Meccanica (2011) 46:1007–1021
DOI 10.1007/s11012-010-9357-1

Optimal synthesis of function generator of four-bar linkages based on distribution of precision points

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Received: 9 July 2009 / Accepted: 13 September 2010 / Published online: 5 October 2010

Abstract This article describes a method for optimizing the synthesis of a four-bar linkage mechanism to generate a definite mathematic function. In this research, the objective function was defined based on the least squares of error between the generated function and the desired function. Because of non-linearity of the objective function and the constraint, SQP method has been used to find the optimal mechanism. This method of optimization is a gradient-based method and its result is more trustable than heuristic methods. In this study, five precision points have been used to synthesize four-bar linkages; therefore, by using this amount of precision points, a set of non-linear equations was obtained for mechanism synthesis. Unlike the previous researches that dimensional parameters are used as optimization variables, precision points distribution was used in current research. The main innovation of this paper is presentation of some directions to have estimative prediction for distribution of precision points in optimal mechanism which maybe useful for designers. These directions were obtained based on the method that presented in current work and regard to shape of desired function and its first and second derivative.

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Keywords Four bar linkage · Mechanism · Precision points · Optimization · SQP method

1 Introduction

Optimization of mechanisms is one of the notable subjects in mechanism design. Because of non-linearity of objective functions and optimization constraints and lack of explicit relations for their gradients, these optimization problems are very complex [1]. Optimization of mechanisms for generating a specific path or specific function has been investigated in previous researches.

Freudenstein [2, 3] and Fox [4] performed initial studies on this subject. Sutherland and Siddall [5] optimized the four-bar linkage mechanism based on the multifactor objective function. They studied optimization of three kinds of mechanisms: RRRR, RRRP and RGGR, and they utilized a penalty function for using constraints of optimization. In this research, the objective function has been defined as the sum of three factors, consisting of the least squares of errors for the desired function, which compare to the generated function, the pressure angle of the follower link, and the controlling dimension of the mechanism. By using the weighted approximation's method, Levitskii et al. [6] presented an estimated solution for the optimal mechanism which is able to generate a certain function by means of five precision points.

Rao [7, 8] optimized 4R mechanisms by using the Freudenstein relation, where he minimized the least squares of error based on three precision points to generate a desired function. Sun [9] decreased the error of a generated four-bar mechanism using a quadratic interpolation method. He demonstrated that the accuracy of this method is greater than the method used by Rao [8]. A research similar to Rao's [8] has been implemented by Chen [10], optimizing the generating function of four-bar linkage. He used the Mudguardt's method and improved the error value of optimal mechanism. Guj et al. [11] used an optimization technique based on the penalty function method to control the inertial force in the four bar linkages. They claimed that the results of their study could be applied in optimizing the high speed mechanism.

Freudenstein and Soylemez [12] also optimized the transmission force in the spatial four-bar mechanism. Angeles and Gosselin [13] presented a method for optimizing the transmission defect in planar and spherical mechanisms. Angeles et al. [14, 15] optimized the coupler curve in the four-bar linkage mechanism using the Newton-Gauss method. However, the response of the generated form was not valid in their work. Choi et al. [16] presented a specific to optimize and to decrease the generated path of a four-bar linkage; they also considered the offset of joints. They showed that the accuracy of this approximation method in most cases is comparable with Monte Carlo algorithm. Yao and Angeles [17] optimized the generated path of four-bar linkages using the contour method. They obtained a set of equations that deduces the state of the objective function (least squares of error), and by solving these equations, they acquired suitable solutions for optimization. Vasiliu and Yonnou [18] optimized a planar mechanism to generate a desired path using a neural network method. They indicated that this method has a lower cost and it is faster and has a higher convergence than other methods. Soong and Yao [19] optimized kinematics and dynamics by using MOST software (the multifactor objective function optimization software). Zhou et al. [20] utilized a genetic algorithm and optimized the adjustable four-bar linkage. Their objective function was defined as the sum of squares of errors between the generated path and the desired path. Due to using the genetic algorithm method, there was no need any initial guess for optimization, and they found global solutions. Moreover, optimizing the generated function of Ackermann steering linkage has been studied by Simionescu and Beale [21]. They assumed that the coupler link is elastic with variable length and also found optimal results for the mechanism's reference angle.

Marin et al. [22] decreased the occupied space of four-bar linkage to generate a specific path using a combined gross-fine method. They indicated that the cost of computing for this method is not very high, and they obtained global solutions. Zhou and Cheung [23] optimized Multi-Phase motion generation in adjustable four-bar linkages using the genetic algorithm method, and they showed that, similar to in their previous work, they did not require an initial guess and were able to generate global solutions. Sancibrian et al. [24] optimized the generated mechanism path using a gradient-based method. They used a particular formulation to find an analytical relation for the gradient of the objective function. They satisfied the existence constraint and the optimization was performed effectively via calculating the gradients.

Erkaya and Uzmay [25] optimized the generated path of the four-bar mechanism using the neuro-genetic algorithm (NN-GA) and, similar to Choi's study [16], they considered the effect of the offset of joints. Bulatovic and Dordevic [26] optimized the generated path for the four-bar mechanism using differential evolution and the method of variable controlled deviations. In their research, the desired path was a straight horizontal, and they showed that the amount of variation of the shape function has decreased to 0.0001% and the variation of the position function has increased to 1.5%. Nariman-Zadeh et al. [27] optimized the four-bar mechanism for generating a specific path using a multifactor objective function related to Patro synthesis and a genetic algorithm. Alfaro [1] optimized the generated path for 4R four-bar linkage by utilizing simulated annealing method. He considered five points in his study for designing the mechanism and showed that more constraints can be satisfied beside the existence constraints.

