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SOME RESULTS ON THE POWER GRAPH OF GROUPS

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Abstract. The aim of this paper is to identify complete power graphs of groups and compute their clique number and show that power graphs are perfect. Moreover, automorphism of the power graph of cyclic groups is determined here.

1. Introduction

For a given group G, we may define the power graph $\mathcal{P}(G)$ as a graph with vertex set G, in which two distinct vertices x and y are adjacent if one of them is a power of the other one. Cameron and Ghosh [1] prove that finite abelian groups with isomorphic power graphs must be isomorphic. All of the standard notations come from [4].

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2. Complete graph and Automorphism group of power graph

In this section, we shall characterize all groups with complete and compute automorphism of the power graph of cyclic groups. Let us start with the following simple lemma.

Lemma 2.1. *Let G be a torsion group. If x and y are adjacent vertices in $\mathcal{P}(G)$, then either $|x|$ divides $|y|$ or $|y|$ divides $|x|$. The converse is true whenever G is a cyclic group.*

Remark 2.2. For a finite nontrivial group G, Chakrabarty et. all, proved that $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of prime power order ([2, Theorem 2.12]). In the next theorem, we investigate a result for infinite groups. It is a well-known result that a group is locally cyclic if and only if it is isomorphic to a subgroup of $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ (see [5]).

Theorem 2.3. *Let G be an infinite group. Then $\mathcal{P}(G)$ is complete if and only if $G \cong \mathbb{Z}_{p^\infty}$ for some prime p.*

For any graph Γ and $v \in V(\Gamma)$, the set of neighbours of the vertex v in Γ is denoted by $N_\Gamma(v)$, or briefly by $N(v)$. Furthermore, we define $N[v] = N(v) \cup \{v\}$.

Lemma 2.4. *If $x, y \in V(\mathcal{P}(\mathbb{Z}_n))$ and $|x| = |y|$, then $N[x] = N[y]$.*

Theorem 2.5. $\mathrm{Aut}(\mathcal{P}(\mathbb{Z}_n))$ *has a subgroup isomorphic to* $S_{\varphi(n)+1} \times \prod_{d|n, d\neq 1,n} S_{\varphi(d)}$.

Proof. Let $\{d_1, d_2, \ldots, d_r\}$ be the set of all divisors of n different from 1 and n. Let $A_0 = \{g \in \mathbb{Z}_n : |g| = n\} \cup \{\overline{0}\}$ and $A_i = \{g \in \mathbb{Z}_n : |g| = d_i\}$, for $i = 1, 2, \ldots, r$. Then by Lemma 2.1, the induced subgraphs of $\mathcal{P}(\mathbb{Z}_n)$ on A_i's are complete of order $\varphi(d_i)$, for $i = 1, 2 \ldots, r$ and induced subgraph of $\mathcal{P}(\mathbb{Z}_n)$ on A_0 is a complete graph of order $\varphi(n)+1$. Hence, by Lemma 2.4, every bijection $\alpha : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ such that $\alpha|_{A_i} : A_i \longrightarrow A_i$ is a permutation for $i = 0, 1, 2, \ldots, r$, is an automorphism of $\mathcal{P}(\mathbb{Z}_n)$. $\square$

Corollary 2.6. *Let n be a natural number such that for every $x, y \in \mathbb{Z}_n$, $\deg(x) \neq \deg(y)$ whenever $|x| \neq |y|$. Then*

$$\mathrm{Aut}(\mathcal{P}(\mathbb{Z}_n)) \cong S_{\varphi(n)+1} \times \prod_{d|n, d\neq 1,n} S_{\varphi(d)}.$$

As an example if $G = \mathbb{Z}_{12}$ and $x = \overline{2}$, $y = \overline{6}$ are elements of G. Then $|x| = 6$ and $|y| = 2$, and $deg(x) = deg(y) = 9$. However, there is no automorphism of $\mathcal{P}(\mathbb{Z}_n)$ which maps x to y. Therefore, we have $\mathrm{Aut}(\mathcal{P}(\mathbb{Z}_{12})) \cong S_{\varphi(12)+1} \times \prod_{d|12, d\neq 1,12} S_{\varphi(d)}$. The same happens for $\mathbb{Z}_{24}$.

Conjecture 2.7. For every natural number n,

$$\mathrm{Aut}(\mathcal{P}(\mathbb{Z}_n)) \cong S_{\varphi(n)+1} \times \prod_{d|n, d\neq 1,n} S_{\varphi(d)}.$$

3. Clique number

This section is devoted to a consideration of clique number and chromatic number of power graphs. It is clear that $\chi(\Gamma) \geq \omega(\Gamma)$ for every graph Γ and also a graph Γ is perfect if $\chi(\Gamma_1) = \omega(\Gamma_1)$ for all induced subgraphs Γ_1 of Γ. Due to the strong perfect graph theorem given by Chudnovsky et. al. in [3], a finite graph Γ is perfect if and only if neither Γ nor $\overline{\Gamma}$ contains an odd cycle of length at least 5 as an induced subgraph. Utilizing this fact, we can prove that the power graph of each group is perfect.

Theorem 3.1. *The power graph of a finite group is perfect.*

Theorem 3.2. *Let G be a finite group. Then*

$$\omega(\mathcal{P}(G)) = \max\{\omega(\mathcal{P}(\mathbb{Z}_n)) : n \in \pi_e(G)\},$$

where $\pi_e(G) = \{|x| : x \in G\}$.

