

S-duality invariant dilaton couplings at order α'^3

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ABSTRACT: The Riemann curvature correction to the type II supergravity at eight-derivative level is given schematically as $(t_8 t_8 + \frac{1}{8} \epsilon_{10} \epsilon_{10}) R^4$ at tree-level. The replacement of the generalized Riemann curvature in $t_8 t_8 R^4$, proposed by Gross and Sloan, produces various NS-NS couplings which are invariant under T-duality. Recently, using the combination of S-duality and T-duality transformations on these couplings, we have found groups of couplings which are invariant under the S-duality transformation. In this paper, we have examined the couplings involving the dilaton with direct scattering amplitude calculations of four NS-NS vertex operators in the superstring theory and found exact agreement.

The coupling $\epsilon_{10} \epsilon_{10} R^4$ is a total derivative term at four-field level. The σ -model beta function approach implies the presence of this term at the tree-level. By examining the sphere-level scattering amplitude of five gravitons, we have also confirmed the presence of this term in the tree-level effective action.

KEYWORDS: String Duality, Superstrings and Heterotic Strings

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1 Introduction

One of the nonperturbative symmetries in superstring theory is the S-duality of type IIB superstring theory [1–6]. It relates the type IIB theory at weak (strong) coupling to the type IIB at strong (weak) coupling. At low energy, this is the symmetry of type IIB supergravity which contains the following bosonic couplings in the Einstein frame [7]:

$$S \supset \frac{1}{2} \int d^{10}x \sqrt{-G} \left[R + \frac{1}{4} \text{Tr}(\mathcal{M}_{,\mu} \mathcal{M}_{,\mu}^{-1}) - \frac{1}{12} \mathcal{H}_{\mu\nu\rho}^T \mathcal{M} \mathcal{H}_{\mu\nu\rho} - \frac{1}{4} |\tilde{F}_{(5)}|^2 \right] - \frac{1}{8} \int C_{(4)} \mathcal{H}^T \mathcal{N} \mathcal{H} \quad (1.1)$$

where the five-form field strength is $\tilde{F}_{(5)} = dC_{(4)} + \frac{1}{2} \mathcal{B}^T \mathcal{N} \mathcal{H}$, and the self-duality condition $\tilde{F}_{(5)} = * \tilde{F}_{(5)}$ is imposed by hand.¹ The metric and the R-R four-form are invariant under the $\text{SL}(2, R)$ transformations. The B-field and the R-R two-form transform as doublet [8, 9], i.e.,

$$\mathcal{B} \equiv \begin{pmatrix} B \\ C_{(2)} \end{pmatrix} \rightarrow (\Lambda^{-1})^T \begin{pmatrix} B \\ C_{(2)} \end{pmatrix}; \quad \Lambda = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{SL}(2, R) \quad (1.2)$$

Since the parameters of the $\text{SL}(2, R)$ transformations are constant, their field strengths $\mathcal{H} = d\mathcal{B}$ also transform as doublet. The dilaton and the R-R scalar transform nonlinearly as $\tau \rightarrow \frac{p\tau+q}{r\tau+s}$ where the complex scalar field is defined as $\tau = C_{(0)} + ie^{-\phi}$. The matrix \mathcal{M} defined in terms of the dilaton and the R-R scalar, i.e.,

$$\mathcal{M} = e^\phi \begin{pmatrix} |\tau|^2 & C_{(0)} \\ C_{(0)} & 1 \end{pmatrix} \quad (1.3)$$

then transforms as [10]

$$\mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T \quad (1.4)$$

¹We use only subscript indices and the repeated indices are contracted with the inverse of metric. Our conventions also set the gravitational coupling constant $\kappa = 1$.

The derivatives of the matrix \mathcal{M} also transform as (1.4). The matrix \mathcal{N} which is defined as

$$\mathcal{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.5}$$

can be written as $\Lambda\mathcal{N}\Lambda^T$. Using these matrices and the transformation (1.2), one observes that the supergravity action (1.1) is invariant under the $SL(2, R)$ transformations. The stringy behaviors of the superstring theory which are encoded in both higher derivative corrections and the genus corrections, however, are not captured by the above classical action. Moreover, the $SL(2, R)$ transformations of the classical theory are broken to the $SL(2, Z)$ transformations. Therefore, the stringy corrections must be invariant under the $SL(2, Z)$ transformations.

The higher derivative corrections to the supergravity (1.1) start at the eight-derivative level, and were first found from the sphere-level four-graviton scattering amplitude [11, 12] as well as from the σ -model beta function approach [13, 14]. The result in the Einstein frame is

$$S \supset \frac{\gamma\zeta(3)}{3.2^7} \int d^{10}x e^{-3\phi/2} \sqrt{-G} \left(t_8 t_8 R^4 + \frac{1}{8} \epsilon_{10} \epsilon_{10} R^4 \right) \tag{1.6}$$

where $\gamma = \frac{\alpha'^3}{2^5}$ and we have used the normalization of the tensor t_8 as in (3.5). The couplings given by $t_8 t_8 R^4$ have nonzero contribution at four-graviton level, so they were found from the sphere-level S-matrix element of four graviton vertex operators [11, 12], whereas the couplings given by $\epsilon_{10} \epsilon_{10} R^4$ have nonzero contribution at five-graviton level [15] which have not yet been confirmed by the sphere-level S-matrix element of five graviton vertex operators. However, the presence of this term in the tree-level effective action was dictated by the σ -model beta function approach [13, 14]. In this paper, among other things, we will show that the sphere-level scattering amplitude of five gravitons confirms the presence of $\epsilon_{10} \epsilon_{10} R^4$ in the tree-level effective action.

Unlike the two-derivative level supergravity action (1.1) which is invariant under the S-duality without adding loop or nonperturbative effects, the presence of the dilaton factor in the eight-derivative level indicates that the action (1.6) needs loop and nonperturbative corrections to become S-duality invariant. The $SL(2, Z)$ invariant form of the action (1.6) has been conjectured in [16]–[33] to be

$$S \supset \frac{\gamma}{3.2^8} \int d^{10}x E_{(3/2)}(\tau, \bar{\tau}) \sqrt{-G} \left(t_8 t_8 R^4 + \frac{1}{8} \epsilon_{10} \epsilon_{10} R^4 \right) \tag{1.7}$$

where $E_{(3/2)}(\tau, \bar{\tau})$ is the $SL(2, Z)$ invariant non-holomorphic Eisenstein series which has the following weak-expansion [16]:

$$E_{(3/2)}(\tau, \bar{\tau}) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + 8\pi\tau_2^{1/2} \sum_{m \neq 0, n \geq 1} \left| \frac{m}{n} \right| K_1(2\pi |mn| \tau_2) e^{2\pi i mn \tau_1} \tag{1.8}$$

where $\tau = \tau_1 + i\tau_2$ and K_1 is the Bessel function. The Eisenstein series which appear in various 16 fermion couplings are completely determined by supersymmetry and the structure of the moduli space [22]. Linear supersymmetry transformations then relates

those couplings to the R^4 couplings [22]. The above expansion shows that there are no perturbative corrections beyond the tree level and one-loop level, but there are an infinite number of D-instanton corrections. By explicit calculation, it has been shown in [27] that there is no two-loop correction to the action (1.7). The coupling $\epsilon_{10}\epsilon_{10}R^4$ at one-loop level has been confirmed in [34, 35] by explicit calculation of torus-level S-matrix element of five graviton vertex operators.

