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The Coiflet–Galerkin method for linear Volterra integral equations

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ABSTRACT

This paper deals with the application of the Wavelet–Galerkin method based on Coiflets as a basis for solving linear Volterra integral equations (VIEs). The main contribution of this work is that some new connection coefficients are introduced and a suitable algorithm is developed for their solution; once they have been computed they can be stored and applied to any linear VIE. The convergence properties of the Coiflet–Galerkin method are analyzed. Some test examples are presented to illustrate the performance of the method with respect to the error norms and CPU time.

1. Introduction

In recent years, wavelet bases have been used in the numerical approximation of different problems arising in many fields of science and engineering. Wavelet bases, in conjunction with collocation-type methods, have been used for solving Fredholm and Volterra integral equations, see [1–6]. The numerical solution of integro-differential equations by using CAS wavelets is the subject of [7]. Wavelets have also been considered in the approximation of some nonlinear problems, namely nonlinear VIEs [8], nonlinear integro-differential equations [9] and nonlinear Fredholm integral equations [10]. An algorithm based on interpolating polynomial approximation is discussed in [11] for two-dimensional Fredholm equations.

In the Wavelet–Galerkin method wavelet bases are used with the well-known Galerkin procedure in the place of other conventional bases like Legendre or Chebyshev bases. The Daubechies–Galerkin method was used for solving Fredholm integral equations in [12,13]; Daubechies wavelets have also been applied to VIEs [14] and integro-differential equations [15]. Wavelet–Galerkin algorithms for boundary integral equations are considered in [16]. A reference for the application of the Galerkin method to nonlinear VIEs is [17] and weakly singular Fredholm integral equations are studied by Gao in [18].

In this work we investigate the application of the wavelet Galerkin method based on Coiflets as a basis for solving VIEs. In connection with this method, we introduce some new connection coefficients which have not been evaluated earlier and we propose a suitable algorithm for their evaluation. The organization of the paper is as follows. In Section 2 we consider the Coifman wavelet systems and give a summary of their properties. In Section 3 we introduce the Coiflet–Galerkin method for the approximation of a linear VIE of the second kind. Then some new connection coefficients are encountered and their evaluation is the issue of Section 4. In Section 5 we discuss the convergence properties of the method. Some simple numerical examples are provided in Section 6 to illustrate the capabilities of the method concerning the error norms and CPU time.

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2. Coifman wavelets (Coiflets)

In this section we review some properties of the Coifman wavelet systems. Coiflets are orthogonal wavelets for which the *scaling* function ϕ and the *mother wavelet* ψ have several vanishing moments. The scaling function $\phi(x)$ is defined implicitly by the dilation equation

$$\phi(x) = \sum_k a_k \phi(2x - k), \tag{2.1}$$

where the $a_k \in \mathbb{C}$ are called scaling or recursion coefficients. In the case of a compact support the summation above involves only a finite number of terms. Exact values of the scaling coefficients for some Coiflets of different genuses are presented in [19]. The Coifman scaling function ϕ satisfies the orthogonality conditions

$$\langle \phi(x), \phi(x-k) \rangle = \int \phi(x)\phi(x-k)dx = \delta_{0,k}, \quad k \in \mathbb{Z}, \tag{2.2}$$

where $\delta_{0,k}$ is the Kronecker delta function. Consider the family of orthogonal functions

$$\phi_{n,l}(x) = 2^{n/2}\phi(2^n x - l), \quad n, l \in \mathbb{Z}. \tag{2.3}$$

For each n the set $\{\phi_{n,l}(x)\}_{l\in\mathbb{Z}}$ generates a space $V_n \subset L^2(\mathbb{R})$ such that $V_n \subset V_{n+1}$. Moreover, the set $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R})$. Denoting by P_n the orthogonal projection operator in $L^2(\mathbb{R})$ to the subspace V_n:

$$(P_n f)(x) = \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}} f(x)\phi_{n,k}(x)dx \right) \phi_{n,k}(x), \tag{2.4}$$

then $P_n f$ converges to f in the L^2 norm.

Definition 2.1. An orthonormal wavelet system with compact support is called a Coifman wavelet system of degree r if the moments of ϕ and ψ satisfy

$$\mathrm{Mom}(\phi)_l = \int x^l \phi(x)dx = \delta_{0,l}, \quad \mathrm{Mom}(\psi)_l = \int x^l \psi(x)dx = 0, \quad l = 0, \ldots, r. \tag{2.5}$$

We shall need the following results [20].

Theorem 2.1. *For a Coifman wavelet system of degree r with scaling function $\phi(x)$ and scaling vector α, assume α has finite length. If $f(x) \in C_0^r(\mathbb{R})$ then the wavelet orthogonal projection P_n satisfies*

$$\|f(x) - (P_n f)(x)\|_{L^2} \leqslant C 2^{-nr}, \tag{2.6}$$

where and C depends only on $f(x)$ and the scaling vector α.

Theorem 2.2. *Under the same conditions as in the previous theorem, define the wavelet sampling approximation of the function $f(x)$ at level $n \in \mathbb{N}$ as*

$$(S_n f)(x) = 2^{-n/2} \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2^n}\right) \phi_{n,k}(x), \tag{2.7}$$

where $\phi_{n,k}(x) = 2^{n/2}\phi(2^n x - k)$. We have

$$\|f(x) - (S_n f)(x)\|_{L^2} \leqslant C 2^{-nr}, \tag{2.8}$$

where C depends only on $f(x)$ and the scaling vector α.

This result can be extended to higher dimensions, for as follows: For $f(x_1, \ldots, x_m) \in C_0^r(\mathbb{R}^m)$, let

$$(S_n f)(x_1, \ldots, x_m) = 2^{-\frac{nm}{2}} \sum_{k_1,\ldots,k_m\in\mathbb{Z}} f\left(\frac{k_1}{2^n}, \ldots, \frac{k_m}{2^n}\right) \phi_{n,k_1}(x_1), \ldots, \phi_{n,k_m}(x_m). \tag{2.9}$$

Similarly to (2.8), we have

$$\|f(x_1, \ldots, x_m) - (S_n f)(x_1, \ldots, x_m)\|_{L^2} \leqslant C 2^{-nr}. \tag{2.10}$$

3. Problem approximation

Consider the following VIE of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad 0 \leqslant t \leqslant x \leqslant 1, \tag{3.1}$$

where u is an unknown function and $f \in L^2[0,1],\ K \in L^2([0,1]\times[0,1])$ are explicitly known. In this section we define the Coiflet–Galerkin method applied to Eq. (3.1) and introduce some new connection coefficients.