Shen et al. [28] have demonstrated a new formulation to design a mechanism including optimal motion generator. In addition, Acharyya and Mandal [29] have optimized the generated path of plane four-bar linkage using the heuristic methods consisting of genetic algorithm (GE), particle swarm optimization technique (PSO) and differential evolution (DE). They have illustrated that the result of DE method are better than other methods. They also accomplished the crank-rocker four-bar mechanisms of shape optimization to generate a certain path. They obtained the optimum results by the constraints of Grashof and mechanism's presence. Avilés et al. [30] presented an improved method to optimize the dimensional synthesis of planner linkages. They used an elastic strain-energy function to minimize the actuating power of mechanism. Avilés et al. [31] also optimized the generalized rigid-body guidance dimensional synthesis of linkages using finite element method. They developed a second order method to guarantee the convergence and decrease the computational cost. They affirmed that the presented method is useful for other purpose such as trajectory, function generation, rigid-body guidance, and combined synthesis of planner linkages. Peng and Sodhi [32] developed the optimal synthesis of multi phase path generation of adjustable planner four-bar linkages. They optimized the path generation of this mechanism based on the link length structural error. Therefore, the design variable and results in compact error functions is decreased using this technique.

In this study, a specific method has been used to minimize the global error of the generated function of four-bar linkage, and a desired function has been generated using a Freudenstein relation based on five precision points. For this reason, the synthesis equations are non-linear and were solved using the Newton-Gauss method. Unlike the previous research, which used dimensional parameters (ratio of linkage's length) as optimization variables, precision point distribution was used in our current research.

Here, we try to achieve a better global accuracy between generated and desired functions. The results of current research are suitable especially for function generator of mechanisms in which the position of precision points is not commonly prescribed. Therefore, this method is not appropriate for pass generation of mechanisms where precision points are usually placed exactly at special positions.

This approach is one of the important innovations of the present investigation because, based on the shape of the desired function and its derivatives, a more accurate prediction exists for precision point distribution in an optimal mechanism. Generally for optimization of designing of some mechanical elements, designer can act pre-referee based on type of equipment and objective function. For example in disk cam designing, if optimization objective function is pressure angle or Hertzian stress, designer knows that they can decrease these two parameters by increasing radius of base circle. So far, for designing four bar linkages to generate specific function, because of nonlinear behavior and large number of optimization variables, designer could not specify length of linkages and positions of each links relate to each other to optimize generated function. Hence, they used dimensional parameters as optimization variables and different methods of optimization and computational analysis have been used. In this paper, we tried to show that it is possible to estimate distribution of precision points for optimal mechanism based on the shape of desired function. For this purpose, the effect of the slope and its variation on the distribution of precision points and the value of error related to the optimal mechanism have been studied with some standard examples which presented in Sect. 4.

In the current research, the least squares of errors between the desired function and the generated function is defined as the objective function. Because of non-linearity of the objective function and the constraint, an SQP method has been used for optimization. Also, to generate physical results, the existence constraint has been used. Optimization of generated function and generated path of four bar linkage are one of the most complex and difficult optimization problems [1] and using any gradient-based methods for this purpose is so difficult. Due to these reasons and according to literature review, using the heuristic methods such as genetic algorithm, neural network, simulated annealing and ... were be prevalent in previous researches and application of gradient-based methods has been limited. Also, it is important to know that the result of heuristic methods is not unique so the result of gradient-based method is more valuable. To the best knowledge of authors, there are no reports about the using any SQP method for solving this problem.

2 Objective function and constraint

In this paper, the Freudenstein equation has been used for mechanism synthesis. This equation can be used for generating a specific function with five precision points [33]:

$$K_1\cos(\theta_4+\psi)-K_2\cos(\theta_2+\phi)+K_3 = \cos(\theta_2+\phi-\theta_4-\psi). \quad (1)$$

In above relation, ψ and φ are the reference angles of crank and follower links relative to the ground; θ_2 and

θ_4 are the angles of crank and follower links in comparison to their reference angles. Figure 1 shows these angles, with K_1, K_2 and K_3 being non-dimension constants that relate to mechanism dimensional parameters [33]:

$$K_1 = \frac{d}{a}, \tag{2a}$$

$$K_2 = \frac{d}{c}, \tag{2b}$$

$$K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}. \tag{2c}$$

In these relations, a, b, c and d display the length of links (see Fig. 1). We can evaluate the desired function ($f(x)$) in the interval of $x_l \le x \le x_u$ by utilizing the 4R four-bar mechanism. For generating a function with this mechanism, we assume that the rotation of crank and follower at special stroke angles (Θ_2, Θ_4) is

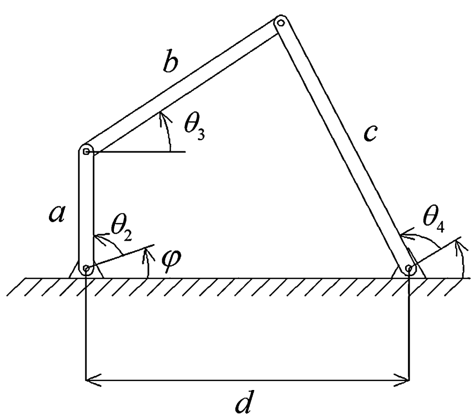

Fig. 1 The schematic plan of the 4R four-bar linkage

proportional to x and $f(x)$. Figure 2. shows the indicators of generated function and its variable in terms of the rotation of the follower and crank link. Here, the generated function and its variable are related proportionally to the linkage angles. The generated function ($\hat{f}(x)$) is completely coincident with $f(x)$ at the five points. These points are named precision (accuracy) points, and, for other points at the desired function $(x_i, f(x_i))$, precision points in terms of crank and follower angles are determined by the following relations [33]:

$$\theta_{2,i} = \left(\frac{x_i - x_l}{x_u - x_l}\right)\Theta_2, \tag{3a}$$

$$\theta_{4,i} = \left[\frac{f(x_i) - f(x_l)}{f(x_u) - f(x_l)}\right]\Theta_4. \tag{3b}$$

We can obtain a set of equations with five unknown quantities by substituting five suitable precision points $((\theta_{2,i}, \theta_{4,i}),\ i = 1, 2, \ldots, 5)$ into the Freudenstein equation (see (1)). The design of mechanism is completed by solving this set of equations for K_1, K_2, K_3, ψ and φ. Here, the Newton-Gauss method was used to solve this set of non-linear equations.