The B-field and dilaton couplings have been added to $t_8t_8R^4$ by extending the Riemann curvature to the generalized Riemann curvature [36]²

$$\bar{R}_{ab}{}^{cd} = R_{ab}{}^{cd} - \eta_{[a}{}^{[c}\phi_{,b]}{}^{d]} + e^{-\phi/2}H_{ab}{}^{[c,d]} \quad (1.9)$$

where the bracket notation is $H_{ab}{}^{[c,d]} = \frac{1}{2}(H_{ab}{}^{c,d} - H_{ab}{}^{d,c})$, and comma denotes the partial derivative. The resulting couplings are then invariant under T-duality [35, 37, 48]. Imposing the invariance of these couplings under the combination of S-duality and linear T-duality, some new S-duality invariant actions at eight-derivative level have been found in [38]. The S-duality invariant action which includes, among other things, the couplings of two gravitons and two dilatons is [38]

$$S \supset \frac{\gamma}{2} \int d^{10}x E_{(3/2)}(\tau, \bar{\tau}) \sqrt{-G} \left[-\frac{1}{16} \text{Tr}[\mathcal{M}_{,qr}\mathcal{M}_{,qr}^{-1}] R_{hkmn}^2 - \frac{1}{2} \text{Tr}[\mathcal{M}_{,ph}\mathcal{M}_{,pr}^{-1}] R_{hkmn} R_{mnkr} \right. \\ \left. - \frac{1}{2} \text{Tr}[\mathcal{M}_{,mr}\mathcal{M}_{,ph}^{-1}] R_{hkmn} R_{npkr} - \frac{1}{2} \text{Tr}[\mathcal{M}_{,mh}\mathcal{M}_{,pr}^{-1}] R_{hkmn} R_{npkr} \right] \quad (1.10)$$

The S-duality invariant action which includes, among other things, the couplings of two B-fields and two dilatons is [38]³

$$S \supset \frac{\gamma}{8} \int d^{10}x E_{(3/2)}(\tau, \bar{\tau}) \sqrt{-G} \left[\frac{1}{6} \text{Tr}[\mathcal{M}_{,nh}\mathcal{M}_{,nk}^{-1}] \mathcal{H}_{mpr,h}^T \mathcal{M} \mathcal{H}_{mpr,k} \right. \\ \left. - \frac{1}{2} \text{Tr}[\mathcal{M}_{,nh}\mathcal{M}_{,km}^{-1}] \mathcal{H}_{mpr,k}^T \mathcal{M} \mathcal{H}_{npr,h} - \frac{1}{2} \text{Tr}[\mathcal{M}_{,hm}\mathcal{M}_{,kn}^{-1}] \mathcal{H}_{mpr,k}^T \mathcal{M} \mathcal{H}_{npr,h} \right] \quad (1.11)$$

The S-duality invariant action which includes, among other things, the couplings of one dilaton, one graviton and two B-fields is [38]

$$S \supset \frac{\gamma}{8} \int d^{10}x E_{(3/2)}(\tau, \bar{\tau}) \sqrt{-G} \left[4\mathcal{H}_{hqr,n}^T \mathcal{M}_{,rm} \mathcal{H}_{knp,h} R_{mpkq} - 4\mathcal{H}_{npr,h}^T \mathcal{M}_{,mk} \mathcal{H}_{hnq,r} R_{kqmp} \right. \\ \left. - 4\mathcal{H}_{npr,m}^T \mathcal{M}_{,qh} \mathcal{H}_{mpr,k} R_{krhn} + 4\mathcal{H}_{npr,h}^T \mathcal{M}_{,qm} \mathcal{H}_{knr,h} R_{mrkp} - 2\mathcal{H}_{npr,h}^T \mathcal{M}_{,mh} \mathcal{H}_{npr,k} R_{mrkq} \right. \\ \left. + 2\mathcal{H}_{mnq,h}^T \mathcal{M}_{,rk} \mathcal{H}_{npq,k} R_{prhm} - 2\mathcal{H}_{mnp,k}^T \mathcal{M}_{,rm} \mathcal{H}_{npq,h} R_{qrhk} \right] \quad (1.12)$$

The S-duality invariant action containing four dilatons can also be found from the couplings of four H s in $t_8t_8\bar{R}^4$. The $\text{SL}(2, R)$ invariant form of the coupling $t_8t_8(\partial H)^4$

²Note that the normalizations of the dilation and B-field here are $\sqrt{2}$ and 2 times the normalization of the dilaton and B-field in [36], respectively.

³Note that the normalization of the Eisenstein series and the normalization of the form field strengths here are different from the corresponding terms in [38], i.e., $E_{(3/2)}|_{\text{here}} = 2\zeta(3)E_{3/2}|_{\text{there}}$, $H|_{\text{here}} = dB$ and $H|_{\text{there}} = \frac{1}{2}dB$, $F|_{\text{here}} = dC$ and $F|_{\text{there}} = \frac{1}{2}dC$.

requires the coupling $t_8 t_8 (\partial F)^4$ for the R-R three-form field strengths. Then following the linear T-duality prescription given in [38], one can find the couplings of four R-R one-form field strengths which is $3 \cdot 2^5 (F_{n,m} F_{n,m})^2$. Then using the following expression:

$$\text{Tr}[\mathcal{M}_{,hk} \mathcal{M}_{,mn}^{-1}] = -2e^{2\phi} F_{h,k} F_{m,n} - 2\phi_{,hk} \phi_{,mn} \quad (1.13)$$

one may find the S-duality invariant coupling

$$S \supset \frac{\gamma}{32} \int d^{10}x E_{(3/2)}(\tau, \bar{\tau}) \sqrt{-G} \left[a (\text{Tr}[\mathcal{M}_{,nm} \mathcal{M}_{,nm}^{-1}])^2 + b \text{Tr}[\mathcal{M}_{,nm} \mathcal{M}_{,hk}^{-1}] \text{Tr}[\mathcal{M}_{,hk} \mathcal{M}_{,nm}^{-1}] \right] \quad (1.14)$$

where the constants a and b satisfy $a + b = 1$. The combination of the dilaton couplings in (1.14) and (1.10), and the four graviton couplings in (1.7) should give the coupling $t_8 t_8 \hat{R}^4$ where \hat{R} is the first two terms on the right hand side of (1.9). In the string frame, the dilaton term in (1.9) disappears [37], so the Einstein frame coupling $t_8 t_8 \hat{R}^4$ transforms to $t_8 t_8 R^4$ in the string frame. Hence, there is no dilaton coupling in the string frame.

In this paper, we are going to confirm the actions (1.10), (1.11), (1.12) and (1.14) by comparing the four NS-NS couplings in them with the direct calculation of the S-matrix element of four NS-NS vertex operators in the type IIB superstring theory.

The outline of the paper is as follows: we begin in section 2 by reviewing the tree level and one-loop level scattering amplitude of four NS-NS vertex operators in the type IIB superstring theory. In subsection 2.1, we show that the couplings of two dilatons and two gravitons extracted from the S-duality invariant action (1.10) are exactly reproduced by the corresponding superstring theory S-matrix element. In subsections 2.2, 2.3, and 2.4, we show that the dilaton couplings in actions (1.14), (1.11), and (1.12), respectively, are reproduced by the appropriate S-matrix elements in the superstring theory. In section 3, using the observation that the sphere-level S-matrix element in RNS formalism must contain terms which are proportional to at least two Mandelstam variables [45, 47], we show that the massless poles and the contact terms of five-graviton amplitude in the field theory calculated from the Hilbert-Einstein term in (1.1) and from the couplings $t_8 t_8 R^4 + \frac{\alpha}{2} \epsilon_{10} \epsilon_{10} R^4$, satisfy this condition when the constant $\alpha = \frac{1}{4}$. This confirms the tree-level action (1.6).