Let $\phi(x)$ be a Coiflet scaling function of degree r and support $[p,q]$; hence the function $\phi_{n,i}(x) = 2^{n/2}\phi(2^n x - i)$ has the support $[2^{-n}(p+i), 2^{-n}(q+i)]$. Since $\{\phi_{n,i}(x)\}_{i\in\mathbb{Z}}$ is an orthonormal basis for the space V_n, then any function $u_n(x) \in V_n$ can be represented by the scaling function series.

$$u_n(x) = \sum_{i=-\infty}^{\infty} u_{n,i}\phi_{n,i}(x), \tag{3.2}$$

where the $u_{n,i}$ are unknown coefficients. Due to the compact support of $\phi_{n,i}(x)$ and since we look for an approximate solution of (3.1) in the interval $[0,1]$, then (3.2) reduces to:

$$u_n(x) = \sum_{i=1-q}^{2^n-p-1} u_{n,i}\phi_{n,i}(x). \tag{3.3}$$

Until the end of this section by $\sum$, we mean $\sum_{1-q}^{2^n-p-1}$. Approximating u by u_n in (3.1) we obtain:

$$\sum_i u_{n,i}\phi_{n,i}(x) = f(x) + \sum_i u_{n,i}\int_0^x K(x,t)\phi_{n,i}(t)dt, \qquad 0 \leqslant t \leqslant x \leqslant 1. \tag{3.4}$$

In order to solve (3.4) by the Galerkin procedure, we use (2.7) and (2.9) to approximate the functions $f(x)$ and $K(x,t)$ in the interval $[0,1]$, as follows:

$$f(x) \approx 2^{-\frac{n}{2}}\sum_k f\left(\frac{k}{2^n}\right)\phi_{n,k}(x), \tag{3.5}$$

$$K(x,t) \approx 2^{-n}\sum_{j,l} K\left(\frac{j}{2^n},\frac{l}{2^n}\right)\phi_{n,j}(x)\phi_{n,l}(t). \tag{3.6}$$

In order to evaluate the first and the last few values of $f(\frac{k}{2^n})$ and $K(\frac{j}{2^n},\frac{l}{2^n})$ in (3.5) and (3.6) we need to extend these functions smoothly, and this will be discussed in the Section 3.1. Using the expressions (3.5) and (3.6) into (3.4) yields

$$\sum_i u_{n,i}\phi_{n,i}(x) = 2^{-\frac{n}{2}}\sum_k f\left(\frac{k}{2^n}\right)\phi_{n,k}(x) + 2^{-n}\sum_{i,j,l} K\left(\frac{j}{2^n},\frac{l}{2^n}\right)u_{n,i}\phi_{n,j}(x)\int_0^x \phi_{n,i}(t)\phi_{n,l}(t)dt. \tag{3.7}$$

We next multiply both sides of (3.7) by $\phi_{n,m}(x),\ m = 1-q,\ldots,2^n-p-1$, and then integrate over $[0,1]$, to obtain

$$\sum_i u_{n,i}\int_0^1 \phi_{n,i}(x)\phi_{n,m}(x)dx = 2^{-\frac{n}{2}}\sum_k f\left(\frac{k}{2^n}\right)\int_0^1 \phi_{n,k}(x)\phi_{n,m}(x)dx + 2^{-n}\sum_{i,j,l} K\left(\frac{j}{2^n},\frac{l}{2^n}\right)u_{n,i}\int_0^1 \phi_{n,j}(x)\phi_{n,m}(x) \times \int_0^x \phi_{n,i}(t)\phi_{n,l}(t)dtdx, m = 1-q,\ldots,2^n-p-1. \tag{3.8}$$

Define the following connection coefficients

$$\Gamma_{k,l}(x) = \int_0^x \phi(y-k)\phi(y-l)dy, \tag{3.9}$$

$$\Lambda_{i,j}^{k,l} = \int_0^{2^n} \phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy. \tag{3.10}$$

Then the linear system (3.8) reduces to

$$\sum_{i=1-q}^{2^n-p-1} u_{n,i}\Gamma_{i,m}(2^n) = 2^{-\frac{n}{2}}\sum_{k=1-q}^{2^n-p-1} f\left(\frac{k}{2^n}\right)\Gamma_{k,m}(2^n) + 2^{-n}\sum_{i,j,l=1-q}^{2^n-p-1} K\left(\frac{j}{2^n},\frac{l}{2^n}\right)u_{n,i}\Lambda_{j,m}^{i,l}, \quad (m = 1-q,\ldots,2^n-p-1). \tag{3.11}$$

Hence, the unknown parameters $u_{n,i} = u(\frac{i}{2^n})$ can be obtained by solving a linear system of the form

$$A\mathbf{u} = \mathbf{b}, \tag{3.12}$$

where

$$\begin{aligned} A_{m,i} &= \Gamma_{i,m}(2^n) - 2^{-n}\sum_{j,l=1-q}^{2^n-p-1} K\left(\tfrac{j}{2^n},\tfrac{l}{2^n}\right)\Lambda_{j,m}^{i,l}, \\ b_m &= 2^{-\frac{n}{2}}\sum_{k=1-q}^{2^n-p-1} f\left(\tfrac{k}{2^n}\right)\Gamma_{k,m}(2^n), \quad (m,i = 1-q,\ldots,2^n-p-1). \end{aligned} \tag{3.13}$$

We emphasize that the connection coefficients $\Gamma_{k,l}(x)$ and $\Lambda_{i,j}^{k,l}$ can be computed once and applied in approximating the solution of any VIE of the second kind.

3.1. Smooth extension of functions

Let $f \in C^m[0,1]$. This function can be extended to $[-\delta,1], 0<\delta\leqslant\frac{1}{m}$, by the reflection formula (see, e.g. [21,22])

$$f(x)=\sum_{j=0}^{m} c_j f(-jx) \quad -\delta\leqslant x<0, \tag{3.14}$$

where the $c_j,\ j=0,\dots,m$ are chosen such that a C^m-smooth joining takes place at $x=0$, that is, if $\lim_{x\to 0} f^{(k)}(x)=f^{(k)}(0)$, which is equivalent to

$$\sum_{j=0}^{m}(-j)^k c_j=1, \quad k=0,\dots,m. \tag{3.15}$$

The values of c_j can be obtained by solving the system (3.15) and they can be used in the extension of f onto $[0,1+\delta]$:

$$f(x)=\sum_{j=0}^{m} c_j f(1-j(x-1)), \quad 1<x\leqslant 1+\delta, \tag{3.16}$$

We thus obtain an extended function $f\in C^m[-\delta,1+\delta]$ and it holds

$$\max_{-\delta\leqslant x\leqslant 1+\delta}|f(x)|\leqslant\sum_{j=0}^{m}|c_j|\max_{0\leqslant x\leqslant 1}|f(x)|=(2^{m+1}-1)\max_{0\leqslant x\leqslant 1}|f(x)|.$$

The function $K(x,t)$ can also be extended in a similar way.

4. Evaluation of the connection coefficients

Here we present an adequate algorithm for evaluating the new connection coefficients introduced in the previous section. These coefficients have not been evaluated earlier and may be useful in the investigation of a similar class of VIEs. So this may be considered as the main contribution of this work.

4.1. Evaluation of $\Gamma_{k,l}(x)=\int_0^x\phi(y-k)\phi(y-l)dy$

Assume that the support of the scaling function $\phi(.)$ is $[p,q]$, therefore the support of function $\phi(.-k)$ is $[p+k,q+k]$ and, in general, $\phi_{n,k}(.)$ has the compact support $[2^{-n}(p+k),2^{-n}(q+k)]$. It is easy to verify the following properties of $\Gamma_{k,l}(x)$:

$$\Gamma_{k,l}(x)=\Gamma_{l,k}(x), \tag{4.1}$$

$$\Gamma_{k,l}(x)=\Gamma_{k-l,0}(x-l)-\Gamma_{k-l,0}(-l)=\Gamma_{l-k,0}(x-k)-\Gamma_{l-k,0}(-k), \tag{4.2}$$

$$\Gamma_{k-i,l-i}(-i)=-\Gamma_{k,l}(i), \tag{4.3}$$

$$\Gamma_{k,l}(x+i)=\Gamma_{k-i,l-i}(x)-\Gamma_{k-i,l-i}(-i)=\Gamma_{k-i,l-i}(x)+\Gamma_{k,l}(i), \tag{4.4}$$

$$\Gamma_{k,l}(x)=0, \quad \text{if} \quad |k-l|\geqslant q-p, \tag{4.5}$$

$$\Gamma_{k,l}(x)=0, \quad \text{if } x\geqslant 0 \text{ and } (k\leqslant -q \text{ or } l\leqslant -q \text{ or } k\geqslant x-p \text{ or } l\geqslant x-p), \tag{4.6}$$

$$\Gamma_{k,l}(x)=0, \quad \text{if } x\leqslant 0 \text{ and } (k\geqslant -p \text{ or } l\geqslant -p \text{ or } k\leqslant x-q \text{ or } l\leqslant x-q). \tag{4.7}$$

$$\Gamma_{k,l}(x)=\frac{1}{2}\sum_{i,j=p}^{q} a_i a_j \Gamma_{2k+i,2l+j}(2x), \tag{4.8}$$