If the segment $[x_l, x_u]$ is divided by n equidistant points, then the generated function ($\hat{f}(x)$) can be obtained using a designed mechanism and relation (1). This generated function has an error comparable with the desired function ($f(x)$), and the objective function has been defined as the sum of squares of errors:

$$F(x_i) = \sum_{j=1}^{n} \left(f(x_j) - \hat{f}(x_i, x_j)\right)^2. \tag{4}$$

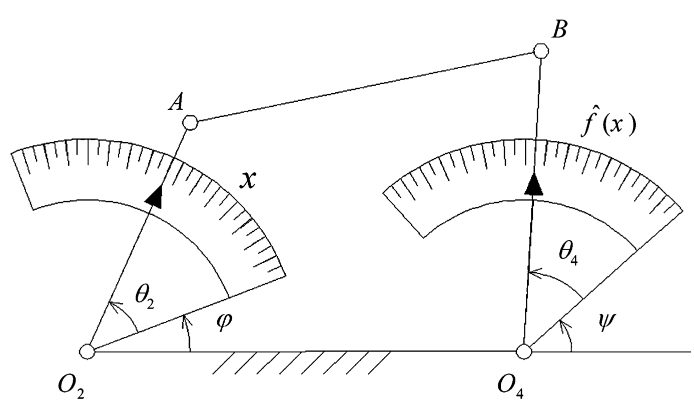

Fig. 2 Indicators of generated function and its variable in terms of the rotation of the follower and crank link

The above definition for objective function is useful to increase the global accuracy of desired function via optimization process which has been used already by other researchers [5, 7–10, 17]. In relation (4), F is the objective function, x_i $(i = 1, 2, \ldots, 5)$ are precision points, and x_j introduces n arbitrary points for evaluating the generated function. In this study, suppose that $n = 1000$. According to relation (4), the value of the objective function can be minimized by utilizing an optimal distribution of precision points (x_i). It is also important to know that the objective function is related to the Freudenstein equation using relation (3). To achieve a physical solution, it is essential to satisfy the optimization constraint. A suitable explicit relation between the angle of follower link (θ_4) and crank link (θ_2) is shown as follow [34]:

$$\tan\left(\frac{\theta_4 + \psi}{2}\right) = \frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C} \tag{5}$$

where

$$A = \sin(\theta_2 + \phi), \tag{6a}$$

$$B = \frac{d}{a} + \cos(\theta_2 + \phi), \tag{6b}$$

$$C = \frac{d}{c}\cos(\theta_2 + \phi) + \frac{a^2 - b^2 + c^2 + d^2}{2ac}. \tag{6c}$$

According to relation (5), if the value under the radical is positive, then a mechanism will exist [34]:

$$A^2 + B^2 - C^2 \geq 0. \tag{7}$$

By substituting values of A, B and C into the relation (7), the constraint of the mechanism will be obtained:

$$\begin{aligned} &\sin^2(\theta_2 + \phi) + [k_1 + \cos(\theta_2 + \phi)]^2 \\ &\quad - [k_2 \cos(\theta_2 + \phi) + k_3]^2 \geq 0. \end{aligned} \tag{8}$$

3 Optimization algorithm

In this study, an SQP method has been used for minimizing the objective function. SQP methods are commonly used for optimizing non-linear problems [35]. In this study, a method that was designed by Biggs [36], Hann [37] and Powell [38, 39] has been used, and it is similar to the Newton method.

If $F(x_i)$ is the objective function and $G(x_i) \leq 0$ is an optimization constraint, then, according to the SQP method, the quadratic sub-problem can be defined as follow:

$$\begin{aligned} &\min \frac{1}{2} d^T H_i d + \nabla F(x_i)^T d \\ &\text{s.t.: } \nabla G(x_i)^T d + G(x_i) = 0. \end{aligned} \tag{9}$$

The result of this sub-problem is used to find the new optimization iteration. Therefore, if d is the direction of iteration and α is the step length, then optimization variables $\vec{x} = [x_1, x_2, \ldots, x_5]$ in the next iteration will be obtained from the following relation [35]:

$$\vec{x}_{k+1} = \vec{x}_k + \alpha_k d_k. \tag{10}$$

The SQP method contains three main parts:

- Updating the Hessian matrix
- Solving the quadratic program
- Linear search and executing the evaluation function

In this study, the BFGS formula was used for updating the Hessian matrix. Broyden [40], Fletcher [41], Goldfarb [42] and Shanno [43] derived the following formula:

$$H_{k+1} = H_k + \frac{q_k q_k^T}{q_k^T s_k} - \frac{H_k^T H_k}{s_k^T H_k s_k} \tag{11}$$

where

$$s_k = \vec{x}_{k+1} - \vec{x}_k, \tag{12a}$$

$$\begin{aligned} q_k &= \nabla F(\vec{x}_{k+1}) + \lambda.\nabla G(\vec{x}_{k+1}) \\ &\quad - [\nabla F(\vec{x}_k) + \lambda.\nabla G(\vec{x}_k)]. \end{aligned} \tag{12b}$$

In relation (12b), λ is estimation for Lagrange coefficients.

In this study, for solving the QP equation, the projection method (Active Set Strategy) has been used [44, 45]. To control the step length (α_k), an evaluation function has been used. In this study, the evaluation function used was introduced by Han [37] and Powell [38, 39]:

$$\psi(x) = F(x) + r.\max\{0, G(x)\} \tag{13}$$

where r is a penalty coefficient that obtains [35]:

$$r = \frac{\|\nabla F(x)\|}{\|\nabla G(x)\|}. \tag{14}$$

Due to the lack of exact relations for the gradient of the objective function and the constraint, the central second order finite difference method is used for their calculation.

4 Results and discussion

In this section, the optimal synthesis of four-bar linkage has been investigated. Here, the proofs for validation of optimization results are presented first, and the studies of the effect of slope value and its variations on precision point distribution are shown last. In addition, for all cases of this study, it is assumed that the distribution of precision points is uniform and the optimization parameters are equal to unit ($K_1 = K_2 = K_3 = \varphi = \psi = 1$) at the first step of optimization.