2 Four-point amplitude

The sphere-level scattering amplitude of four massless NS-NS vertex operators with polarization tensors ε^{ab} in covariant formalism is given by [11]

$$\mathcal{A} = -i \frac{\gamma}{2} e^{-2\phi} \frac{\Gamma(-s/8) \Gamma(-t/8) \Gamma(-u/8)}{\Gamma(1+s/8) \Gamma(1+t/8) \Gamma(1+u/8)} \varepsilon_1^{a_1 b_1} \varepsilon_2^{a_2 b_2} \varepsilon_3^{a_3 b_3} \varepsilon_4^{a_4 b_4} K_{a_1 a_2 a_3 a_4} K_{b_1 b_2 b_3 b_4} \quad (2.1)$$

There is also a factor of delta function $\delta^{10}(k_1 + k_2 + k_3 + k_4)$ imposing conservation of momentum. The Mandelstam variables $s = -4\alpha' k_1 \cdot k_2$, $t = -4\alpha' k_1 \cdot k_3$, $u = -4\alpha' k_2 \cdot k_3$

satisfy $s + t + u = 0$, and

$$\begin{aligned}
 K_{a_1 a_2 a_3 a_4} = 4 [& -k_2 \cdot k_1 k_3 \cdot k_1 \eta_{a_1 a_4} \eta_{a_2 a_3} - k_2 \cdot k_1 k_2 \cdot k_3 \eta_{a_1 a_3} \eta_{a_2 a_4} - k_2 \cdot k_3 k_3 \cdot k_1 \eta_{a_1 a_2} \eta_{a_3 a_4} \\
 & + k_2 \cdot k_1 \eta_{a_1 a_4} (k_1)_{a_2} (k_1)_{a_3} + k_3 \cdot k_1 \eta_{a_1 a_4} (k_1)_{a_2} (k_1)_{a_3} - k_2 \cdot k_1 \eta_{a_2 a_4} (k_1)_{a_3} (k_2)_{a_1} \\
 & - k_3 \cdot k_1 \eta_{a_2 a_4} (k_1)_{a_3} (k_2)_{a_1} + k_3 \cdot k_1 \eta_{a_1 a_4} (k_1)_{a_2} (k_2)_{a_3} - k_3 \cdot k_1 \eta_{a_2 a_4} (k_2)_{a_1} (k_2)_{a_3} \\
 & + k_2 \cdot k_1 \eta_{a_1 a_2} (k_1)_{a_3} (k_2)_{a_4} + k_3 \cdot k_1 \eta_{a_1 a_2} (k_1)_{a_3} (k_2)_{a_4} + k_3 \cdot k_1 \eta_{a_1 a_2} (k_2)_{a_3} (k_2)_{a_4} \\
 & - k_2 \cdot k_1 \eta_{a_3 a_4} (k_1)_{a_2} (k_3)_{a_1} - k_3 \cdot k_1 \eta_{a_3 a_4} (k_1)_{a_2} (k_3)_{a_1} + k_2 \cdot k_1 \eta_{a_2 a_4} (k_2)_{a_3} (k_3)_{a_1} \\
 & - k_2 \cdot k_1 \eta_{a_2 a_3} (k_2)_{a_4} (k_3)_{a_1} + k_2 \cdot k_1 \eta_{a_1 a_4} (k_1)_{a_3} (k_3)_{a_2} + k_3 \cdot k_1 \eta_{a_3 a_4} (k_2)_{a_1} (k_3)_{a_2} \\
 & + k_2 \cdot k_1 \eta_{a_1 a_3} (k_2)_{a_4} (k_3)_{a_2} - k_2 \cdot k_1 \eta_{a_3 a_4} (k_3)_{a_1} (k_3)_{a_2} + k_2 \cdot k_1 \eta_{a_1 a_3} (k_1)_{a_2} (k_3)_{a_4} \\
 & + k_3 \cdot k_1 \eta_{a_1 a_3} (k_1)_{a_2} (k_3)_{a_4} - k_3 \cdot k_1 \eta_{a_2 a_3} (k_2)_{a_1} (k_3)_{a_4} + k_3 \cdot k_1 \eta_{a_1 a_2} (k_2)_{a_3} (k_3)_{a_4} \\
 & + k_2 \cdot k_1 \eta_{a_1 a_3} (k_3)_{a_2} (k_3)_{a_4}] \tag{2.2}
 \end{aligned}$$

The on-shell conditions are $k_i^2 = k_i \cdot \varepsilon_i = \varepsilon_i \cdot k_i = 0$. The polarization tensor is symmetric and traceless for graviton, antisymmetric for B-field and for dilaton it is

$$\varepsilon^{ab} = \frac{\phi}{\sqrt{8}} (\eta^{ab} - k^a \ell^b - k^b \ell^a) \tag{2.3}$$

where ℓ^a is an auxiliary vector which satisfies $k \cdot \ell = 1$ and ϕ is the dilaton polarization which is one. In equation (2.2) we have used the conservation of momentum to write the amplitude in terms of momentum k_1, k_2, k_3 . We have also used the on-shell conditions to rewrite $k_1 \cdot \varepsilon_4 = -k_2 \cdot \varepsilon_4 - k_3 \cdot \varepsilon_4$, similarly for $\varepsilon_4 \cdot k_1$. We have normalized the amplitude (2.1) to be consistent with the normalization factor in the couplings (1.6). The factor of i corresponds to that of the analogous field theory amplitudes calculated in Minkowski space.

The tree-level coupling $t_8 t_8 R^4$ has been found in [12] from the amplitude (2.1) by expanding the gamma functions at low energy

$$\frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)} = -\frac{2^9}{stu} - 2\zeta(3) + \dots \tag{2.4}$$

The first term corresponds to the massless poles in the four-point function which are reproduced by the Einstein-Hilbert action [39], and the second term,

$$\Delta \mathcal{A} = i\gamma\zeta(3) e^{-2\phi} \varepsilon_1^{a_1 b_1} \varepsilon_2^{a_2 b_2} \varepsilon_3^{a_3 b_3} \varepsilon_4^{a_4 b_4} K_{a_1 a_2 a_3 a_4} K_{b_1 b_2 b_3 b_4} \tag{2.5}$$

is the tree-level α'^3 -correction corresponding to four NS-NS states.

The torus-level scattering amplitude of four massless NS-NS vertex operators has been calculated in [40], i.e.,

$$\mathcal{A}^{1\text{-loop}} = i\gamma\pi I(s, t, u) \varepsilon_1^{a_1 b_1} \varepsilon_2^{a_2 b_2} \varepsilon_3^{a_3 b_3} \varepsilon_4^{a_4 b_4} K_{a_1 a_2 a_3 a_4} K_{b_1 b_2 b_3 b_4} \tag{2.6}$$

where $I(s, t, u)$ is a function which includes massive poles, massive double poles and threshold branch cuts [41–43]. It has the following low energy expansion [34, 44]:

$$I(s, t, u) = \frac{\pi}{3} + O(\alpha') \tag{2.7}$$

which results the following one-loop correction at order α'^3 :

$$\Delta\mathcal{A}^{1\text{-loop}} = 2i\gamma\zeta(2)\varepsilon_1^{a_1b_1}\varepsilon_2^{a_2b_2}\varepsilon_3^{a_3b_3}\varepsilon_4^{a_4b_4}K_{a_1a_2a_3a_4}K_{b_1b_2b_3b_4} \quad (2.8)$$

The amplitudes (2.5) and (2.8) are in the string frame, i.e., $G_{ab}^s = e^{\phi/2}G_{ab}$. Using the fact that $\sqrt{-G}t_8t_8R^4$ in (1.7) transforms to $e^{-\phi/2}\sqrt{-G^s}t_8t_8R^4$ in the string frame, one observes that the dilaton factors as well as the factors $\zeta(3)$ and $\zeta(2)$ in the Eisenstein series in (1.7) are consistent with the amplitudes (2.5) and (2.8). As a result, if the dilaton actions (1.10), (1.11), (1.12) and (1.14) at tree-level are consistent with sphere-level scattering amplitude (2.5), then these couplings at one-loop level would be consistent with the torus-level amplitude (2.8). So in the following subsections, we compare these actions with the amplitude (2.5).

