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T-duality of D-brane action at order α' in bosonic string theory

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Abstract: In bosonic string theory, it is known that the Buscher rules for the T-duality transformations receive quantum corrections at order α'. In this paper, we use the consistency of the gravity couplings on the D-brane effective action at order α', with the above T-duality transformations to find the B-field and the dilaton couplings. We show that these couplings are fully consistent with the corresponding disk-level S-matrix elements in string theory.

Keywords: D-branes, Bosonic Strings, String Duality

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1 Introduction and results

One of the most fantastic dualities of string theory is T-duality [1–5]. It relates the bosonic string theory compactified on a circle with radius ρ to the same theory compactified on another circle with radius α'/ρ. It also relates the D_p-brane of the theory to the D_{p-1}-brane and D_{p+1}-brane, depending on whether the original D_p-brane is along or orthogonal to the circle, respectively. The bulk action S which is given by the closed string field theory, must be invariant under the T-duality [6, 7], and the D_p-brane action S_{D_p} which is given by the cubic string field theory [8], must be covariant under the T-duality, i.e.,

$$S \xrightarrow{T} S\,, \qquad S_{D_p} \xrightarrow{T} S_{D_{p\pm1}}\,. \tag{1.1}$$

An appropriate framework for incorporating the T-duality into these actions is the double field theory formalism in which the fields depend both on the usual spacetime coordinates and on the winding coordinates [9–11]. These actions can be expanded at low energy, i.e.,

$$S = \sum_{n=0}^{\infty}(\alpha')^n S^{(n)}\,, \qquad S_{D_p} = \sum_{n=0}^{\infty}(\alpha')^n S^{(n)}_{D_p}\,, \tag{1.2}$$

and then information about $S^{(n)}$ and $S^{(n)}_{D_p}$ can be found from the α'-expansion of S-matrix elements [12–15] and from T-duality [16, 17].

The bulk effective action of bosonic string theory includes various couplings of the closed string tachyon τ, graviton $G_{\mu\nu}$, dilaton ϕ and antisymmetric B-field. Because of the tachyon, the perturbative bosonic string theory is unstable. In this paper we assume that the tachyon freezes at $\tau = 0$. With this assumption, the leading α'-order terms of the bulk effective action in the bosonic string theory are given by the following couplings

$$S^{(0)} = \frac{1}{2\kappa^2}\int d^D x e^{-2\phi}\sqrt{-G}\left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]. \tag{1.3}$$

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The heterotic and the superstring theories have the same couplings as well as some other couplings involving the other massless fields in their corresponding supermultiplets [18]. The above action is invariant under the standard Buscher rules for the T-duality transformations (see e.g., [10]).

The next-to-leading α'-order terms of the bulk effective action have been found in [12] from the corresponding sphere-level S-matrix elements. T-duality, however, is not manifest in these couplings. It has been shown in [16, 19] that by using proper field redefinitions, one can change the couplings at order α' into a manifestly T-dual invariant form. However, the T-duality transformations are the standard Buscher rules plus some α'-corrections [16]. The T-dual invariant action is $S^{(0)}+\alpha' S^{(1)}$ where the action $S^{(0)}$ is given in (1.3) and $S^{(1)}$ is

$$\begin{aligned} S^{(1)}=&\frac{\lambda_0}{2\kappa^2}\int d^D x e^{-2\phi}\sqrt{-G}\Big[-R^2_{GB}+16\Big(R^{\mu\nu}-\frac{1}{2}G^{\mu\nu}R\Big)\partial_\mu\phi\partial_\nu\phi-16\nabla^2\phi(\partial\phi)^2+16(\partial\phi)^4\\ &+\frac{1}{2}\Big(R_{\mu\nu\lambda\rho}H^{\mu\nu\alpha}H^{\lambda\rho}{}_{\alpha}-2R^{\mu\nu}H^2_{\mu\nu}+\frac{1}{3}RH^2\Big)-2\Big(\nabla^\mu\partial^\nu\phi H^2_{\mu\nu}-\frac{1}{3}\nabla^2\phi H^2\Big)-\frac{2}{3}(\partial\phi)^2H^2\\ &-\frac{1}{24}H_{\mu\nu\lambda}H^\nu{}_{\rho\alpha}H^{\rho\sigma\lambda}H_\sigma{}^{\mu\alpha}+\frac{1}{8}H^2_{\mu\nu}H^{2\,\mu\nu}-\frac{1}{144}(H^2)^2\Big]\,, \end{aligned} \tag{1.4}$$

where $\lambda_0=-\frac{1}{4}$ for the bosonic string theory, $\lambda_0=-\frac{1}{8}$ for the heterotic theory and $\lambda_0=0$ for the superstring theory. In above action, $H^2_{\mu\nu}=H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta}$ and $R^2_{GB}=R^2_{\mu\nu\lambda\sigma}-4R^2_{\mu\nu}+R^2$ is the Gauss-Bonnet combination of the curvature squared terms which does not change the graviton propagator in (1.3). The T-duality invariance of the above action is such that the action $S^{(1)}$ itself is not fully invariant under the standard T-duality transformation. It produces some extra terms. The extra terms however are canceled with the transformation of the action $S^{(0)}$ under the α'-corrected T-duality [16].