4.1 Validation the optimization results

In current research, the validation of optimization results has been evaluated by two methods. At the first step, the optimization results have been compared by results of some researchers for designing a mechanism which is able to pass the nine certain desired points optimally. Rao [8], Sun [9] and Chen [10] used the optimization methods of Marquardt, quadratic interpolation and improved least squares for this purpose respectively. They used three precision points and design the mechanism design by solving a set of linear equation. As a result, the analysis leads to determining just the K_1, K_2 and K_3 values and not to obtaining the mechanism's reference angle (ψ and ϕ). Due to necessity a specific relation for desired function in current research, a quadratic function is interpolated on these nine points. Thus, the relation of interpolation function is obtained on $x \in [0, 1]$ and $f(x) \in [0, 1]$. Here, it is assumed that the stroke angle of crank link is 320° and stroke angle of follower link is 322°. The obtained relation based on least squares method is as follow:

$$f(x) = -1.002962x^4 + 2.000303x^3 - 0.865303x^2 + 0.872479x - 0.003185. \tag{15}$$

The initial guess for the distribution of precision points is the uniform distribution $x_{p,0} = [0, 1/4, 1/2, 3/4, 1]$ and the primary suppositions for all of designing parameters is set to one ($k_1 = k_2 = k_3 = \varphi = \psi = 1$).

In Table 1, desired points and the deviation value in the current research and Rao [8], Sun [9] and Chen's [10] studies have been presented. According to the table, the maximum error is exceeded of eight degree in all prior researches while it is lower than 0.7 degree in current work. Also, in previous studies, minimum value of total square error related to Sun [9] is equal to 178.24 but in the current research, the value of total square error has been decreased to 0.6626. Unlike the previous works, we used five precision points for mechanism design by solving Freudenstein's nonlinear set of equations. Due to using five accuracy points in mechanism's design, the obtained generated function in the present research has a good accordance with the desired points. In Table 1, the error that related to primary supposition of optimization process (uniform distribution of precision points) is presented. Here, the empty cells in the table are related to precision points which their error is equal to zero. According to the data of the Table, the error value in the other points is significantly high which has led to an increase in total square error up to the large value 369.13. Here, the optimization in 10 main steps and 96 sub steps has reduced the mechanism's total square error to 0.6626. On the other word, the optimization process succeeded to decline the mechanism's errors to 557 times comparing to the primary supposition. The optimization results are completely physically and the minimum value of left hand side of relation (8) is positive and equal to 0.6903. The results of optimization (consisting of designed optimal mechanism and accuracy points' distribution) had been presented in Table 2.

In this article, the optimization's results have been validated in other situations. Furthermore, for optimal designing of Ackermann's mechanism, the results of current study and the results of Simionescu and Beale [21] is demonstrated. Ackermann's mechanism is an efficient tool for proper steering the car in turns. In Fig. 3, the mechanism has been shown. To proper steering the car's wheels while passing a turn, the angles of linkages connected to steering wheels should be satisfied the equation as follow:

$$\theta_{OA}(\theta_I) = \frac{1}{\cot\theta_I + 1/(W_b/W)_t} \tag{16}$$

where, θ_I is the turning angle of inner wheel, θ_{OA} is the outer angle of outer wheel, W_b is the wheelbase and W_t is the wheel track of the vehicle.

As the mechanism is designed for turning in equal conditions to right and left; the AB and CD linkages'

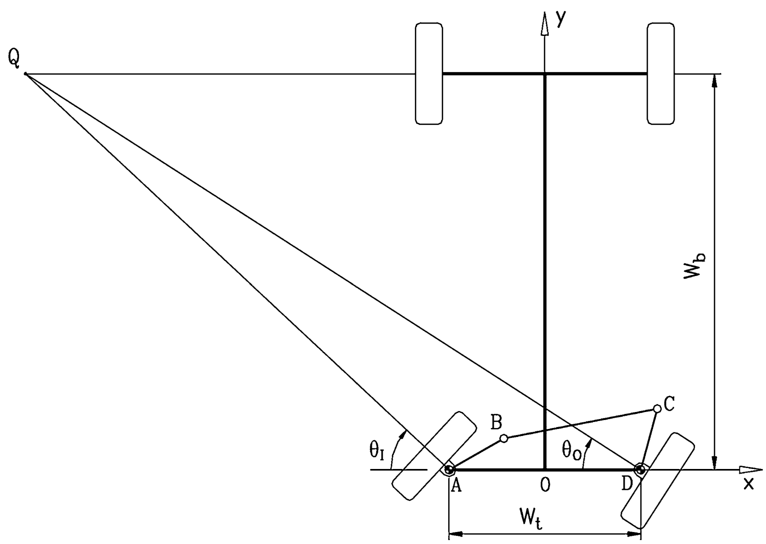

Fig. 3 Ackermann steering linkage [21]

Table 1 Deviation of generated points from desired points in current work and Rao's [8], Sun's [9] and Chen's [10] research

Desired points (degree)		Deviation of from desired function (degree)				
θ_2	θ_4	Rao [8]	Sun [9]	Chen [10]	Current research	
					Initial guess	Optimum mechanism
0.0	0.0	0.00	−0.73	−1.00	–	−0.0352
40.0	28.0	−0.19	−0.58	−0.20	−1.5059	+0.0344
80.0	61.6	+3.20	+2.17	+1.60	–	−0.0046
120.0	96.0	+3.51	+1.54	0.00	+13.6354	+0.0102
160.0	129.0	−2.50	−5.62	−8.00	–	+0.0020
200.0	170.0	−4.88	−8.78	−12.00	−13.0249	+0.0922
240.0	225.0	+4.28	+0.50	−4.10	–	+0.0461
280.0	274.0	+6.72	+3.89	0.00	+3.3596	−0.2254
320.0	322.0	+8.27	+6.80	+4.00	–	+0.7737
$\lvert\delta\theta_4\rvert_{\max}$		8.27	8.78	12.00	13.6354	0.7737
$\sum\lvert\delta\theta_4\rvert$		33.55	30.60	30.90	31.5258	1.2239
$\sum\delta\theta_4^2$		184.53	178.24	244.41	369.1268	0.6626

length are the same and also in the case of ψ and Φ reference angles, we have $\psi + \phi = 180°$. Thus, to increase the precision point's numbers and enhance the mechanism's accuracy, it has been assumed that BC linkage is an elastic one with a variation length during the turning. Here, we assumed the AB and CD linkages' length are equal to 0.16, $W_b/W_t = 1.9$ and $\theta_{I_{\max}} = 40°$. The maximum error has being evaluated for 60 points which is equal to 6.85% in Simionescu and Beale's [21] studies and 5.27% in the recent research. The main parameter in designing the mechanism is the reference angle ψ that has been obtained 109.73893 in the research of Simionescu and Beale [21] and 109.74008 in the current studies, in optimum conditions.

