerkin weak form for elastostatics*

As is well-known the equilibrium of a body that occupies the region Ω bounded by Γ can be stated mathematically as follows:

$$\nabla.\sigma + b = 0 \text{ in } \Omega \qquad (10)$$

$$u = \bar{u} \text{ on } \Gamma_u, \quad \sigma.n = \bar{t} \text{ on } \Gamma_t \qquad (11)$$

where σ, b , u, n, are the stress tensor, the body forces, the displacement field and the unit outward normal to the boundary Γ, respectively. While $\bar{t}$ and $\bar{u}$ represent the given traction and displacements on the portion Γ_t and Γ_u of the boundary, respectively. The variational or weak form of the Eqs. (10) and (11) is

$$\int_{\Omega} \sigma^T \delta\varepsilon d\Omega - \int_{\Omega} b^T \delta u d\Omega - \int_{\Gamma_t} t^T \delta u d\Gamma - \delta W_u = 0, \qquad (12)$$

where δ is the variational operator, ε the strain tensor, and δW_u represents a term that is introduced to enforce the essential boundary conditions. The explicit form of δW_u depends on the method by which the essential boundary conditions are applied. We use Lagrange multipliers to apply the essential boundary conditions [8]. Then, δW_u is defined as

$$\delta W_u = \int_{\Gamma_u} \delta\lambda.(u - \bar{u}) d\Gamma + \int_{\Gamma_u} \delta u^T.\lambda d\Gamma, \qquad (13)$$

where λ is the Lagrange multiplier that is expressed by

$$\lambda(x) = N_I(s)\lambda_I, \; x \in \Gamma_u, \quad \delta\lambda(x) = N_I(s)\delta\lambda_I, \; x \in \Gamma_u, \qquad (14)$$

where $N_I(s)$ is a Lagrange interpolant and s the arc length along the boundary. For elasticity problems, the strain can be expressed as

$$\varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) \qquad (15)$$

and the stress–strain relationship is

$$\sigma = D : \varepsilon. \qquad (16)$$

Substituting Eq. (8) (which is the approximation function of the MLS) and Eqs (13)-(16) into Eq. (12), we have

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{q} \end{Bmatrix} \qquad (17)$$

where

$$[K]_{ij} = \int_{\Omega} B_i^T D B_j d\Omega \qquad (18)$$

$$G_{ik} = -\int_{\Gamma_u} \phi_i N_k d\Gamma \qquad (19)$$

$$\boldsymbol{f} = \int_{\Omega} \Phi_I \boldsymbol{b} \, d\Omega + \int_{\Gamma_t} \Phi_I \bar{\boldsymbol{t}} \, d\Gamma \qquad (20)$$

$$q = -\int_{\Gamma_u} N_k \bar{u} \, d\Gamma \qquad (21)$$

and

$$[B]_i = \begin{bmatrix} \phi_{i,x} & 0 \\ 0 & \phi_{i,y} \\ \phi_{i,y} & \phi_{i,x} \end{bmatrix} \qquad (22)$$

$$N_k = \begin{bmatrix} N_k & 0 \\ 0 & N_k \end{bmatrix} \qquad (23)$$

$$D = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \text{ for plain stress} \qquad (24)$$

In which a comma designates a partial derivative with respect to the indicated spatial variable; E and υ are Young’s modulus and Poisson’s ratio, respectively.

2.3. *Modeling of Geometric Discontinuity*

Geometric discontinuities such as cracks and joints can be modeled by the EFG method in different ways such as: the visibility criterion, the diffraction method and the transparency method [9,10]. The visibility criterion is employed in the present paper. This criterion considers each geometric boundary (external or internal edges, cracks, holes, etc.) as an opaque surface. To determine the nodes that need to be considered in the domain of influence of a point, which are required for construction of the shape functions, all are connected to the point of interest. Such straight lines can be imagined to be a ray of light. If the ray encounters an opaque surface, such as the boundary of the body or an interior discontinuity, it terminates and the corresponding node is not included in the domain of influence. As an example, in Fig. 1 the domains of influence of two typical points I and J which are located near the joint, are shown as shaded areas. All the nodes included in the region as defined above, are considered as those belonging to the domain of influence and are used to calculate the shape functions.

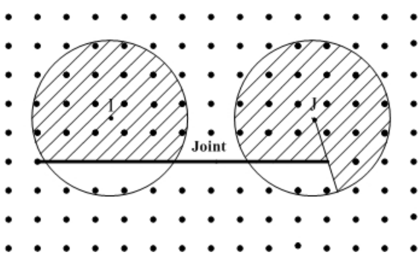


Fig. 2. Domain of influence of nodes I and J located near a joint [10].