Proof. Suppose that $C = \{x_1, \dots, x_m\} \subseteq V(\mathcal{P}(G))$ induces a complete subgraph. Clearly $\langle C\rangle$ is an abelian subgroup. Let $C_i = \{S_{p_i}(\langle x\rangle) : x \in C\}$, where $S_{p_i}(\langle x\rangle)$ is the Sylow p_i-subgroup of $\langle x\rangle$. Also, let $|S_{p_i}(\langle x_{k_i}\rangle)| = \max\{|S_{p_i}(\langle x\rangle)| : x \in C\}$. Since $\langle S_{p_i}(x_s), S_{p_i}(x_t)\rangle$ is cyclic for every $1 \leq s, t \leq m$, it is easy to see that $\langle C_i\rangle = S_{p_i}(\langle x_{k_i}\rangle)$ is a cyclic group. Therefore $\langle C\rangle = \langle C_1, \dots, C_m\rangle$ is a cyclic group and the result follows. $\square$

In what follows, we shall give a precise formula for the clique number of a finite cyclic group.

Lemma 3.3. *Let $n = p_1^{\lambda_1} \dots p_r^{\lambda_r}$ be a natural number and $m = \lambda_1 + \dots + \lambda_r$. Let $\mathcal{S}$ be the set of all $(m+1)$-tuples $(d_0, d_1, \dots, d_m)$ such that $n = d_0 > d_1 > \dots > d_m = 1$ is a chain of divisors of n and d_{i-1}/d_i is a prime for all $i = 1, \dots, m$. Let also $f : \mathcal{S} \longrightarrow \mathbb{N}$ be a map defined by $f(d_0, \dots, d_m) = \varphi(d_0) + \dots + \varphi(d_m)$. Then f takes its maximum value at $(d_0, \dots, d_m)$ if and only if $d_{i-1}/d_i \leq d_i/d_{i+1}$ for all $0 < i < m$, and it is unique with this property.*

Theorem 3.4. *Let $n = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_r^{\lambda_r}$ with $p_1 < p_2 < \cdots < p_r$. Then*

$$\begin{aligned}\omega(\mathcal{P}(\mathbb{Z}_n)) = \varphi(n) + \varphi\left(\frac{n}{p_1}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\lambda_1}}\right) + \varphi\left(\frac{n}{p_1^{\lambda_1} p_2}\right) \\ + \cdots + \varphi\left(\frac{n}{p_1^{\lambda_1} p_2^{\lambda_2}}\right) + \cdots + \varphi\left(\frac{n}{p_1^{\lambda_1} p_2^{\lambda_2} \dots p_r}\right) \\ + \cdots + \varphi\left(\frac{n}{p_1^{\lambda_1} p_2^{\lambda_2} \dots p_r^{\lambda_r - 1}}\right) + \varphi(1),\end{aligned}$$

where φ is the Eulerian function.

Proof. Let Y be a clique in $\mathcal{P}(\mathbb{Z}_n)$. We first observe that if $y \in V(Y)$, then $ry \in V(Y)$ whenever $\gcd(r, |y|) = 1$. Hence the elements of $V(Y)$ can be partitioned into sets each of which contains elements of the same order. Thus $V(Y) = Y_{h_1} \cup Y_{h_2} \cup \cdots \cup Y_{h_k}$, where Y_{h_i} possesses of all elements of order h_i and $|Y_{h_i}| = \varphi(h_i)$. By Lemma 2.1 and the fact that Y is a complete subgraph of $\mathcal{P}(\mathbb{Z}_n)$, it follows that for each $i, j \leq m$, either $h_i | h_j$ or $h_j | h_i$. Without loss of generality, we assume that $h_1 | \cdots | h_k$. On the other hand, for any chain of positive divisors $l_1, \dots, l_t$ of n such that $l_1 | \cdots | l_t$, we can find a clique of size $\sum_{i=1}^{t} \varphi(l_i)$ in $\mathcal{P}(\mathbb{Z}_n)$. Now, suppose that X is a maximal clique in $\mathcal{P}(\mathbb{Z}_n)$. Then $V(X) = X_{d_0} \cup X_{d_1} \cup \cdots \cup X_{d_m}$ ($d_0 = n$), where X_{d_i} possesses of all elements of order d_i, $|X_{d_i}| = \varphi(d_i)$ and $d_m | \cdots | d_0$. Since X is a maximal clique, then $|V(X)| = \sum_{i=0}^{m} \varphi(d_i) = \omega(\mathcal{P}(\mathbb{Z}_n))$. Therefore, by Lemma 3.3, $m = \lambda_1 + \cdots + \lambda_r$ and $d_i = \max\{h : h | d_{i-1}, h < d_{i-1}\}$. The proof is complete. □

References

1. P.J. Cameron and S. Ghosh, *The power graph of a finite group*, Discrete Math. 311 (2011), 1220–1222.
2. I. Chakrabarty, S. Ghosh and M. K. Sen, *Undirected power graphs of semigroups*, Semigroup Forum. 78 (2009), 410–426.
3. M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, *The strong perfect graph theorem*, Ann. Math. 164 (2006), 51–229.
4. Chris. Godsil and Gordon Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.
5. Schmidt. Roland, *Subgroup lattices of groups*, 2nd Edition, Walter de Gruyter, Berlin, 1994.