4.2 Example

Here, the SQP method was used to generate an optimal logarithm function ($f(x) = \ln(x)$) as an example.

Table 2 Specifications of optimal mechanism and distribution of precision points for generating special pass that defined in Table 1

Specifications of optimized mechanism		Distribution of precision points			
		Initial guess		Optimum mechanism	
		$x_{i,0}$	$\theta_{2i,0}$	x_i	θ_{2i}
K_1	0.155138	0.00	0.00	0.009262	2.96384
K_2	0.265037	0.25	80.00	0.193889	62.04448
K_3	0.418168	0.50	160.00	0.456439	140.06048
φ(rad)	1.005352	0.75	240.00	0.769083	246.10656
ψ(rad)	2.414792	1.00	320.00	0.931941	298.22112

Table 3 Specifications of optimal mechanism and distribution of precision points for generating logarithm function

Specifications of optimized mechanism		Precision points in initial and optimum mechanisms			
		$x_{i,0}$	x_i	$x^*_{i,0}$	x^*_i
K_1	1.0023	0.1500	0.1718	0.0000	0.8500
K_2	0.6006	0.7921	0.3153	0.2500	0.0644
K_3	0.8980	1.4341	0.7357	0.5000	0.2280
φ(rad)	0.3464	2.0762	1.5481	0.7500	0.5444
ψ(rad)	2.0190	2.7183	2.4382	1.0000	0.8909

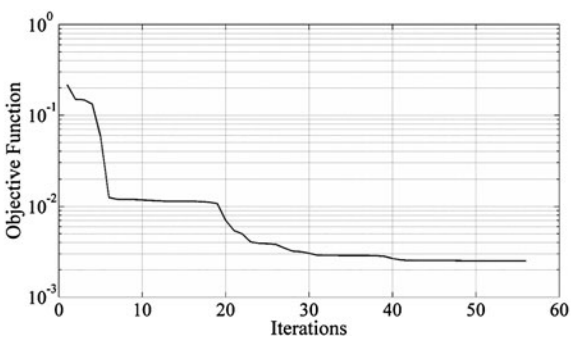

Fig. 4 History of optimization process for generating logarithm function

For this purpose, we optimize the mechanism in the domain of $x \in [0.15, e]$ and the stroke angles of crank and follower are $\pi/4$ and $\pi/3$, respectively.

The optimization process was performed during 56 main steps and 482 sub steps, and mean square of error has decreased from 0.2163 to 0.0025. The history of the optimization process is shown in Fig. 4; thus, after 40 steps, the optimization method was able to find the optimal result.

Table 3 represents specifications of the optimal mechanism. At optimal conditions, the left hand side of relation (8) is positive and equal to 1.5267, satisfying the constraint for existence of the mechanism. Relation (17) shows the non-dimension distribution:

$$x_i^* = \frac{(x_i - x_l)}{(x_u - x_l)}. \tag{17}$$

Table 3 illustrates the values of non-dimension points at first guess ($x^*_{i,0}$) and at the optimal condition (x^*_i). When we compare these two distributions, it is clear that the concentration of points is higher on left hand side of the domain for the optimal mechanism.

The structure of initial and optimal mechanism for generating the logarithm function is presented in Fig. 5(a) and 5(b). Here, the structures are shown at the initial situation of linkages ($\theta_2 = \theta_4 = 0$). The length of links are obtained by supposing a unit value for ground link ($d = 1$) and using the relation (2). Figure 5(c) illustrates the follower angle (θ_4) versus the crank angle (θ_2) to generate the logarithm function which is obtained proportionally to the desired function and its variable (refer to the relation (3)). The deviation between follower angle of initial ($\theta_{4,IM}$) and optimal mechanism ($\theta_{4,OM}$) is presented in Fig. 5(d). According to the figure, the maximum deviation between optimal and initial mechanism is about 1.168°