2.1 Two-dilaton-two-graviton amplitude

It has been shown in [45] that the amplitude (2.5) is zero for one dilaton and three gravitons, and for three dilatons and one graviton. They are consistent with the fact that it is impossible to construct $SL(2, R)$ invariant couplings which have such components. However, the amplitude (2.5) is non-zero for two dilatons and two gravitons. The result is [45]

$$\begin{aligned} \Delta\mathcal{A} = i\frac{\gamma\zeta(3)}{2}e^{-2\phi} & [16(k_2.k_3)^2(k_3.k_1)^2\text{Tr}[\varepsilon_3.\varepsilon_4] + 16(k_2.k_1)^2k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2 \\ & + 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2 + 16(k_3.k_1)^2k_1.\varepsilon_3.k_1k_2.\varepsilon_4.k_2 \\ & + 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_2 + 32(k_3.k_1)^2k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_2 \\ & + 16(k_3.k_1)^2k_2.\varepsilon_3.k_2k_2.\varepsilon_4.k_2 + 32k_2.k_1k_3.k_1k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3 \\ & + 32(k_3.k_1)^2k_1.\varepsilon_3.k_2k_2.\varepsilon_4.k_3 + 32(k_3.k_1)^2k_2.\varepsilon_3.k_2k_2.\varepsilon_4.k_3 \\ & + 16(k_3.k_1)^2k_2.\varepsilon_3.k_2k_3.\varepsilon_4.k_3 + 32(k_2.k_1)^2k_3.k_1k_1.\varepsilon_3.\varepsilon_4.k_2 \\ & + 64k_2.k_1(k_3.k_1)^2k_1.\varepsilon_3.\varepsilon_4.k_2 + 32(k_3.k_1)^3k_1.\varepsilon_3.\varepsilon_4.k_2 \\ & + 32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_2 + 32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_2 \\ & + 32k_2.k_1(k_3.k_1)^2k_2.\varepsilon_3.\varepsilon_4.k_3 + 32(k_3.k_1)^3k_2.\varepsilon_3.\varepsilon_4.k_3] \phi_1\phi_2 \quad (2.9) \end{aligned}$$

where ϕ_1 and ϕ_2 are the polarizations of the dilatons which are one. However, for clarity we keep them. We have also divided the amplitude in [45] by 2 because the normalization of the dilaton in section 1 is $\sqrt{2}$ times the normalization of the dilaton vertex operator.

Now we have to compare the above S-matrix element with the corresponding couplings in (1.10). Using the expression (1.13), one finds the following couplings of two dilatons and two Riemann curvatures at tree-level in the Einstein frame:

$$\begin{aligned} S \supset \frac{\gamma\zeta(3)}{2} \int d^{10}x \sqrt{-G} e^{-3\phi/2} & \left[\frac{1}{4}\phi_{,qr}^2 R_{hkmn}^2 + 2\phi_{,ph}\phi_{,pr} R_{hkmn} R_{mnkr} \quad (2.10) \right. \\ & \left. + 2\phi_{,mr}\phi_{,ph} R_{hkmn} R_{npkr} + 2\phi_{,mh}\phi_{,pr} R_{hkmn} R_{npkr} \right] \end{aligned}$$

Using the perturbation $G_{ab} = \eta_{ab} + 2h_{ab}$ where h_{ab} is the graviton, the Riemann curvature at the linear order of the graviton becomes

$$R_{abcd} = h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac} \quad (2.11)$$

Transforming the couplings (2.10) to the string frame, one finds the overall dilaton factor to be $e^{-2\phi}$ as in the string amplitude (2.9). Using conservation of momentum and on-shell relations to write the momentum space couplings in terms of the momentum k_1, k_2, k_3 and to rewrite $k_1 \cdot \epsilon_4 = -k_2 \cdot \epsilon_4 - k_3 \cdot \epsilon_4$, we have checked explicitly that the couplings of two dilatons and two gravitons in (2.10) are exactly the same as the couplings in (2.9).

2.2 Four-dilaton amplitude

The amplitude (2.5) for four dilatons has been found in [45] to be

$$\Delta\mathcal{A} = i \frac{\gamma\zeta(3)}{4} e^{-2\phi} [16((k_2 \cdot k_1)^2 + k_2 \cdot k_1 k_3 \cdot k_1 + (k_3 \cdot k_1)^2)^2] \phi_1 \phi_2 \phi_3 \phi_4 \quad (2.12)$$

where we have also divided the amplitude in [45] by 4 because the normalization of the dilaton in section 1 is $\sqrt{2}$ times the normalization of the dilaton vertex operator.

On the other hand, using the expression (1.13), one finds the following coupling of four dilatons at tree-level in (1.14):

$$S \supset \frac{\gamma\zeta(3)}{4} \int d^{10}x e^{-3\phi/2} \sqrt{-G} [\phi_{,nh} \phi_{,nh}]^2 \quad (2.13)$$

In the string frame, one finds the overall dilaton factor to be $e^{-2\phi}$ as in the string amplitude. Transforming the above coupling to the momentum space, i.e., labeling the dilatons by the particle labels 1,2,3,4 and adding 24 permutations, one finds exactly the momentum space couplings (2.12) after using conservation of momentum and the on-shell relation $k_i \cdot k_i = 0$. This confirms the constants a and b in (1.14) to satisfy $a = 1 - b$. The constant b , on the other hand, can be fixed by comparing the couplings of two dilatons and two R-R one-form field strengths in (1.14) with the corresponding scattering amplitude in string theory [46].

2.3 Two-dilaton-two-B-field amplitude

The scattering amplitude (2.5) for two dilatons and two B-fields has been found in [45] to be

$$\begin{aligned} \Delta\mathcal{A} = i \frac{\gamma\zeta(3)}{8} e^{-2\phi} & \left[2k_3 \cdot k_1 (2(k_1 \cdot \epsilon_3 \cdot \epsilon_4 \cdot k_2 + k_2 \cdot \epsilon_3 \cdot \epsilon_4 \cdot k_2 + k_2 \cdot \epsilon_3 \cdot \epsilon_4 \cdot k_3) + k_3 \cdot k_1 \text{Tr}[\epsilon_3 \cdot \epsilon_4]) \right. \\ & \left. + (16k_3 \cdot k_1 k_3 \cdot k_2 - 8(k_2 \cdot k_1)^2) (-2k_1 \cdot \epsilon_3 \cdot k_2 k_2 \cdot \epsilon_4 \cdot k_3 + k_2 \cdot k_1 (2k_1 \cdot \epsilon_3 \cdot \epsilon_4 \cdot k_2 + k_3 \cdot k_1 \text{Tr}[\epsilon_3 \cdot \epsilon_4])) \right] \phi_1 \phi_2 \end{aligned} \quad (2.14)$$

where we have also divided the amplitude in [45] by 8 because the normalization of the B-field in section 1 is twice the normalization of the B-field vertex operator, and the normalization of the dilaton in section 1 is $\sqrt{2}$ times the normalization of the dilaton vertex operator.