Relation (4.2) is obtained by choosing $y-l=t$ or $y-k=t$ in the definition of $\Gamma_{k,l}(x)$. Relations (4.5)–(4.7) are valid due to the compact support of functions $\phi(y-k)$ and $\phi(y-l)$. The two-scale relation (4.8) follows from applying (2.1) to $\Gamma_{k,l}(x)$. Other relations can also be obtained in a straightforward way.

Considering (4.2), if one can evaluate the values of $\Gamma_{k,0}(x)$ for each k and x, then the values of $\Gamma_{k,l}(x)$, for each $k,\ l$ and x can be easily obtained. Now, we discuss some properties of $\Gamma_{k,0}(x)$

$$\Gamma_{k,0}(x)=0, \quad \text{if} \quad |k|\geqslant q-p, \tag{4.9}$$

$$\Gamma_{k,0}(x) = 0, \quad \text{if} \quad x \geqslant 0 \quad \text{and} \quad (k \leqslant -q \quad \text{or} \quad k \geqslant x - p), \tag{4.10}$$

$$\Gamma_{k,0}(x) = 0, \quad \text{if} \quad x \leqslant 0 \quad \text{and} \quad (k \geqslant -p \quad \text{or} \quad k \leqslant x - q), \tag{4.11}$$

$$\Gamma_{k,0}(x) = \Gamma_{k,0}(q), \quad \text{if} \quad x \geqslant q, \tag{4.12}$$

$$\Gamma_{k,0}(x) = \Gamma_{k,0}(p), \quad \text{if} \quad x \leqslant p, \tag{4.13}$$

$$\Gamma_{k,0}(q) - \Gamma_{k,0}(p) = \delta_{k,0}. \tag{4.14}$$

Relations (4.9)–(4.11) result directly from (4.5)–(4.7), while equations (4.12) and (4.13) hold because of the compact support of the function $\phi(y)$. We thus see that the unknown values of $\Gamma_{k,0}(x)$ are limited to $x \in [p,q]$ and $|k| \leqslant q - p - 1$. To find these values we apply the recursive relation (4.8) and (4.2) to $\Gamma_{k,0}(x)$ to obtain

$$\Gamma_{k,0}(x) = \frac{1}{2}\sum_{i,j=p}^{q} a_i a_j \left(\Gamma_{2k+i-j,0}(2x-j) - \Gamma_{2k+i-j,0}(-j)\right), \quad |k| \leqslant q-p-1 \quad \&p \leqslant x \leqslant q. \tag{4.15}$$

We are interested in determining the unknown values of $\Gamma_{k,0}(x)$ for integers k and x; applying relations (4.9)–(4.14), to (4.15) for integers k and x yields a non-homogeneous system of linear equations which can be easily solved. Some values of $\Gamma_{k,0}(x)$ are displayed in Table 1.

In order to evaluate the integrals $\Lambda_{i,j}^{k,l}$ defined by (3.11) three new parameter connection coefficients are introduced in the next section.

4.2. Evaluation of integrals $\Upsilon_r^{k,l} = \int_{\mathbb{R}} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy$

Consider the following three parameter connection coefficients:

$$\Upsilon_r^{k,l} = \int_{\mathbb{R}} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy. \tag{4.16}$$

From now on by the integral sign $\int$, we mean $\int_{\mathbb{R}}$. Now, we discuss some properties of integrals $\Upsilon_r^{k,l}$

$$\Upsilon_r^{k,l} = 0, \quad \text{if} \quad |r| \geqslant q-p \quad \text{or} \quad |k-l| \geqslant q-p. \tag{4.17}$$

$$\Upsilon_r^{k,l} = \frac{1}{4}\sum_{k_1,\ldots,k_4=p}^{q} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\left(\Upsilon_{2r+k_3-k_4}^{2k+k_1-k_4,2l+k_2-k_4} + \delta_{2r+k_3-k_4,0}\Gamma_{2k+k_1,2l+k_2}(k_4)\right). \tag{4.18}$$

Relation (4.17) holds because of the support of functions $\phi(y)$ and $\phi(y-r)$ and (4.5). In order to prove (4.18), we apply the two-scale relations (2.1) and (4.8) to (4.16) and then use (4.4):

$$\begin{aligned}\Upsilon_r^{k,l} &= \frac{1}{2}\sum_{k_1,\ldots,k_4=p}^{q} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\int \phi(2y-k_4)\phi(2y-2r-k_3)\Gamma_{2k+k_1,2l+k_2}(2y)dy \\ &= \frac{1}{4}\sum_{k_1,\ldots,k_4=p}^{q} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\int \phi(t)\phi(t-2r-k_3+k_4)\Gamma_{2k+k_1,2l+k_2}(t+k_4)dt \\ &= \frac{1}{4}\sum_{k_1,\ldots,k_4=p}^{q} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\int \phi(t)\phi(t-2r-k_3+k_4)(\Gamma_{2k+k_1-k_4,2l+k_2-k_4}(t) + \Gamma_{2k+k_1,2l+k_2}(k_4))dt.\end{aligned}$$

The last equality gives (4.18). In the next lemma we prove a useful relation for evaluating the values of $\Upsilon_r^{k,l}$.

Lemma 4.1. *If* $|r-l| \geqslant q-p$ *then*

$$\Upsilon_r^{k,l} = \Gamma_{k,l}(p+l)\Gamma_{r-p,-p}(l) - \Gamma_{k,l}(q+l)\Gamma_{r-q,-q}(l), \tag{4.19}$$

Table 1
Some values of $\Gamma_{k,0}(x)$ for Coiflet of genus 2 with the support $[p,q] = [-3,4]$.

x	k	$\Gamma_{k,0}(x)$	x	k	$\Gamma_{k,0}(x)$	x	k	$\Gamma_{k,0}(x)$	x	k	$\Gamma_{k,0}(x)$
−3	−3	0.2407e−3	−1	−3	−0.5645e−3	1	−2	0.9644e−2	2	−1	−0.4024e−1
	−2	0.9608e−2		−2	−0.1046e−1		−1	−0.3887e−1		0	0.361381
	−1	−0.4023e−1		−1	−0.128276		0	0.355720		1	0.089409
	0	−0.638238		0	−0.625523		1	0.128276		2	−0.1514e−1
	1	0.8805e−1		1	0.8733e−1		2	−0.2556e−1		3	−0.8053e−3
	2	−0.5495e−2		2	−0.5495e−2		3	−0.1086e−3		4	0.1809e−3

and if $|r-k| \geqslant q-p$ *then*

$$\Upsilon_r^{k,l} = \Gamma_{k,l}(p+k)\Gamma_{r-p,-p}(k) - \Gamma_{k,l}(q+k)\Gamma_{r-q,-q}(k). \tag{4.20}$$