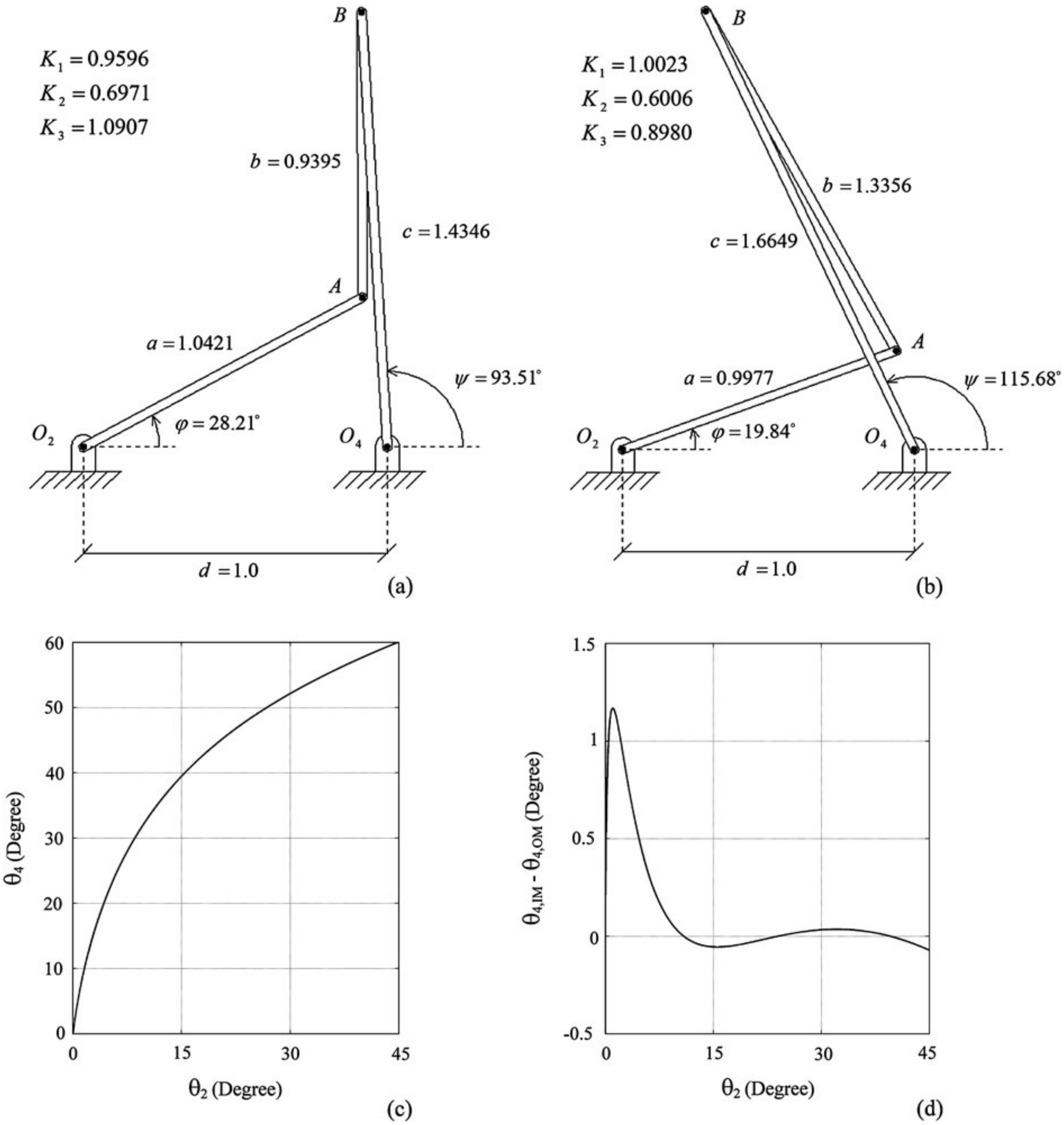

Fig. 5 Descriptive figures of mechanisms and their related diagrams, (**a**) initial mechanism, (**b**) optimum mechanism, (**c**) desired functions in terms of linkage angle and (**d**) deviation between follower angle of initial and optimal mechanism

at $\theta_2 = 0.991^\circ$. The standard deviation, mode, range and median value of deviation between crank and follower angles are equal to 0.282, -0.072, 1.240 and 0.0135 respectively.

In Fig. 6, the amount of deviation between the generated function and the logarithm function (desired function) is shown. According to this figure, the accuracy at $x < 0.7$ is very low in the initial mechanism, but the optimization process reduced this deviation. Because of the high value of the logarithm function's slope at $x < 0.7$, this deviation has appeared in the initial mechanism. In this figure, asymptotes introduce the position of precision points in the initial and optimal mechanisms.

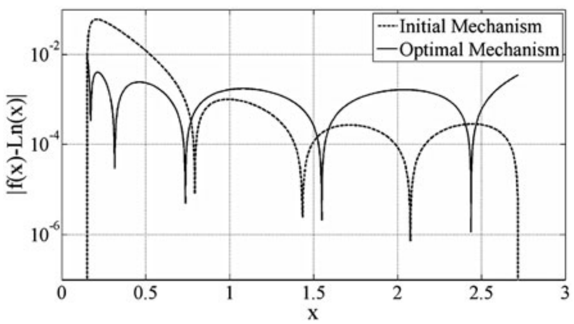

Fig. 6 Deviation of generated function from logarithm function for initial and optimum mechanism

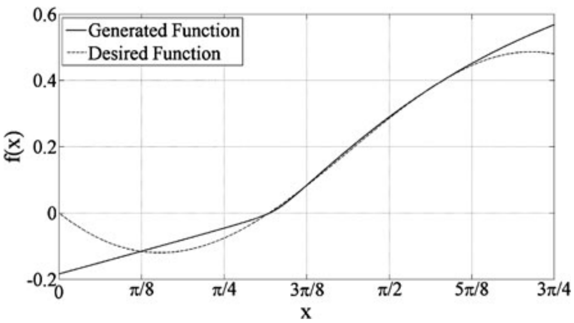

Fig. 7 Diagrams of generated and desired function with $f(x) = 0.5(x-1)\sin x$

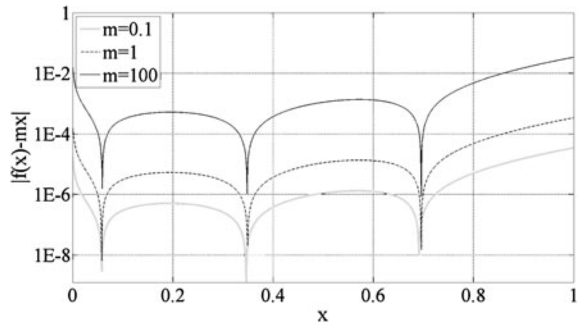

Fig. 8 Deviation of optimum generated function from linear function for different slopes

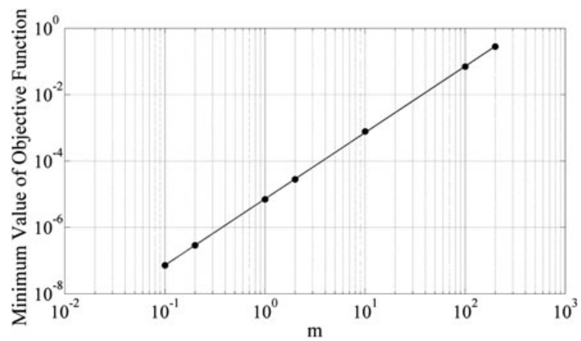

Fig. 9 Minimum amount of objective function in terms of slope of linear function

Here, another complicated example is presented with following relation as the desired function:

$$f(x) = 0.5(x-1)\sin x.$$

The optimization is done in the domain of $x \in [0, 3\pi/4]$ and the stroke angles of crank and follower are supposed 80° and 90°, respectively. The uniform distribution is supposed as the initial guess. The desired function has two extremum positions in this domain. Due to the complicated form of the desired function, the optimization process could not decrease the value of the objective function less than 2.01667. Figure 7 shows diagrams of the optimal generated function and the desired function. According to the figure, the deviation is high near to the extremum positions.