The above amplitude should be compared with the corresponding couplings in (1.11). Using the expression (1.13), and

$$\mathcal{H}^T \mathcal{M} \mathcal{H} = e^{-\phi} H H + \dots \quad (2.15)$$

where dots refer to the other terms which involve the R-R fields, one finds the following couplings of two dilatons and two B-fields:

$$\begin{aligned}
 S \supset \frac{\gamma\zeta(3)}{8} \int d^{10}x \sqrt{-G} e^{-5\phi/2} \left[-\frac{2}{3} \phi_{,nh} \phi_{,nk} H_{mpr,h} H_{mpr,k} + 2\phi_{,hn} \phi_{,km} H_{mpr,k} H_{npr,h} \right. \\
 \left. + 2\phi_{,hm} \phi_{,kn} H_{mpr,k} H_{npr,h} \right] \quad (2.16)
 \end{aligned}$$

The B-field strength is $H_{abc} = B_{ab,c} + B_{ca,b} + B_{bc,a}$. In the string frame, one again finds the overall dilaton factor to be $e^{-2\phi}$. Using conservation of momentum and on-shell relations, we have checked explicitly that the momentum space couplings of two dilatons and two B-fields in above equation are exactly the same as the couplings in (2.14).

2.4 One-dilaton-one-graviton-two-B-field amplitude

The scattering amplitude (2.5) for one dilaton, one graviton and two B-fields can be calculated by inserting one of the polarization tensors by (2.3), one by symmetric and traceless polarization and the other two by antisymmetric polarization tensors. We have found that the auxiliary vector ℓ is cancel in the amplitude. The result is

$$\begin{aligned}
 \Delta\mathcal{A} = i \frac{\gamma\zeta(3)}{8} e^{-2\phi} \left[-32(k_2.k_1)^2 k_2.\epsilon_4.k_3 k_1.\epsilon_2.\epsilon_3.k_1 - 32k_2.k_1 k_3.k_1 k_2.\epsilon_4.k_3 k_1.\epsilon_2.\epsilon_3.k_1 \right. \\
 - 32k_2.k_1 k_3.k_1 k_2.\epsilon_4.k_3 k_1.\epsilon_2.\epsilon_3.k_2 - 32(k_2.k_1)^2 k_1.\epsilon_3.k_2 k_1.\epsilon_2.\epsilon_4.k_3 \\
 - 32k_2.k_1 k_3.k_1 k_1.\epsilon_3.k_2 k_1.\epsilon_2.\epsilon_4.k_3 + 32(k_2.k_1)^2 k_1.\epsilon_2.k_3 k_1.\epsilon_3.\epsilon_4.k_2 \\
 + 32k_2.k_1 k_3.k_1 k_1.\epsilon_2.k_3 k_1.\epsilon_3.\epsilon_4.k_2 + 32(k_2.k_1)^2 k_3.\epsilon_2.k_3 k_1.\epsilon_3.\epsilon_4.k_2 \\
 + 32(k_2.k_1)^2 k_1.\epsilon_2.k_1 k_1.\epsilon_3.\epsilon_4.k_3 + 64k_2.k_1 k_3.k_1 k_1.\epsilon_2.k_1 k_1.\epsilon_3.\epsilon_4.k_3 \\
 + 32(k_3.k_1)^2 k_1.\epsilon_2.k_1 k_1.\epsilon_3.\epsilon_4.k_3 + 64(k_2.k_1)^2 k_1.\epsilon_2.k_3 k_1.\epsilon_3.\epsilon_4.k_3 \\
 + 64k_2.k_1 k_3.k_1 k_1.\epsilon_2.k_3 k_1.\epsilon_3.\epsilon_4.k_3 + 32(k_2.k_1)^2 k_3.\epsilon_2.k_3 k_1.\epsilon_3.\epsilon_4.k_3 \\
 + 32k_2.k_1 k_3.k_1 k_1.\epsilon_2.k_3 k_2.\epsilon_3.\epsilon_4.k_2 + 32k_2.k_1 k_3.k_1 k_1.\epsilon_2.k_1 k_2.\epsilon_3.\epsilon_4.k_3 \\
 + 32(k_3.k_1)^2 k_1.\epsilon_2.k_1 k_2.\epsilon_3.\epsilon_4.k_3 + 32k_2.k_1 k_3.k_1 k_1.\epsilon_2.k_3 k_2.\epsilon_3.\epsilon_4.k_3 \\
 - 32(k_2.k_1)^2 k_2.\epsilon_4.k_3 k_3.\epsilon_2.\epsilon_3.k_1 - 32(k_2.k_1)^2 k_1.\epsilon_3.k_2 k_3.\epsilon_2.\epsilon_4.k_2 \\
 - 32(k_2.k_1)^2 k_1.\epsilon_3.k_2 k_3.\epsilon_2.\epsilon_4.k_3 - 32(k_2.k_1)^2 k_3.k_1 k_1.\epsilon_2.\epsilon_3.\epsilon_4.k_3 \\
 - 32k_2.k_1 (k_3.k_1)^2 k_1.\epsilon_2.\epsilon_3.\epsilon_4.k_3 + 32(k_2.k_1)^3 k_1.\epsilon_2.\epsilon_4.\epsilon_3.k_1 \\
 + 64(k_2.k_1)^2 k_3.k_1 k_1.\epsilon_2.\epsilon_4.\epsilon_3.k_1 + 32k_2.k_1 (k_3.k_1)^2 k_1.\epsilon_2.\epsilon_4.\epsilon_3.k_1 \\
 + 32(k_2.k_1)^2 k_3.k_1 k_1.\epsilon_2.\epsilon_4.\epsilon_3.k_2 + 32k_2.k_1 (k_3.k_1)^2 k_1.\epsilon_2.\epsilon_4.\epsilon_3.k_2 \\
 - 32(k_2.k_1)^3 k_1.\epsilon_3.\epsilon_2.\epsilon_4.k_2 - 32(k_2.k_1)^2 k_3.k_1 k_1.\epsilon_3.\epsilon_2.\epsilon_4.k_2 \\
 - 32(k_2.k_1)^2 k_3.k_1 k_2.\epsilon_3.\epsilon_2.\epsilon_4.k_2 - 32(k_2.k_1)^2 k_3.k_1 k_2.\epsilon_3.\epsilon_2.\epsilon_4.k_3 \\
 - 32(k_2.k_1)^2 k_3.k_1 k_3.\epsilon_2.\epsilon_3.\epsilon_4.k_2 - 32(k_2.k_1)^2 k_3.k_1 k_3.\epsilon_2.\epsilon_3.\epsilon_4.k_3 \\
 + 32(k_2.k_1)^3 k_3.\epsilon_2.\epsilon_4.\epsilon_3.k_1 + 32(k_2.k_1)^2 k_3.k_1 k_3.\epsilon_2.\epsilon_4.\epsilon_3.k_1 \\
 \left. - 32(k_2.k_1)^3 k_3.k_1 \text{Tr}[\epsilon_2.\epsilon_3.\epsilon_4] - 32(k_2.k_1)^2 (k_3.k_1)^2 \text{Tr}[\epsilon_2.\epsilon_3.\epsilon_4] \right] \phi_1 \quad (2.17)
 \end{aligned}$$

where ϵ_2 is the polarization of the graviton. We have also divided the amplitude by $4\sqrt{2}$ to be consistent with the normalization of fields in section 1.