3. J-integral Method

The J-integral technique, introduced by Cherepanov [11] and Rice [12], is widely used in rate-independent quasi-static fracture analysis to characterize the energy release rate associated with crack growth and is defined as

$$J_k = \int_{\Gamma} (W n_k - t_j u_{j,k}) ds, \quad k = 1,2 \tag{25}$$

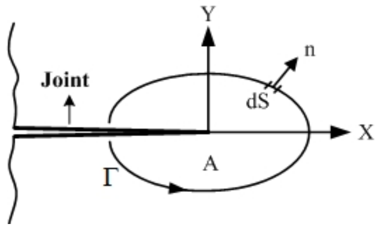


Fig. 3. The J-integral path.

where Γ is an arbitrary counter-clockwise path remote from the crack tip, beginning on the lower crack face and ending on the upper crack face, a generic contour surrounding the crack front (Fig. 1), $W = 1/2\sigma.\varepsilon$ is the strain energy density and $t_j = \sigma_{ij} n_i$ are the tractions evaluated along the contour Γ, with normal unit outward components n_j. For linear elastic material, J is related to the stress intensity factors by

$$J_1 = \frac{K_I^2 + K_{II}^2}{E'}, \quad J_2 = \frac{2K_I K_{II}}{E'} \tag{26}$$

in which $E' = E$ for plane stress state or $E' = E/(1-\upsilon^2)$ for plane strain state [10]. In pure mode I fracture: $K_{II} = 0$ and K_I calculated from:

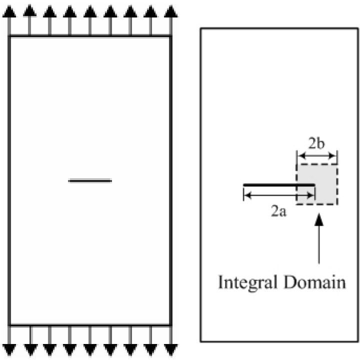
$$J_1 = \frac{K_I^2}{E'} \tag{27}$$

4. Examples

The EFGM method was applied to perform fracture-mechanics analysis of jointed rock mediums. Both single (mode I) and mixed mode (modes I and II) were considered and two examples are presented. The discretization nodes are distributed in a square grid in the domain of problems, except for the regions near the crack tip which a star-shape pattern are used. For numerical integration, a 7×7 Gauss quadrature in the cells around the crack tip and a 4×4 quadrature in the remaining cells are used. To model this problem by EFGM and calculate the SIFs by J-integral, the required programs were developed in MATLAB based on equations (1) to (27).

4.1. *Example 1. Central horizontal joint under mode I*

Consider a center horizontal cracked plate under a tensile uniform load, as shown in Fig. 3(a), with length $L = 200\,mm$, width $W = 100\,mm$ and the crack length of $2a = 24\,mm$. The far-field tension stress is assumed $\sigma = 1\,MPa$. The elastic modulus E and Poison's ratio υ were $E = 72.4 \times 10^3$ MPa and $\upsilon = 0.3$ respectively.



(a) (b)

Fig. 4. (a) Geometry of the Center-Cracked Tension specimen, (b) J-integral domain.

Three different paths, with different number of nodes located on them, are considered to calculate the J-integral (Fig. 3(b)). The obtained values for K_I, via different paths and various numbers of scattered nodes, are shown in Table 1. Aliabadi [13] has also computed the stress intensity factor for this problem and has proposed the following formula:

$$K_I = \sigma\sqrt{\pi a}\,[1 + 0.043(\frac{a}{b}) + 0.491(\frac{a}{b})^2 + 7.125(\frac{a}{b})^3 - 28.403(\frac{a}{b})^4 + 59.583(\frac{a}{b})^5 - 65.278(\frac{a}{b})^6 + 29.762(\frac{a}{b})^7]. \quad (28)$$

By using this polynomial, the value of the mode I stress intensity factor is $K_I = 6.63\,MPa\sqrt{mm}$. This problem was also solved by Javidrad [5], using the finite element method together with displacement and stress extrapolation techniques. He obtained the SIF equal to 6.6 $MPa.\sqrt{mm}$ by displacement extrapolation method and 6 $MPa.\sqrt{mm}$ by stress extrapolation method. The results of element free Galerkin match quite well with other solutions.

Table 1: Mode-I SIF using different domain size and various node numbers

Domain size $2b \times 2b$	K_I $(MPa mm^{1/2})$		
	32 nodes	48 nodes	64 nodes
$a \times a$	6.81	6.80	6.81
$1.2a \times 1.2a$	6.76	6.74	6.73
$1.5a \times 1.5a$	6.78	6.77	6.76

4.2. *Example 2. Inclined joint under mixed mode*

This example involves a central inclined-cracked plate as shown in Fig. 4(a) which is subjected to far-field compression stress $\sigma = 1$ MPa in both sides. The plate has length $L = 200$ mm, width $W = 100$ mm and crack length of $2a = 30$ mm. A plain stress condition was assumed with $E = 50 \times 10^3$ MPa and $\upsilon = 0.25$. The mode I and mode II SIFs were calculate according to Eq. (26). In this example ($\beta = 45^\circ$) stress intensity factors, K_I and K_{II} are equal and can simply determined from Eq. (26) but in general it is necessary to use other methods such as M-integral [14] and decomposition procedure [10,15]. Table 2 shows the values of K_I and K_{II} for different domains and various numbers of scattered nodes on integral domains.