4.3 Linear functions

Here, the effect of the linear function slope ($f(x) = mx$) on precision point distribution in the optimal mechanism has been studied. So, the mechanism has been optimized at different slopes (different values of m) and in the specific domain: $x \in [0, 1]$. Under these conditions, the stroke angles of links (2) and (4) were taken to be $\pi/3$ and $\pi/2$, respectively.

In Table 4, specifications of the optimal mechanism at different slopes have been presented. Table 5 presents the distribution of precision points related to the optimal mechanism; it also shows the minimum of the objective function at different slopes. According to this data for all slopes, two points are out of the domain (note negative values in this table).

Figure 8 shows the absolute value of the difference between the generated function and the desired function in terms of slope.

Figure 9 represents the objective function minimum at different values of the slope (related to the optimal mechanism). Relation (18) represents the value of the objective function in the optimal mechanism in terms of the slope of the linear function:

$$F_{optimum} = 7.204668 \times 10^{-6} m^{1.99808}, \tag{18}$$

$$\left.\frac{F_2}{F_1}\right|_{optimum} \approx \left(\frac{m_2}{m_1}\right)^2. \tag{19}$$

Table 4 Specifications of optimal mechanisms for generating linear function ($f(x) = mx$)

Slope of line (m)	K_1	K_2	K_3	φ(rad)	ψ(rad)	Minimum amount of constraint
0.1	0.0687026	0.4045662	0.6641363	3.1449366	3.1466087	0.7220227
0.2	0.0696455	0.4066361	0.6630092	3.1428742	3.1435184	0.7236283
1	0.0696848	0.4067226	0.6629619	3.1428520	3.1434852	0.7236825
2.5	0.0695646	0.4064559	0.6631085	3.1428779	3.1435238	0.7235230
100	0.0696838	0.4067205	0.6629630	3.1428652	3.1435050	0.7236756

Table 5 Distribution of precision points in optimal mechanisms for generating linear function ($f(x) = mx$)

Slope of line (m)	Distribution of accuracy points					Minimum of objective function
	x_1	x_2	x_3	x_4	x_5	
0.1	−0.254290	−0.139506	0.0583280	0.3461491	0.6910749	7.242E–8
0.2	−0.254291	−0.139508	0.0583281	0.3461492	0.6910750	2.896E–7
1	−0.254370	−0.141507	0.0579972	0.3490415	0.6953933	7.005E–6
2.5	−0.253673	−0.141683	0.0598560	0.3469035	0.6790528	4.887E–5
100	−0.255335	−0.141245	0.0592072	0.3481816	0.6754180	0.07012225

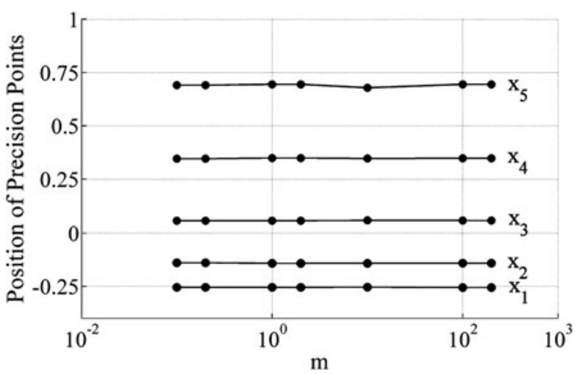

Fig. 10 Distribution of precision points in terms of slope of linear function

In relation (19), F_1 and F_2 are the values of the objective function in the optimal mechanism for two different slopes: m_1 and m_2. According to this relation, the minimum value of the objective function is proportional to the second order of generated function's slope.

Figure 10 shows precision point distributions at different values of slope. Here, the distributions of precision points are independent from slopes. According to the figure, in order to optimizing the generated function, two of precision points are located at out of domain which is an interesting matter in optimal mechanism design.

4.4 Polynomial function

In this part of the study, the effect of slope variation of the desired function on optimization has been studied. Here, several polynomial functions have been used. These functions have the same domain and range: $x \in [0, 0.5]$ and $f(x) \in [0, 0.25]$.

The relation of these polynomial functions is defined below:

$$f(x) = x^n + \left(\frac{0.25 - 0.5^n}{0.5^{n-1}}\right)x^{n-1}. \tag{20}$$

In order to generating these functions, the stroke angles of links (2) and (4) were taken to be $2\pi/3$ and $\pi/8$, respectively. In Fig. 11, these polynomial functions are shown and, according to this figure, the function slope decreases in the left hand side of the domain and increases in the right hand side with an increase of polynomial order. Table 6 shows the specifications of the optimal mechanism; thus, according to the last column data, the existence constraint is satisfied for all mechanisms. Table 7 shows the distribution of precision points for the optimal mechanism, and with increasing order of the desired function, the error value increases.