Proof. Considering (4.2), (4.12) and (4.13), if $y \leqslant p+l$ then $\Gamma_{k,l}(y) = \Gamma_{k,l}(p+l)$ and if $y \geqslant q+l$ then $\Gamma_{k,l}(y) = \Gamma_{k,l}(q+l)$, hence

$$\begin{aligned}
\Upsilon_r^{k,l} &= \int_{-\infty}^{p+l} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy + \int_{p+l}^{q+l} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy + \int_{q+l}^{\infty} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy \\
&= \Gamma_{k,l}(p+l)\int_{-\infty}^{p+l} \phi(y)\phi(y-r)dy + \Gamma_{k,l}(q+l)\int_{q+l}^{\infty} \phi(y)\phi(y-r)dy + \int_{p+l}^{q+l} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy \\
&= \Gamma_{k,l}(p+l)\int_{p}^{p+l} \phi(y)\phi(y-r)dy + \Gamma_{k,l}(q+l)\int_{q+l}^{q} \phi(y)\phi(y-r)dy + \int_{p+l}^{q+l} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy \\
&= \Gamma_{k,l}(p+l)\Gamma_{r-p,-p}(l) - \Gamma_{k,l}(q+l)\Gamma_{r-q,-q}(l) + \int_{p+l}^{q+l} \phi(y)\phi(y-r)\Gamma_{k,l}(y)dy,
\end{aligned}$$

where the two last relations result from the compact support of function $\phi(y)$ and an application of (4.4). Therefore by considering the support of functions $\phi(y)$ and $\phi(y-r)$, if $|r-l| \geqslant q-p$ the relation (4.19) holds.

On the other hand, it can be easily verified that if $y \leqslant p+k$ then $\Gamma_{k,l}(y) = \Gamma_{k,l}(p+k)$ and if $y \geqslant q+k$ then $\Gamma_{k,l}(y) = \Gamma_{k,l}(q+k)$; therefore, by similar arguments as in the previous case (4.20) is easily obtained. □

By applying the recursive relation (4.18), and taking into account (4.17), (4.19) and (4.20), the values of the integrals $\Upsilon_r^{k,l}$ can be evaluated for any integers k, l and r. In Table 2 some values of $\Upsilon_r^{k,l}$ are included.

4.3. Evaluation of integrals $\Lambda_{i,j}^{k,l} = \int_0^{2^n} \phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy$

The integrals $\Lambda_{i,j}^{k,l}$ satisfy the following properties

$$\Lambda_{i,j}^{k,l} = \Lambda_{j,i}^{k,l} = \Lambda_{i,j}^{l,k} = \Lambda_{j,i}^{l,k}, \tag{4.21}$$

$$\Lambda_{i,j}^{k,l} = 0 \quad \text{if} \quad |k-l| \geqslant q-p \quad \text{or} \quad |i-j| \geqslant q-p, \tag{4.22}$$

$$\Lambda_{i,j}^{k,l} = 0 \quad \text{if} \quad (i \leqslant -q \text{ or } j \leqslant -q \text{ or } i \geqslant 2^n - p \text{ or } j \geqslant 2^n - p), \tag{4.23}$$

$$\Lambda_{i,j}^{k,l} = 0 \quad \text{if} \quad (k \leqslant -q \text{ or } l \leqslant -q \text{ or } k \geqslant 2^n - p \text{ or } l \geqslant 2^n - p). \tag{4.24}$$

Relation (4.21) follows easily from (4.1) and (3.11). Relation (4.22) is valid due to (4.5) and the fact that the supports of the functions $\phi(y-i)$ and $\phi(y-j)$ do not intersect when $|i-j| \geqslant q-p$. Relation (4.23) also holds because of the compact support of functions $\phi(y-i)$ and $\phi(y-j)$. By (4.6) and considering that in (3.11) $y \in [0, 2^n]$ relation (4.24) can be obtained.

Lemma 4.2. *For the integrals* $\Lambda_{i,j}^{k,l}$, *if* $|i-l| \geqslant q-p$ *or* $|j-l| \geqslant q-p$ *then*

$$\Lambda_{i,j}^{k,l} = \Gamma_{k,l}(p+l)\Gamma_{i,j}(p+l) + \Gamma_{k,l}(q+l)\big(\Gamma_{i,j}(2^n) - \Gamma_{i,j}(q+l)\big), \tag{4.25}$$

and if $|i-k| \geqslant q-p$ *or* $|j-k| \geqslant q-p$ *then*

$$\Lambda_{i,j}^{k,l} = \Gamma_{k,l}(p+k)\Gamma_{i,j}(p+k) + \Gamma_{k,l}(q+k)\big(\Gamma_{i,j}(2^n) - \Gamma_{i,j}(q+k)\big). \tag{4.26}$$

Table 2
Some values of integrals $\Upsilon_r^{k,l}$ for Coiflet of genus 2 with the support $[p,q] = [-3,4]$.

r	k	l	$\Upsilon_r^{k,l}$	r	k	l	$\Upsilon_r^{k,l}$	r	k	l	$\Upsilon_r^{k,l}$	r	k	l	$\Upsilon_r^{k,l}$
−4	−2	−5	−8.7602e−8	0	−1	−2	0.5965e−3	2	1	0	0.3575e−3	3	2	0	0.7761e−5
		−4	−0.2596e−5			−1	−0.1447e−1			1	0.1991e−1			1	−0.1840e−3
		−3	0.6616e−5		2	1	−0.1209e−2		2	1	0.1607e−2			2	−0.1247e−3
		−2	−0.5771e−4			2	0.7897e−3			2	0.1392e−1	4	3	2	0.1323e−4
−3	1	−2	−0.1008e−5	1	0	−3	0.3082e−4		3	2	−0.1849e−2			3	−0.1463e−5
		−1	−1.3839e−7			−2	0.1848e−2			3	0.2009e−3	5	4	4	−7.3428e−8
		0	−0.3082e−4			−1	−0.3025e−2		4	1	0.1824e−5		5	3	1.9412e−8
		1	0.4266e−5			0	0.5373e−1			2	0.9163e−4			4	1.3452e−8

Proof. As indicated in the proof of Lemma 4.1, we have

$$\begin{aligned}\Lambda_{i,j}^{k,l} &= \int_0^{2^n} \phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy \\ &= \Gamma_{k,l}(p+l)\int_0^{p+l}\phi(y-i)\phi(y-j)dy + \int_{p+l}^{q+l}\phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy + \Gamma_{k,l}(q+l)\int_{q+l}^{2^n}\phi(y-i)\phi(y-j)dy \\ &= \Gamma_{k,l}(p+l)\Gamma_{i,j}(p+l) + \Gamma_{k,l}(q+l)(\Gamma_{i,j}(2^n)-\Gamma_{i,j}(q+l)) + \int_{p+l}^{q+l}\phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy,\end{aligned}$$

if $p+i \geqslant q+l$ or $p+j \geqslant q+l$ or $q+i \leqslant p+l$ or $q+j \leqslant p+l$, the integral in the last equation vanishes and (4.25) holds. Relation (4.26) can also be proved by similar arguments. □

Taking into account the above relations, one can determine which unknown values of the integrals $\Lambda_{i,j}^{k,l}$ are reduced to a bounded region related to the indices $i,\ j,\ k$ and l. In Fig.1 we have plotted this region with respect to the indices i and j and the grey part indicates where the unknown values of $\Lambda_{i,j}^{k,l}$ are located. The same figure can be plotted for the region where indices k and l can vary.