The following equations are used to obtain the theoretical SIFs [16]:

$$K_I = \sigma_v \sqrt{\pi a}\,\sin^2\beta$$
$$K_{II} = -\sigma_v \sqrt{\pi a}\,\sin\beta\cos\beta \quad (29)$$

In the above formulation $a =$ half the flaw length, β is the flaw inclination angle. Using these equations, solutions are:

$K_I = 3.43\, MPa.\sqrt{mm}$, $K_{II} = -3.43\, MPa.\sqrt{mm}$.

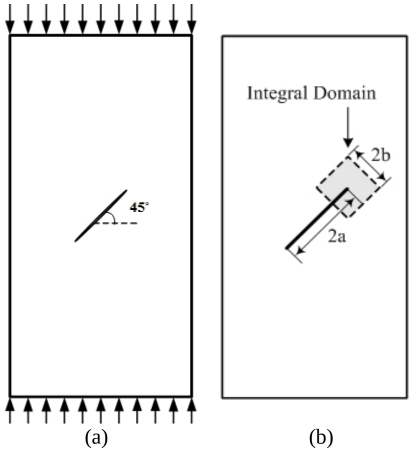


Fig. 5. (a) Geometry of the center-inclined-cracked plate, (b) J-integral domain.

The results of element free Galerkin show good agreement with analytical solutions.

Table 2: Mode-I SIF using different domain size and various node numbers.

Domain size $2b \times 2b$	$K_I\ (MPa.mm^{1/2})$		
	29 nods	41 nods	61 nods
$a \times a$	3.54	3.56	3.58
$1.2a \times 1.2a$	3.62	3.63	3.64
$1.5a \times 1.5a$	3.74	3.76	3.77

Table 3: Mode-II SIF using different domain size and various node numbers.

Domain size $2b \times 2b$	$K_{II}\ (MPa.mm^{1/2})$		
	29 nods	41 nods	61 nods
$a \times a$	3.54	3.56	3.58
$1.2a \times 1.2a$	3.62	3.63	3.64
$1.5a \times 1.5a$	3.74	3.76	3.77

5. Conclusions

The element free Galerkin method was used to determine the stress intensity factors in jointed rock medium together with the J-integral technique. The assumption of linear elastic fracture is adopted and the visibility criterion was used to model rock fractures. The Lagrange multipliers method was employed to enforce the boundary conditions. To evaluate the proposed model, two examples were considered and mode I and mode II SIFs were calculated in cracked rock samples. This study shows that the results of the EFGM are in a very good agreement with other solutions obtained by either analytical or the finite elements. It seems that the element free Galerkin method can be used as a significant tool in rock fracture mechanics.

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Dear Dr. B. Hassani,

On behalf of the organizing committee and Iranian Society for Rock Mechanics, I would like to take this opportunity to invite you to participate in the ISRM International Symposium 2008, 5^{th} Asian Rock Mechanics Symposium (ARMS5), which will be held in Tehran, Iran. From November 21 to December 5, 2008, the ISRM Board, Commissions, and Council Meetings, the Symposium and a series of Technical activities will be organized.

A large number of papers were approved. The technical programme will include a series of keynote speakers by well known experts.

Apart from presentations and discussions, a post conference technical cum cultural tour is in schedule for you to visit historical and technical sites at various attractive places around the country.

Thus, you are requested to book your Air line ticket(s) as early as possible. Please make sure to forward your flight schedule, in order to enable us to make necessary arrangements for your arrival and hotel accommodations.

It may not be out of place to mention here that visiting Iran is a unique experience, the experience of being in the cradle of a great culture and civilization. Boasting several millennia of recorded history, Iran enjoys a great legacy of ruins and hallowed stonework. The Iranian style of architecture is distinctive and creative. Persepolis and the rare impressive edifices of Esfahan, as well as the Air Traps of the central desert region, are hallmarks of the unique Iranian architecture. If you simply want to walk in the footsteps of some of history's most outstanding figures, this historic land is blessed with several of the best.

It is an honor for us to serve you in the best possible way to make your stay pleasant, interesting and memorable with ever lasting events here in Iran, the Land of Hospitality. I will appreciate your positive and favorable response to our invitation and I look forward to seeing you in Iran.

With warmest regards,

Dr. Abbas Majdi,

Symposium Chairman, and
President, Iranian Society for Rock Mechanics

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