Table 6 Specifications of optimal mechanisms for generating polynomial function

Degree of function (n)	K_1	K_2	K_3	φ(rad)	ψ(rad)	Minimum amount of constraint
2	2.1359	0.8660	1.6115	1.3646	3.5186	0.7343
3	1.7646	0.9516	1.5054	1.3721	3.3603	0.2808
4	1.4838	0.9998	1.3324	1.4085	3.2796	0.1234
5	1.1726	1.0098	1.0767	1.3841	3.2331	0.0253
6	1.3856	1.0052	1.3521	1.4736	3.1749	0.0183
7	1.2499	1.0027	1.2312	1.4443	3.1605	0.0103

Table 7 Distribution of precision points in optimal mechanisms for generating polynomial function

Degree of function (n)	Distribution of accuracy points					Minimum of objective function
	x_1	x_2	x_3	x_4	x_5	
2	0.0230	0.1144	0.2480	0.3819	0.4757	1.0720E–5
3	0.0263	0.1279	0.2658	0.3927	0.4780	4.7917E–5
4	0.0312	0.1465	0.2877	0.4042	0.4802	2.6253E–4
5	0.0230	0.1144	0.2480	0.3819	0.4757	1.0720E–5
6	0.0441	0.1910	0.3340	0.4276	0.4849	0.0011
7	0.0493	0.2096	0.3517	0.4362	0.4867	0.0015

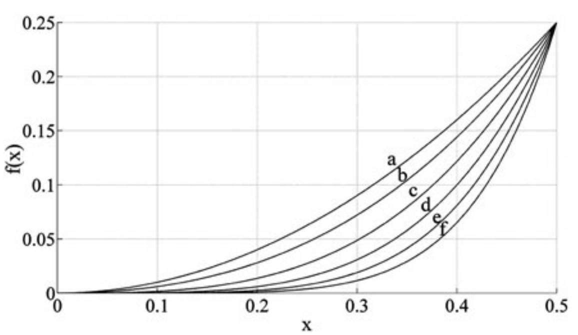

Fig. 11 Diagrams of polynomial function: a—degree two, b—degree three, c—degree four, d—degree five, e—degree six and f—degree seven ($f(x) = x^n + ax^{n-1}$, $a = (0.25 - 0.5^n)/0.5^{n-1}$)

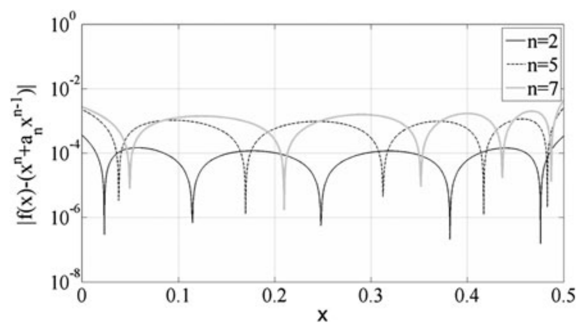

Fig. 12 Deviation of optimum generated function from polynomial function

Figure 12 shows the deviation of generated functions compared to polynomial functions of orders two, five and seven. Here, asymptotes introduce the position of precision points. According to this figure, with increasing function order, the deviation will be increased along the domain because, by increasing the polynomial function order, the slope in the left hand side of the domain will be decreased and that in the right hand side of domain will be increased. Thus, increasing the degree of polynomial functions raises the growth value of the objective function in optimal mechanisms, where this is shown in Fig. 13. According to this figure, unlike at higher orders, the variation of the objective function is sharp at lower orders.

Figure 14 shows the distribution of precision points in the optimal mechanism in terms of the order of the polynomial function. According to this figure, the distribution of precision points is close to the right hand side of domain at increased orders of the polynomial. This result is clear for points such as x_2, x_3, x_4.

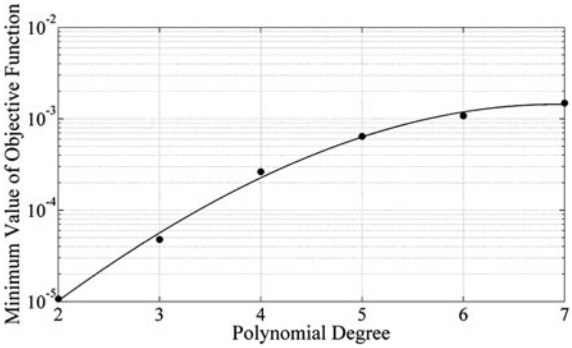

Fig. 13 Minimum amount of objective function in terms of the degree of polynomial function

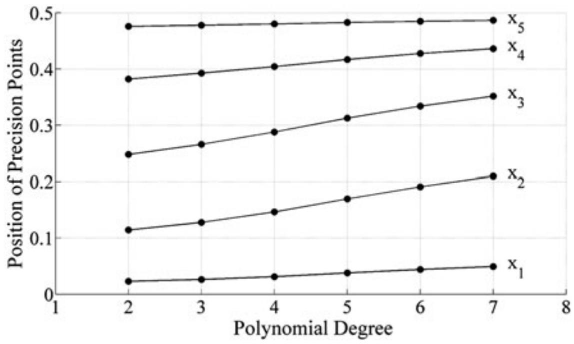

Fig. 14 Distribution of precision points in terms of the degree of polynomial function

5 Conclusion

In this study, a gradient-base SQP method has been used for minimizing the objective function. Due to high order non-linearity of objective function and constraint and lack of specific relations for their gradient, applying this method for optimizing the function generator of linkage is too complex. In each step of the optimization, the Freudenstein equation has been used for synthesis of the mechanism, and the Newton-Gauss method has been used for solving a set of non-linear equations related to this synthesis.

The important results of this research are:

- The optimization process noticeably increases the global accuracy in different mechanisms. In the current research, the error has decreased to the order of 10^{-4}–10^{-2}.
- In order to optimizing the global accuracy of four-bar linkages, the position of precision points is not necessary to locate in the calculating domain. On the other word, depend to the shape of desired function; it is possible to obtain the optimal mechanism in which some precision points located at the out of calculating domain. For instance, in linear functions that have been studied in Sect. 4.3, two precision points are located at the out of the function's domain (see Fig. 12).
- If the desired function has different slopes, the optimization process will shift the distribution of the precision points from the low slope to the high slope region. On the other word, the concentration of the precision points is higher in the sharp slope regions in the optimal mechanism (refer to Sect. 4.4 and especially Fig. 14). This matter is approximately similar to the mesh generation for solving the partial differential equations in which the concentration of the calculating grids must be higher in the high gradient region to obtain more accurate numerical solution.
- For linear functions, the distribution of precision points is independent of the value of the slope, and the value of error in the mechanism is related to order two of the line slope.

By considering the above results, designers can be had a perception for precision point's distribution of optimal linkage based on the shape of desired function and its slop.

Acknowledgements This paper is presented based on a researching project which is granted by Shahrood University of Technology. Therefore, the authors appreciate for their financial support.

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