The above amplitude should be compared with the corresponding couplings in (1.12). Using the expansion

$$\mathcal{H}_{hqr,n}^T \mathcal{M}_{,rm} \mathcal{H}_{knp,h} = -e^{-\phi} \phi_{,rm} H_{hqr,n} H_{knp,h} + \dots \quad (2.18)$$

where dots refer to the other terms which involve the R-R fields, one finds the following couplings at the tree-level in the Einstein frame:

$$S \supset \frac{\gamma \zeta(3)}{8} \int d^{10}x \sqrt{-G} e^{-5\phi/2} \left[-8\phi_{,rm} H_{hqr,n} H_{knp,h} R_{mpkq} + 8\phi_{,mk} H_{npr,h} H_{hnq,r} R_{kqmp} \right. \\ \left. + 8\phi_{,qh} H_{npq,m} H_{mpr,k} R_{krhn} - 8\phi_{,qm} H_{npq,h} H_{knr,h} R_{mrkp} + 4\phi_{,mh} H_{npq,h} H_{npr,k} R_{mrkq} \right. \\ \left. - 4\phi_{,rk} H_{mnq,h} H_{npq,k} R_{prhm} + 4\phi_{,rm} H_{mnp,k} H_{npq,h} R_{qrhk} \right] \quad (2.19)$$

In the string frame, the overall dilaton factor becomes $e^{-2\phi}$. We have checked explicitly that the momentum space of the above couplings are exactly the same as the couplings in (2.17).

3 Five-point amplitude

The coupling $\epsilon_{10}\epsilon_{10}R^4$ in (1.6) is a total derivative term at four graviton level [15], so its presence in the effective action can not be confirmed by the S-matrix element of four gravitons. The σ -model beta function approach implies the presence of this term at the tree-level [13, 14]. To confirm this term directly from the sphere-level S-matrix calculations, one needs to study the scattering amplitude of five gravitons. In this section we are going to show that the sphere-level scattering amplitude of five gravitons confirms the presence of the coupling $\epsilon_{10}\epsilon_{10}R^4$ at the tree-level. The one-loop calculation which confirms the presence of this term in (1.7), appears in [34, 35]

To confirm the tree-level couplings (1.6) by the sphere-level S-matrix element of five graviton vertex operators, one has to first calculate explicitly the latter amplitude and then expand it at low energy to find massless poles and contact terms. The massless poles and the contact terms of the string theory amplitude should be the same as the massless poles and the contact terms of the corresponding Feynman amplitude in the field theory. In the field theory side, the eight-derivative massless poles are produced by the Hilbert-Einstein term in the supergravity (1.1) and by the four-graviton couplings in $t_8 t_8 R^4$. The eight-derivative contact terms of the S-matrix element should also be the same as the five-graviton couplings in (1.6).

We are not going to calculate the S-matrix element explicitly in the string theory side, however, we use the observation made in [45, 47] that indicates the sphere-level S-matrix elements should have no term with zero or one Mandelstam variable $k_i \cdot k_j$. In other words, the terms in the S-matrix elements should have at least two Mandelstam variables. This is resulted from the fact that in the disk-level scattering amplitude of open string gauge bosons, two of the vertex operators must be in (-1) -picture and all other must be in 0-picture. As a result, the open string amplitude has no term with zero Mandelstam variable. Using the KLT prescription, one finds that the closed string amplitudes at sphere-level have at least two Mandelstam variables.

The scattering amplitude of five on-shell gravitons in field theory side is given by the following Feynman amplitude:

$$A = \sum_{i=1}^{10} A_i + V_{R^4}(1, 2, 3, 4, 5) \quad (3.1)$$

where the massless poles A_i are given as

$$\begin{aligned} A_1 &= V_{R^4}(1, 2, 3, h)_{mn} G_{mn,hk} V_R(h, 4, 5)_{hk} \\ A_2 &= V_{R^4}(1, 2, 4, h)_{mn} G_{mn,hk} V_R(h, 3, 5)_{hk} \\ A_3 &= V_{R^4}(1, 4, 3, h)_{mn} G_{mn,hk} V_R(h, 2, 5)_{hk} \\ A_4 &= V_{R^4}(4, 2, 3, h)_{mn} G_{mn,hk} V_R(h, 1, 5)_{hk} \\ A_5 &= V_{R^4}(1, 2, 5, h)_{mn} G_{mn,hk} V_R(h, 3, 4)_{hk} \\ A_6 &= V_{R^4}(1, 3, 5, h)_{mn} G_{mn,hk} V_R(h, 2, 4)_{hk} \\ A_7 &= V_{R^4}(3, 2, 5, h)_{mn} G_{mn,hk} V_R(h, 1, 4)_{hk} \\ A_8 &= V_{R^4}(1, 4, 5, h)_{mn} G_{mn,hk} V_R(h, 2, 3)_{hk} \\ A_9 &= V_{R^4}(2, 4, 5, h)_{mn} G_{mn,hk} V_R(h, 1, 3)_{hk} \\ A_{10} &= V_{R^4}(3, 4, 5, h)_{mn} G_{mn,hk} V_R(h, 1, 2)_{hk} \end{aligned} \quad (3.2)$$

Our notation for the vertexes is that h appears for the off-shell graviton whereas the particle number appears for the on-shell graviton. For example, in the vertex $V_R(h, 4, 5)_{hk}$ one of the gravitons is off-shell and the other two gravitons are on-shell with particle labels 4 and 5. The propagator and the vertex $V_R(h, 4, 5)_{hk}$ are read from the supergravity (1.1)

$$\begin{aligned} G_{mn,hk} &= -\frac{i}{2k \cdot k} \left(\eta_{hm} \eta_{kn} + \eta_{hn} \eta_{km} - \frac{1}{4} \eta_{hk} \eta_{mn} \right) \quad (3.3) \\ V_R(h, 4, 5)_{hk} &= -2i \left[-\frac{1}{4} k_4 \cdot \varphi_5 k_5 \cdot \varphi_4 \rho_4 \cdot \rho_5 \eta_{hk} - \frac{1}{4} k_4 \cdot \rho_5 k_5 \cdot \varphi_4 \rho_4 \cdot \varphi_5 \eta_{hk} - \frac{1}{4} k_4 \cdot \varphi_5 k_5 \cdot \rho_4 \varphi_4 \cdot \rho_5 \eta_{hk} \right. \\ &\quad + \frac{3}{4} k_4 \cdot k_5 \rho_4 \cdot \varphi_5 \varphi_4 \cdot \rho_5 \eta_{hk} - \frac{1}{4} k_4 \cdot \rho_5 k_5 \cdot \rho_4 \varphi_4 \cdot \varphi_5 \eta_{hk} + \frac{3}{4} k_4 \cdot k_5 \rho_4 \cdot \rho_5 \varphi_4 \cdot \varphi_5 \eta_{hk} \\ &\quad - \frac{1}{2} \rho_4 \cdot \varphi_5 \varphi_4 \cdot \rho_5 (k_4)_h (k_4)_k - \frac{1}{2} \rho_4 \cdot \rho_5 \varphi_4 \cdot \varphi_5 (k_4)_h (k_4)_k - \frac{1}{2} \rho_4 \cdot \varphi_5 \varphi_4 \cdot \rho_5 (k_4)_k (k_5)_h \\ &\quad - \frac{1}{2} \rho_4 \cdot \rho_5 \varphi_4 \cdot \varphi_5 (k_4)_k (k_5)_h - \frac{1}{2} \rho_4 \cdot \varphi_5 \varphi_4 \cdot \rho_5 (k_5)_h (k_5)_k - \frac{1}{2} \rho_4 \cdot \rho_5 \varphi_4 \cdot \varphi_5 (k_5)_h (k_5)_k \\ &\quad + \frac{1}{2} k_4 \cdot \varphi_5 \varphi_4 \cdot \rho_5 (k_4)_k (\rho_4)_h + \frac{1}{2} k_4 \cdot \rho_5 \varphi_4 \cdot \varphi_5 (k_4)_k (\rho_4)_h + \frac{1}{2} k_5 \cdot \varphi_4 \rho_4 \cdot \varphi_5 (k_5)_k (\rho_5)_h \\ &\quad + \frac{1}{2} k_5 \cdot \rho_4 \varphi_4 \cdot \varphi_5 (k_5)_k (\rho_5)_h + \frac{1}{2} k_4 \cdot \varphi_5 k_5 \cdot \varphi_4 (\rho_4)_k (\rho_5)_h - \frac{1}{2} k_4 \cdot k_5 \varphi_4 \cdot \varphi_5 (\rho_4)_k (\rho_5)_h \\ &\quad + \frac{1}{2} k_4 \cdot \varphi_5 \rho_4 \cdot \rho_5 (k_4)_k (\varphi_4)_h + \frac{1}{2} k_4 \cdot \rho_5 \rho_4 \cdot \varphi_5 (k_4)_k (\varphi_4)_h - k_4 \cdot \rho_5 k_4 \cdot \varphi_5 (\rho_4)_k (\varphi_4)_h \\ &\quad + \frac{1}{2} k_4 \cdot \varphi_5 k_5 \cdot \rho_4 (\rho_5)_k (\varphi_4)_h - \frac{1}{2} k_4 \cdot k_5 \rho_4 \cdot \varphi_5 (\rho_5)_k (\varphi_4)_h + \frac{1}{2} k_5 \cdot \varphi_4 \rho_4 \cdot \rho_5 (k_5)_k (\varphi_5)_h \\ &\quad + \frac{1}{2} k_5 \cdot \rho_4 \varphi_4 \cdot \rho_5 (k_5)_k (\varphi_5)_h + \frac{1}{2} k_4 \cdot \rho_5 k_5 \cdot \varphi_4 (\rho_4)_k (\varphi_5)_h - \frac{1}{2} k_4 \cdot k_5 \varphi_4 \cdot \rho_5 (\rho_4)_k (\varphi_5)_h \\ &\quad \left. - k_5 \cdot \rho_4 k_5 \cdot \varphi_4 (\rho_5)_k (\varphi_5)_h + \frac{1}{2} k_4 \cdot \rho_5 k_5 \cdot \rho_4 (\varphi_4)_k (\varphi_5)_h - \frac{1}{2} k_4 \cdot k_5 \rho_4 \cdot \rho_5 (\varphi_4)_k (\varphi_5)_h \right] \end{aligned}$$