Now we want to find a recursive relation to evaluate the unknown values of integrals $\Lambda_{i,j}^{k,l}$. Applying (2.1) and the two-scale relation (4.8) to (3.11), we have

$$\Lambda_{i,j}^{k,l} = \frac{1}{4}\sum_{k_1,k_2,k_3,k_4=p}^{q} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\int_0^{2^{n+1}}\phi(y-2i-k_1)\phi(y-2j-k_2)\Gamma_{2k+k_3,2l+k_4}(y)dy. \quad (4.27)$$

The upper limit of the integral on the right-hand side of (4.27) is 2^{n+1}, hence there is not in general a recursive relation for $\Lambda_{i,j}^{k,l}$. To overcome this difficulty, we consider the following cases. Fig. 1 can also be helpful in better understanding these cases.

Case 1. If $i or j \in [-p, 2^n - q]$, then for each k and l

$$\Lambda_{i,j}^{k,l} = \int_0^{2^n}\phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy = \int_{\mathbb{R}}\phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy, \quad (4.28)$$

which follows from the compact support of the functions $\phi(y-i)$ and $\phi(y-j)$. Now by using (4.4), we have

$$\Lambda_{i,j}^{k,l} = \int\phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy = \int\phi(y)\phi(y-(j-i))\big(\Gamma_{k-i,l-i}(y)+\Gamma_{k,l}(i)\big)dy = \Upsilon_{j-i}^{k-i,l-i} + \delta_{j-i,0}\Gamma_{k,l}(i), \quad (4.29)$$

which follows from (4.16) and (2.2).

Case 2. If $(i,j) \in [1-q, -p-1]^2$, then for each k and l, and sufficiently large n, i.e. $n \geqslant \log_2 2(q-p-1)$, relation (4.27) can be written in a recursive form, as

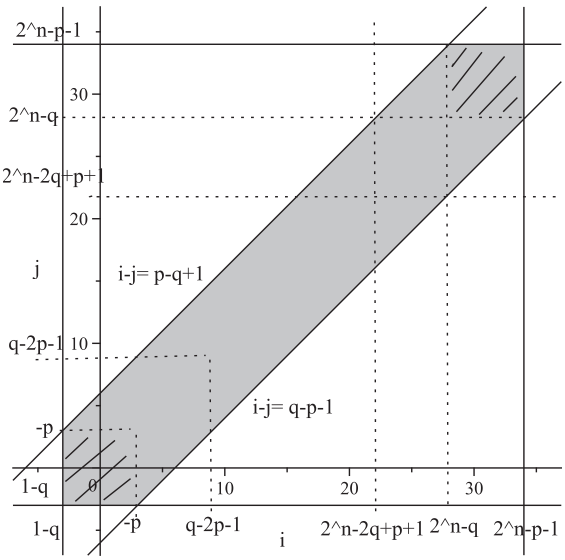

Fig. 1. Region of unknown values of integrals $\Lambda_{i,j}^{k,l}$ related to the indices i and j. (Let $n = 5$).

$$\Lambda_{i,j}^{k,l} = \frac{1}{4}\sum_{k_1,\ldots,k_4=p}^{q} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\Lambda_{2i+k_1,2j+k_2}^{2k+k_3,2l+k_4}. \tag{4.30}$$

This holds because the support of the function $\phi(y-(2i+k_1))$ in (4.27) is $[p+2i+k_1, q+2i+k_1]$ and if $q+2i+k_1 \leqslant 2^n$ then the interval of integration in relation (4.27) can be considered as $[0,2^n]$ instead of $[0,2^{n+1}]$. On the other hand, by Lemma 4.2, one can see that when $(i,j) \in [1-q,-p-1]^2$, if k or $l \geqslant q-2p-1$, the values of $\Lambda_{i,j}^{k,l}$ can be determined. Therefore, relation (4.30) can be applied for $(i,j) \in [1-q,-p-1]^2$, and $(k,l) \in [1-q,q-2p-2]^2$. After evaluating the connection coefficients $\Upsilon_r^{k,l}$ and using them in (4.30), a non-homogeneous system of linear equations is obtained which can be easily solved.

Case 3. If $(i,j) \in [2^n-q+1, 2^n-p-1]^2$, then for each k and l we have

$$\begin{aligned}\Lambda_{i-2^n,j-2^n}^{k-2^n,l-2^n} &= \int_0^{2^n}\phi(y-i+2^n)\phi(y-j+2^n)\big(\Gamma_{k,l}(y+2^n)-\Gamma_{k,l}(2^n)\big)dy \\ &= \int_{2^n}^{2^{n+1}}\phi(t-i)\phi(t-j)\Gamma_{k,l}(t)dt - \Gamma_{k,l}(2^n)\Gamma_{i-2^n,j-2^n}(2^n).\end{aligned} \tag{4.31}$$

Adding (3.11)–(4.31)

$$\Lambda_{i,j}^{k,l} + \Lambda_{i-2^n,j-2^n}^{k-2^n,l-2^n} = \int_0^{2^{n+1}}\phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy - \Gamma_{k,l}(2^n)\Gamma_{i-2^n,j-2^n}(2^n). \tag{4.32}$$

When $i \in [2^n-q+1, 2^n-p-1]$ then the support of function $\phi(y-i)$ is $D \subset [2^n+p-q+1, 2^n-p+q-1]$, now by choosing $n \geqslant \log_2(q-p-1)$, clearly $D \subset [0,2^{n+1}]$, hence (4.32) can be written as follows:

$$\Lambda_{i,j}^{k,l} + \Lambda_{i-2^n,j-2^n}^{k-2^n,l-2^n} = \int \phi(y-i)\phi(y-j)\Gamma_{k,l}(y)dy - \Gamma_{k,l}(2^n)\Gamma_{i-2^n,j-2^n}(2^n),$$

by applying (4.29), we have

$$\Lambda_{i,j}^{k,l} = -\Lambda_{i-2^n,j-2^n}^{k-2^n,l-2^n} + \Upsilon_{j-i}^{k-i,l-i} + \delta_{j-i,0}\Gamma_{k,l}(i) - \Gamma_{k,l}(2^n)\Gamma_{i-2^n,j-2^n}(2^n). \tag{4.33}$$

This means that in this case all values of $\Lambda_{i,j}^{k,l}$ can be obtained directly from the other values which have been previously evaluated. We can restrict more the region where k and l can vary, in fact when $(i,j) \in [2^n-q+1, 2^n-p-1]^2$, if k or $l \leqslant 2^n-2q+p+1$ then from Lemma 4.2 the values of $\Lambda_{i,j}^{k,l}$ can be determined.

Based on the above mentioned cases, all of the unknown $\Lambda_{i,j}^{k,l}$ can be evaluated. Table 3 shows the values of some of these integrals.

5. Convergence analysis

Consider the following VIE of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad x \in I := [0,1], \tag{5.1}$$

or the equivalent operator form

$$u = f + \mathcal{K}u, \quad x \in I := [0,1], \tag{5.2}$$

where

$$(\mathcal{K}\phi)(x) := \int_0^x K(x,t)\phi(t)dt, \quad x \in I,$$

and $f : I \to \mathbb{R}$ and $K : D \times \mathbb{R} \to \mathbb{R}$ (with $D := \{(x,t) : 0 \leqslant t \leqslant x \leqslant 1\}$) are known functions. The problem (5.1) has a unique solution if $g \in C(I)$ and K is continuous for all $(x,t) \in D$ and all u and also satisfies the (uniform) Lipschitz conditions:

$$|K(x,t,u_1) - K(x,t,u_2)| \leqslant l_1|u_1-u_2|,$$

$$|K_x(x,t,u_1) - K_x(x,t,u_2)| \leqslant l_2|u_1-u_2|,$$

for all $x \in I$, $(x,t) \in D$ and $u_1, u_2 \in R$, with l_1, and l_2 independent of u_1 and u_2.