where we have written the graviton polarization as

$$\varepsilon_{hk} = \frac{1}{2}(\rho_h \varphi_k + \rho_k \varphi_h) \quad (3.4)$$

to impose the symmetry of the graviton. The vertex $V_{R^4}(1, 2, 3, h)_{mn}$ should be read from the action (1.6). Since $\epsilon_{10}\epsilon_{10}R^4$ is a total derivative at four graviton level, this vertex must be read from $t_8 t_8 R^4$.

The t_8 tensor was first defined in [11], i.e., the contraction of t_8 with four arbitrary antisymmetric matrices M^1, \dots, M^4 is

$$\begin{aligned} t_{hkmnpqrs} M_{hk}^1 M_{mn}^2 M_{pq}^3 M_{rs}^4 = \\ 8(\text{tr} M^1 M^2 M^3 M^4 + \text{tr} M^1 M^3 M^2 M^4 + \text{tr} M^1 M^3 M^4 M^2) \\ - 2(\text{tr} M^1 M^2 \text{tr} M^3 M^4 + \text{tr} M^1 M^3 \text{tr} M^2 M^4 + \text{tr} M^1 M^4 \text{tr} M^2 M^3) \end{aligned} \quad (3.5)$$

Using this tensor, the coupling $t_8 t_8 R^4$ can be written as [36]⁴

$$\begin{aligned} t_8 t_8 R^4 \equiv t_{\mu_1 \dots \mu_8} t_{\nu_1 \dots \nu_8} R_{\mu_1 \mu_2 \nu_1 \nu_2} R_{\mu_3 \mu_4 \nu_3 \nu_4} R_{\mu_5 \mu_6 \nu_5 \nu_6} R_{\mu_7 \mu_8 \nu_7 \nu_8} \\ = 3 \cdot 2^7 \left[R_{hkmn} R_{krnp} R_{rsqm} R_{shpq} + \frac{1}{2} R_{hkmn} R_{krnp} R_{rspq} R_{shqm} \right. \\ \left. - \frac{1}{2} R_{hkmn} R_{krmn} R_{rspq} R_{shpq} - \frac{1}{4} R_{hkmn} R_{krpq} R_{rsmn} R_{shpq} \right. \\ \left. + \frac{1}{16} R_{hkmn} R_{khpq} R_{rsmn} R_{srpq} + \frac{1}{32} R_{hkmn} R_{khmn} R_{rspq} R_{srpq} \right] \end{aligned} \quad (3.6)$$

Using the linearized Riemann curvature (2.11), one finds the vertex

$$V_{R^4}(1, 2, 3, h)_{mn} = 3 \cdot 2^7 i \gamma \zeta(3) [k_1 \cdot \varphi_2 k_1 \cdot \varphi_3 k_2 \cdot \rho_3 k_2 \cdot \varphi_1 k_3 \cdot \rho_1 k_3 \cdot \rho_2 (k_1)_m (k_1)_n + \dots] \quad (3.7)$$

where dots refer to all other terms which are too many to be able to write them. In the supergravity vertex (3.3), the terms which are proportional to $k_4 \cdot k_5$ produce contact terms when they are multiplied by the propagator. We call them A_i^c . The other terms produce massless poles in (3.1). We call them A_i^{pole} . So the massless poles in (3.1) are

$$A^{\text{pole}} = \sum_{i=1}^{10} A_i^{\text{pole}} \quad (3.8)$$

We have checked that these massless poles are proportional to at least two Mandelstam variables, as expected from the corresponding sphere-level scattering amplitude in the string theory side [45, 47].

On the other hand, the contact terms A_i^c do not satisfy this constraint, i.e., they have terms with no Mandelstam variable. These contact terms however must be combined with the contact term $V_{R^4}(1, 2, 3, 4, 5)$ in (3.1) to satisfy this constraint. So we expect the following contact terms:

$$A^c = \sum_{i=1}^{10} A_i^c + V_{R^4}(1, 2, 3, 4, 5) \quad (3.9)$$

⁴For the generalized Riemann curvature (1.9) which does not have all symmetries of the Riemann curvature, the coupling $t_8 t_8 \bar{R}^4$ has different structure [48].

to be proportional to at least two Mandelstam variables. The five graviton couplings of $t_8 t_8 R^4$ are resulted from two sources. One source is to use (2.11) for all Riemann curvatures and the expansions $G^{ab} = -2h^{ab}$ for one metric and $G^{ab} = \eta^{ab}$ for all the other metrics in the contracted indices. The other source is to use (2.11) for three Riemann curvatures, use the second order gravitons for one of the Riemann curvature, i.e.,

$$\begin{aligned}
 R_{amnb} = & -h_{bm,t}h_{an,t} + h_{ab,t}h_{mn,t} - h_{mn,t}h_{at,b} + h_{bm,t}h_{at,n} - h_{mn,t}h_{bt,a} + h_{an,t}h_{bt,m} \\
 & - h_{at,n}h_{bt,m} + h_{an,t}h_{mt,b} - h_{at,n}h_{mt,b} - h_{ab,t}h_{mt,n} + h_{at,b}h_{mt,n} + h_{bt,a}h_{mt,n} \\
 & + h_{bm,t}h_{nt,a} - h_{bt,m}h_{nt,a} - h_{mt,b}h_{nt,a} - h_{ab,t}h_{nt,m} + h_{at,b}h_{nt,m} + h_{bt,a}h_{nt,m}
 \end{aligned} \tag{3.10}$$

and to use the expansion $G^{ab} = \eta^{ab}$ for all the metrics in the contracted indices. Using the above two sources to calculate the contact terms in $V_{R^4}(1, 2, 3, 4, 5)$, one still finds that the amplitude (3.9) has terms with no Mandelstam variable. So there must be another term beside the coupling $t_8 t_8 R^4$.