To solve (5.1) by Coiflet–Galerkin method, first define

$$v_n(x_1,x_2,\ldots,x_m) = (S_nv)(x_1,x_2,\ldots,x_m),$$

where S_nv is the sampling approximation of function v defined by (2.9). As mentioned in Section 3, we approximate (5.1) by the following problem: find $u_n \in V_n$ such that

$$u_n = f_n + \mathcal{K}_nu_n, \tag{5.3}$$

Table 3
Some values of integrals $\Lambda_{i,j}^{k,l}$ for Coiflet of genus 2 with the support $[p,q] = [-3,4]$.

i	j	k	l	$\Lambda_{i,j}^{k,l}$	i	j	k	l	$\Lambda_{i,j}^{k,l}$	i	j	k	l	$\Lambda_{i,j}^{k,l}$
−1	−2	2	0	0.2238e−5	1	1	0	−1	−0.3822e−1	5	2	4	3	−0.1840e−3
			1	-0.1098e−5				0	0.3460302				4	−0.1246e−3
	−1	1	0	0.1509e−3		1	3	2	−0.1927e−2		4	5	3	−0.1606e−2
			1	0.5395e−3				3	0.8548e−3				5	−0.3251e−1
0	−3	0	−1	−0.2318e−5	2	0	1	0	0.2097e−3	9	9	9	7	−0.1392e−1
			0	0.1526e−4				1	0.1993e−1				8	0.3251e−1
	0	0	−1	−0.8724e−2		2	0	−1	−0.4020e−1				9	0.5
			0	0.6544e−1				0	0.360921			11	10	−0.1927e−2
15	12	14	13	−0.1840e−3	23	19	21	20	0.6615e−5	31	29	13	13	0.3555e−4
			14	−0.1247e−3				21	−0.5770e−4			25	25	0.3554e−4
	15	1	1	0.987285		22	22	21	−0.3025e−2		31	29	28	0.2372e−4
		7	7	1				22	0.5373e−1				29	0.993104
19	18	21	19	−0.1872e−3	28	25	27	26	−0.1840e−3	34	29	16	16	0.17609e−5
			20	0.2305e−3				27	−0.1246e−3		30	3	3	0.1808e−3
	19	2	2	0.999935		28	27	26	0.1958e−2		32	24	24	0.5495e−2
		11	11	1				27	0.979487		34	28	28	0.6513e−4

where

$$(\mathcal{K}_n\phi)(x) := \int_0^x K_n(x,t)\phi(t)dt.$$

Letting

$$r_n := u_n - f_n - \mathcal{K}_n u_n,$$

by the Galerkin method we try to solve

$$(r_n, \phi_{n,m}) = 0, \quad \forall m, \ \phi_{n,m} \in V_n,$$

where $(.,.)$ denotes the usual inner product in L^2-space. As we know

$$(r_n, \phi_{n,m}) = 0 \qquad \text{iff} \quad P_n r_n = 0,$$

where P_n is the orthogonal projection defined in Section 2. Since $P_n r_n = 0$ then

$$P_n(I - \mathcal{K}_n)u_n = P_n f_n,$$

and since $f_n \in V_n$ then we have

$$(I - P_n\mathcal{K}_n)u_n = f_n. \tag{5.4}$$

Theorem 5.1. *Assume that $\mathcal{K}$ is bounded and 1 is not an eigenvalue of the Volterra integral operator $\mathcal{K}$, then u_n can be uniquely determined from (5.4).*

Proof. Since $\mathcal{K}$ is bounded and 1 is not an eigenvalue of that hence $(I - \mathcal{K})^{-1}$ always exists and

$$\begin{aligned} I - P_n\mathcal{K}_n &= (I - \mathcal{K}) + (\mathcal{K} - \mathcal{K}_n) + (I - P_n)\mathcal{K}_n \\ &= (I - \mathcal{K})\{I + (I - \mathcal{K})^{-1}[(\mathcal{K} - \mathcal{K}_n) + (I - P_n)\mathcal{K}_n]\}. \end{aligned}$$

Applying Theorems 2.2 and 2.1 and the techniques used in [23], then it follows that, for sufficiently large n, the operator $(I - P_n\mathcal{K}_n)^{-1}$ exists and is uniformly bounded, therefore (5.4) is uniquely solvable and

$$u_n = (I - P_n\mathcal{K}_n)^{-1} f_n. \qquad \square$$

Define

$$\begin{aligned} (Gv)(x) &:= f(x) + \int_0^x K(x,t)v(t)dt, \\ (G_n v)(x) &:= f_n(x) + \int_0^x K_n(x,t)v(t)dt. \end{aligned}$$

Let $e_n := u - u_n$ be the error corresponding to the Wavelet–Galerkin solution u_n of (5.1). Assuming $(I-G)^{-1}$ always exists and is bounded on $L^2(I)$, we get the following global convergence result for the problem (5.1).

Theorem 5.2. *Assume that $f \in C^r(I)$ and $K \in C^r(D \times \mathbb{R})$ such that the VIE in (5.1) possesses a unique solution $u \in C^r(I)$. Then for an orthogonal Coiffman wavelet system of degree r, the error e_n satisfies*

$$\|e_n\|_{L^2(I)} \leqslant C2^{-nr}.$$

Proof. By (5.1), we have $u = Gu$, so $P_n u = P_n G u$, and on the other hand

$$u_n = P_n(\mathcal{K}_n u_n + f_n) = P_n G_n u_n,$$

hence

$$P_n u - u_n = P_n G u - P_n G_n u_n = P_n(Gu - Gu_n) + P_n(Gu_n - G_n u_n).$$

Therefore

$$e_n + (P_n - I)u = P_n G e_n + P_n(G - G_n)u_n,$$

so

$$(I - P_n G)e_n = (I - P_n)u + P_n(G - G_n)u_n. \tag{5.5}$$

Clearly

$$I - P_n G = (I - G)\{I + (I - G)^{-1}(G - P_n G).\}$$

By the same discussion as in the previous theorem one can conclude that $(I - P_n G)^{-1}$ exists and is uniformly bounded for large enough n. Thus

$$\|e_n\|_{L^2(I)} \leqslant \|(I - P_n G)^{-1}\|_{L^2(I)} \Big(\|(I - P_n)u\|_{L^2(I)} + \|P_n(Gu_n - G_n u_n)\|_{L^2(I)} \Big).$$

Finally, by considering the boundedness of operator P_n and applying Theorems 2.2 and 2.1 the proof can be easily completed. □

6. Numerical results

In this section we consider a sample of numerical examples to show the efficiency of the Coiflet–Galerkin method in solving VIEs. We have used Coiflets of degree $r = 2$ with the support $[p, q] = [-3, 4]$ in V_n, with $n = 6$, to obtain the results in Tables 4–7. The connection coefficients $\Gamma_{k,l}(x)$ and $\Lambda_{i,j}^{k,l}$ are evaluated once and we used them in each example to construct the corresponding linear system (3.13). In the following examples we compare our method with the collocation-type methods presented in [4,5], which use Haar wavelets (in some references they are called rationalized Haar wavelets), and with a Galerkin method based on Daubechies wavelets [14]. We have considered the interval $[0, 1]$ and the L_∞ error norms obtained with the three methods are displayed in Tables 4–7, while Figs. 2–4 show the error graphs in the case of Coiflets. In Table 8 the L_2 norms corresponding to $n = 5$ and $n = 6$ are also shown and, in Table 9, the CPU times associated with each method are gathered.