Now consider the coupling $\epsilon_{10}\epsilon_{10}R^4$. Using the relation $\frac{1}{2}\epsilon_{\mu\nu\mu_1\dots\mu_8}\epsilon^{\mu\nu\nu_1\dots\nu_8} = -8!\delta_{[\mu_1}^{\nu_1}\dots\delta_{\mu_8]}^{\nu_8}$, one may write

$$\begin{aligned}
 \frac{1}{2}\epsilon_{10}\epsilon_{10}R^4 = & 3 \cdot 2^9 \left[-R_{hkmn}R_{pqkr}R_{nrqs}R_{mshp} + R_{hkmn}R_{pqhk}R_{nrqs}R_{mspr} \right. \\
 & + \frac{1}{2}R_{hkmn}R_{pqrs}R_{mrpq}R_{nshk} - \frac{1}{2}R_{hkmn}R_{pqrs}R_{mrhp}R_{nshk} \\
 & - \frac{1}{16}R_{hkmn}R_{mnpq}R_{pqrs}R_{rshk} - \frac{1}{32}R_{hkmn}R_{mnhk}R_{pqrs}R_{rspq} \\
 & - 2R_{hm}R_{npqr}R_{qrns}R_{mshp} + R_{hm}R_{npqr}R_{qrhs}R_{msnp} + 4R_{hm}R_{mnpq}R_{qrhs}R_{psnr} \\
 & \left. + \frac{1}{3}RR_{mnpq}R_{qrns}R_{psmr} - \frac{1}{12}RR_{mnpq}R_{pqrs}R_{rsmn} + \dots \right] \tag{3.11}
 \end{aligned}$$

where dots refer to the terms with more than one Ricci or scalar curvature. Since we are interested in the couplings of five on-shell gravitons in $V_{R^4}(1, 2, 3, 4, 5)$, such terms have no contribution. The Riemann curvature couplings in the first three lines above have two sources in producing the five graviton contact terms, as we have explained in the previous paragraph. The Ricci curvature terms in the fourth line above have also two sources. One source is to replace the Riemann curvature terms with the linear form (2.11), the Ricci curvature with the nonlinear form (3.10) and $G^{ab} = \eta^{ab}$ for all the metrics in the contracted indices. The second source is to replace the Ricci curvature with $R_{ab} \equiv R_{acbc} = -2h^{cd}R_{acbd}$, and then to replace all the Riemann curvatures with the linear form (2.11) and all the contracted metrics with $G^{ab} = \eta^{ab}$. The scalar curvature couplings in the last line above have one source in producing the five on-shell graviton contact terms. The scalar curvature must be replaced by the nonlinear form (3.10), the Riemann curvatures with the linear form (2.11) and all the contracted metrics must be replaced by $G^{ab} = \eta^{ab}$.

Calculating the contact term $V_{R^4}(1, 2, 3, 4, 5)$ from the couplings $t_8 t_8 R^4 + \frac{\alpha}{2}\epsilon_{10}\epsilon_{10}R^4$ as we have explained above, and using the conservation of momentum and the on-shell relations $k_i \cdot k_i = k_i \cdot \epsilon_i = \epsilon_i \cdot k_i = 0$ to write the amplitude in terms of the independent Mandelstam variables $k_1 \cdot k_2, k_1 \cdot k_3, k_2 \cdot k_3, k_2 \cdot k_4, k_3 \cdot k_4$ and in terms of independent variables $k_1 \cdot \epsilon_2, k_1 \cdot \epsilon_3, k_1 \cdot \epsilon_4, k_2 \cdot \epsilon_1, k_2 \cdot \epsilon_3, k_2 \cdot \epsilon_4, k_2 \cdot \epsilon_5, k_3 \cdot \epsilon_1, k_3 \cdot \epsilon_2, k_3 \cdot \epsilon_4, k_3 \cdot \epsilon_5, k_4 \cdot \epsilon_1, k_4 \cdot \epsilon_2, k_4 \cdot \epsilon_3, k_4 \cdot \epsilon_5,$

we have found that the contact terms (3.9) have at least two Mandelstam variables when the constant $\alpha = \frac{1}{4}$. This confirms that the tree-level couplings in (1.6) are consistent with sphere-level scattering amplitude of five gravitons in the RNS formalism. As a further check, one may calculate explicitly the scattering amplitude in the string theory side which contains only terms proportional to at least two Mandelstam variables, and compare it with the field theory (1.6).

Finally, let us mention that the coupling $t_8 t_8 R^4$ is invariant under standard T-duality [35, 37, 48], whereas the coupling $\epsilon_{10} \epsilon_{10} R^4$ is not invariant under the standard linear T-duality in the absence of B-field. To see this, one may consider the dimensional reduction of this coupling on a circle and may consider the following term

$$\epsilon_{\mu\nu y \mu_2 \dots \mu_8} \epsilon_{\mu\nu y \nu_2 \dots \nu_8} R_{y \mu_2 y \nu_2} R_{\mu_3 \mu_4 \nu_3 \nu_4} R_{\mu_5 \mu_6 \nu_5 \nu_6} R_{\mu_7 \mu_8 \nu_7 \nu_8} \quad (3.12)$$

where y is the Killing coordinate. Under the standard linear T-duality $R_{y \mu_2 y \nu_2}$ goes to $-R_{y \mu_2 y \nu_2}$ [45]. So the coupling $\epsilon_{10} \epsilon_{10} R^4$ is not invariant under the linear T-duality. However, this term is a total derivative at four field level so it has no effect in the action at four field level. This indicates that only the action, not the Lagrangian density, must be invariant under T-duality. Moreover, using the fact that the Gauss-Bonnet term $\epsilon_{10} \epsilon_{10} R^4$ is the only coupling which is total derivative at four field level and the observation that the couplings in $t_8 t_8 R^4$ are the most general couplings which are consistent with the standard linear T-duality [48], one concludes that there is no other gravity term in (1.6) at order α'^3 .

The coupling $\epsilon_{10} \epsilon_{10} R^4$, however, must be extended to T-duality invariant form because this term has significant effect at higher levels. In the absence of the B-field, the only way to make it invariant under the standard T-duality is to include couplings which involve dilaton in the string frame. It has been argued in [35] that there is no dilaton couplings in the string frame. However, the argument in [35] is based on the specific form of the effective action of the heterotic string theory at order α' which has no dilaton in the string frame. This action is related to the standard form of the heterotic effective action which is manifestly T-duality invariant and includes various dilaton couplings [49], by some field redefinitions. The field redefinition, however, changes the standard T-duality transformations to non-standard forms. Therefore, we expect the couplings like

$$\epsilon_{\mu\nu\rho\mu_2 \dots \mu_8} \epsilon_{\mu\nu\rho\nu_2 \dots \nu_8} \partial_{\mu_2} \phi \partial_{\nu_2} \phi R_{\mu_3 \mu_4 \nu_3 \nu_4} R_{\mu_5 \mu_6 \nu_5 \nu_6} R_{\mu_7 \mu_8 \nu_7 \nu_8} \quad (3.13)$$

to be included in the type II theory in the string frame, to make the whole action to be invariant under the standard T-duality transformations. It would be interesting to find such dilaton couplings.

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