Example 6.1. Consider the following VIE taken from [5]:

$$u(x) = \cos x + \int_0^x (t - x)\cos(t - x)u(t)dt,$$

where the exact solution is $u(x) = \frac{1}{3}(2\cos(\sqrt{3}x) + 1)$. The errors in the approximate solution obtained with the three different methods are shown in Table 4. We point out that the value $m = 64$ in [4] corresponds approximately to $n = 6$ in [5,14] and in our method.

Example 6.2. Consider the following VIE

$$u(x) = f(x) + \int_0^x \ln(xt + 1)\, u(t)\, dt,$$

where the exact solution is $u(x) = \sqrt{1 - 0.5x}$. An expression for $f(x)$ can be easily obtained e.g. with Maple or Mathematica:

Table 4
The $\|u(x_i) - u_n(x_i)\|_\infty$ for Example 6.1.

x	Haar wavelets	Daubechies wavelets	Coiflets
0	6.1 e−5	3.2 e−9	5.7 e−10
0.2	1.8 e−3	4.1 e−7	4.7 e−8
0.4	1.2 e−3	8.1 e−7	1.0 e−7
0.6	1.6 e−3	1.1 e−6	1.5 e−7
0.8	5.3 e−3	1.3 e−6	1.8 e−7
1	2.3 e−1	1.3 e−6	1.6 e−7

Table 5
The $\|u(x_i) - u_n(x_i)\|_\infty$ for Example 6.2.

x	Haar wavelets	Daubechies wavelets	Coiflets
0	1.9 e−3	1.3 e−6	6.5 e−7
0.2	1.2 e−3	2.2 e−8	4.5 e−7
0.4	4.4 e−4	3.1 e−8	2.3 e−7
0.6	4.7 e−4	4.2 e−8	1.9 e−8
0.8	1.5 e−3	6.2 e−8	1.6 e−7
1	7.1 e−1	9.4 e−8	3.4 e−7

Table 6
$\|u(x_i) - u_n(x_i)\|_\infty$ for Example 6.3(Eq. (6.1)).

x	Haar wavelets	Daubechies wavelets	Coiflets
0	4.9 e−2	1.3 e−2	8.7 e−3
0.2	5.8 e−3	3.4 e−3	1.4 e−3
0.4	5.1 e−2	3.5 e−3	1.5 e−3
0.6	5.8 e−2	3.8 e−3	1.7 e−3
0.8	2.5 e−2	4.3 e−3	1.9 e−3
1	2.5 e−4	1.1 e−2	2.4 e−2

Table 7
$\|u(x_i) - u_n(x_i)\|_\infty$ for Example 6.3 (Eq. (6.2)).

x	Haar wavelets	Daubechies wavelets	Coiflets
0	9.8 e−2	7.28 e−4	4.06 e−4
0.2	4.7 e−2	4.9 e−4	3.04 e−4
0.4	6.1 e−3	3.4 e−5	1.5 e−5
0.6	5.7 e−3	3.1 e−4	1.7 e−4
0.8	4.7 e−2	6.3 e−4	3.5 e−4
1	0	7.5 e−4	4.01 e−4

$$f(x) = \frac{\sqrt{8}}{3x^{\frac{3}{2}}}(2x+1)^{3/2}\left(\operatorname{arctanh}\left(\frac{\sqrt{2x-x^2}}{\sqrt{2x+1}}\right) - \operatorname{arctanh}\left(\frac{\sqrt{2x}}{\sqrt{2x+1}}\right)\right) - \frac{1}{18}\left(6(x-2)\sqrt{4-2x}\ln(1+x^2) + \frac{\sqrt{4-2x}}{x}(-4x^2+23x+12) - \frac{24}{x} - 64\right), \quad 0 < x \leqslant 1.$$

The value $f(0)$ is taken as $\lim_{x\to 0} f(x) = 1$. The numerical results for this example are gathered in Table 5.

In the following example we also test the capability of our procedure to solve VIEs of the first kind.

Example 6.3. We consider a VIE of the first kind from [4]:

$$\int_0^x \frac{\exp(x-t)}{1+x^2} u(t)dt = -\frac{4\pi\cos(4\pi x) + \sin(4\pi x) - 4\pi\exp(x)}{(1+x^2)(1+16\pi^2)}, \tag{6.1}$$

where the exact solution is $u(x) = \sin(4\pi x)$. We first attempted to treat this equation directly and the results obtained are gathered in Table 6, suggesting that it should be possible to use such an approach for solving first kind VIEs directly. We must point out that the choice of collocation points in [4] is not suitable for our case, since the linear system to be solved would be singular; we have chosen $x_i = \frac{i+0.5}{2^n}$ as collocation points to obtain a nonsingular system.

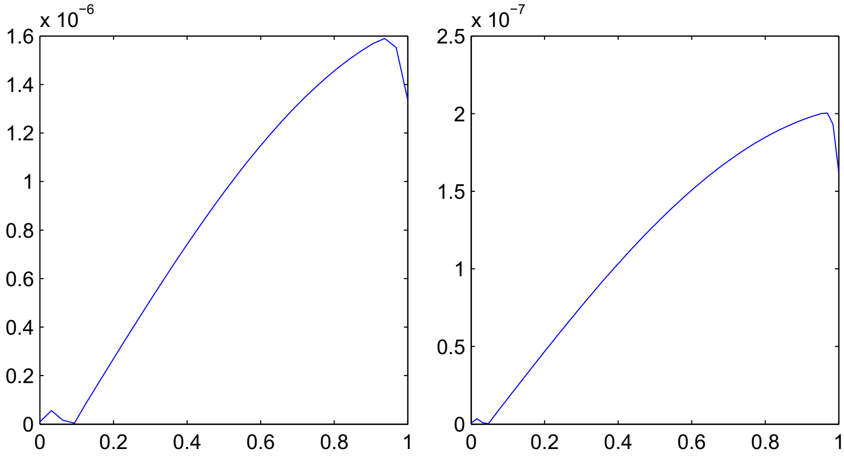

Fig. 2. Error graph (Coiflets) for Example 6.1; left $n = 5$, right $n = 6$.

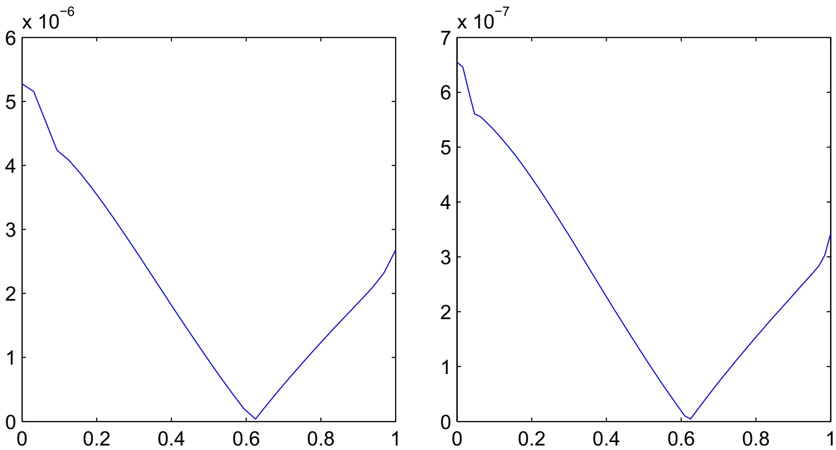

Fig. 3. Error graph (Coiflets) for Example 6.2; left $n = 5$, right $n = 6$.

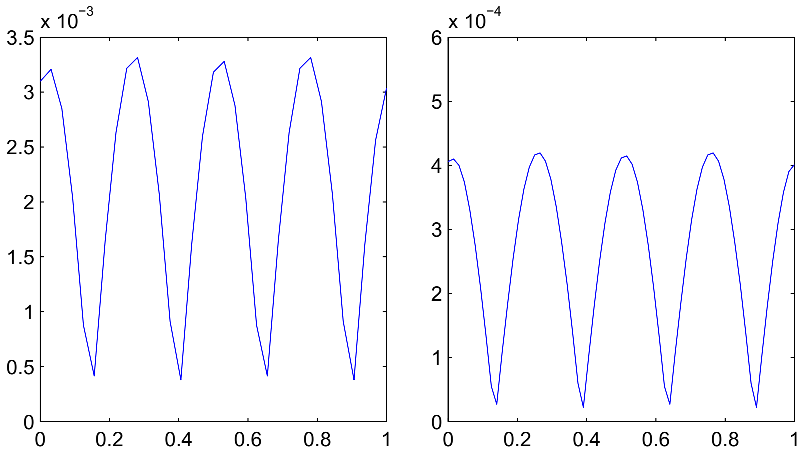

Fig. 4. Error graph (Coiflets) for Eq. (6.2), left $n = 5$; right $n = 6$.

Table 8
The L_2 error norms.

Example	n	Haar	Daubechies	Coiflets
6.1	5	8.9 e−3	1.0 e−5	1.3 e−6
	6	4.5 e−3	1.3 e−6	1.6 e−7
6.2	5	2.7 e−3	7.1 e−7	2.7 e−6
	6	1.4 e−3	9.4 e−8	3.4 e−7
6.3	5	9.9 e−2	5.8 e−3	3.1 e−3
	6	4.9 e−2	7.5 e−4	4.0 e−4

Table 9
CPU time (in seconds).

Example	n	Haar	Daubechies	Coiflets
6.1	5	14	18	7
	6	60	247	28
6.2	5	22	51	7
	6	78	252	27
6.3	5	14	36	6
	6	57	227	25

On the other hand, by using differentiation with respect to x on both sides, (6.1) can be converted into the following second kind VIE

$$u(x) = \bar{g}(x) - \int_0^x e^{x-t}u(t)dt, \tag{6.2}$$

$$\text{with} \quad \bar{g}(x) = \frac{-1}{1+16\pi^2}(-16\pi^2\sin(4\pi x) + 4\pi\cos(4\pi x) - 4\pi e^x)$$

The results obtained for this equation are displayed in Table 7.

In Table 8 the L_2 error norms for Examples 6.1,6.2,6.3 are displayed; in Example 6.3 we have considered the second kind formulation (6.2).

We see that the Haar wavelet method gave worse results in terms of the error norm L_∞ than the other methods; this was to be expected, since it is comparable to a piecewise constant collocation method. Moreover, in many of the examples carried out, the Haar method exhibited a great loss of accuracy near $t = 1$. In terms of CPU time it is faster than the Daubechies wavelet method but slower than the Coifman wavelet method (see Table 9). We now compare the efficiency of the two Wavelet–Galerkin methods. Both methods exhibit the same rate of convergence and, generally, we may say that the differences in the obtained error norms are not very significant.

In [14] a wavelet Galerkin method based on Daubechies wavelets was considered for the solution of linear VIEs. An advantage of using Coiflets instead of Daubechies is the simplicity of the formulas for approximating the given functions $f(x)$ and $K(x,t)$ (cf. (3.5) and (3.6), respectively). This leads to a faster evaluation of the components of system (3.12), due to the smaller number of operations involved. More specifically, examining (3.5) and (3.6), we see that for the Coiflets scheme one needs $(2^n - p + q - 1)$ and $(2^n - p + q - 1)^2$ values of f and K, respectively; for example, by considering $n = 6,\ p = -3,\ q = 4$ the above formulas give 70 values of f and 4900 values of K. In order to make a comparison with the Daubechies algorithm (cf. formulas (3.11) and (3.12) in [14]), we see that $\vartheta(2^n + L - 2)$ and $\vartheta^2(2^n + L - 2)^2$ values, respectively, of f and K, are needed (here ϑ denotes the order of some quadrature rule); for example, by taking $n = 6,\ L = 6,\ \vartheta = 10$ this results in 680 and 462400 evaluations to be performed.

In Table 9, the CPU time used for setting up and solving the linear system (3.12), in the various examples, is shown and compared with the CPU time used by the other methods. In the case of Example 6.3, the second kind VIE (6.2) was considered.

In order to compare the three different methods the same level n was considered. For each of the three wavelet systems, the associated set of functions $\{\phi_{n,i}(x), \quad i \in \mathbb{Z}\}$ is an orthonormal basis for the space V_n (generated by those functions). The $\phi_{n,i}$ can be written in terms of an appropriate scaling function ϕ, whose support is: $[p,q] = [-3,4]$ (Coiflets), $[0, L-1] = [0,5]$ (Daubechies) and $[0,1]$ (Haar). Moreover, since we are looking for an approximate solution in the interval $[0,1]$, we obtain the following representations for any any $u \in V_n$:

$$\text{Coiflets(cf. (3.5))}: \quad u(x) = \sum_{i=1-q}^{2^n-p-1} u_{n,i}\phi_{n,i}(x),$$

$$\text{Daubechies}: \quad u(x) = \sum_{i=2-L}^{2^n-1} u_{n,i}\phi_{n,i}(x),$$

$$\text{Haar}: \quad u(x) = \sum_{i=0}^{2^n-1} u_{n,i}\phi_{n,i}(x).$$

We note that in the three methods under consideration, the dimension of the linear systems to be solved is approximately the same for a fixed n. Indeed, we see from the above observations that, for a fixed n (say $n = 6$), the dimension of the linear system (3.12) is 70×70 ($2^n + q - p - 1 = 70$ where $[p, q] = [-3, 4]$) and the system corresponding to Daubechies wavelets, i.e., system (3.19) in [16], has dimension 68×68 ($2^n + L - 2 = 68$, where $L = 6$); in the case of the Haar wavelet bases the dimension of the corresponding linear system is 64×64 ($2^n = 64$).

7. Conclusions

A Wavelet–Galerkin method based on Coiflets has been proposed for the solution of linear VIEs with continuous kernels. Some new connection coefficients were encountered and suitable algorithms for their evaluation were presented. These coefficients can be stored and used for solving any linear VIE. The convergence properties of the Coiflet Wavelet–Galerkin solution have been examined and several examples were used to test the efficiency of this method when compared with the Daubechies Wavelet–Galerkin method and the Haar collocation procedure. The numerical results indicate that the quality of the approximate solutions obtained by the Coiflet Wavelet–Galerkin is, as expected, much superior to the one obtained with the Haar wavelets; in comparison with the Daubechies–Galerkin scheme solutions the Coiflet based method showed a marginal improvement in the error norms. However, due to the simplicity of its sampling approximation formulas, the Coiflet–Galerkin method is more efficient since it requires much less CPU time for setting up and solving the linear system of equations.

We think that comprehensive studies on the properties and applicability of wavelets based methods are needed. Our ongoing research includes the application of Coiflet–Galerkin method to other classes of Volterra integral equations, including singular kernels.

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