Unlike the leading α'-order action (1.3), the couplings in (1.4) are not unique. One can use field redefinitions of order α' on the action (1.3) to change the couplings in the action (1.4) [12, 20, 21]. For example, using the field redefinition,

$$\phi\to\phi+\delta\phi\,,\qquad G_{\mu\nu}\to G_{\mu\nu}+\delta G_{\mu\nu}\,, \tag{1.5}$$

where

$$\delta\phi=\alpha'\lambda_0 a_1 H^2\,,\qquad \delta G_{\mu\nu}=\alpha'\lambda_0(b_1H^2_{\mu\nu}+b_2\partial_\mu\phi\partial_\nu\phi)\,, \tag{1.6}$$

one can produce the following couplings from (1.3):

$$\begin{aligned} &\frac{\alpha'\lambda_0}{2\kappa^2}\int d^D x e^{-2\phi}\sqrt{-G}\Big[-2a_1H^2\Big(R+4\nabla^2\phi-4(\partial\phi)^2-\frac{1}{12}H^2\Big)\\ &-(b_1H^2_{\mu\nu}+b_2\partial_\mu\phi\partial_\nu\phi)\Big(R^{\mu\nu}+2\nabla^\mu\partial^\nu\phi-\frac{1}{4}H^{2\,\mu\nu}-\frac{1}{2}G^{\mu\nu}\Big[R+4\nabla^2\phi-4(\partial\phi)^2-\frac{1}{12}H^2\Big]\Big)\Big]\,, \end{aligned} \tag{1.7}$$

where a_1, b_1 and b_2 are arbitrary constants. Adding the above terms into the action (1.4) for the following specific values:

$$a_1=-\frac{1}{6}\,,\qquad b_1=-1\,,\qquad b_2=16\,, \tag{1.8}$$

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one finds that the bulk action at order α' simplifies to

$$S^{(1)}=\frac{\lambda_0}{2\kappa^2}\int d^Dx e^{-2\phi}\sqrt{-G}\Big[-R_{GB}^2+\frac{1}{2}R_{\mu\nu\lambda\rho}H^{\mu\nu\alpha}H^{\lambda\rho}{}_{\alpha}+32\nabla^2\phi(\partial\phi)^2-48(\partial\phi)^4 \tag{1.9}$$
$$-\frac{2}{3}(\partial\phi)^2H^2+4\partial^\mu\phi\partial^\nu\phi H_{\mu\nu}^2-\frac{1}{24}H_{\mu\nu\lambda}H^{\nu}{}_{\rho\alpha}H^{\rho\sigma\lambda}H_{\sigma}{}^{\mu\alpha}-\frac{1}{8}H_{\mu\nu}^2H^{2\,\mu\nu}+\frac{1}{144}(H^2)^2\Big],$$

where we have also used the integration by part to write $\nabla^\mu\partial^\nu\phi\partial_\mu\phi\partial_\nu\phi=-\frac{1}{2}\nabla^2\phi(\partial\phi)^2+(\partial\phi)^4$. One may use a different field redefinition to write the above couplings in yet another form. However, in order to keep the Gauss-Bonnet combination unchanged under the field redefinition, we are not allowed to use the field redefinitions $\delta G_{\mu\nu}=\alpha'\lambda_0(b_3G_{\mu\nu}R+b_4R_{\mu\nu})$ and $\delta\phi=\alpha'\lambda_0a_2R$. Moreover, we are not allowed to have the field redefinitions $\delta B_{\mu\nu}\sim\nabla^\alpha H_{\alpha\mu\nu}$ or $\delta\phi\sim\nabla^2\phi$ because these change the B-field or dilaton propagators, respectively. The couplings (1.7) are not invariant under the T-duality. As a result, the above action is not manifestly invariant under the T-duality.

The effective action of D-brane includes various world volume couplings of open string tachyon T, transverse scalar fields Φ^i, gauge field A_a, closed string tachyon, graviton, dilaton and B-field. Duo to the presence of the open string tachyon, the D-branes in bosonic string theory are all unstable. We again assume that the open string tachyon freezes at $T=0$ and the closed string tachyon at $\tau=0$. Then the leading α'-order terms of the D-brane effective action are given by the Dirac-Born-Infeld (DBI) action [22, 23]

$$S_{D_p}^{(0)}=-T_p\int d^{p+1}xe^{-\phi}\sqrt{-det(\tilde{G}_{ab}+\widetilde{B}_{ab})}\,, \tag{1.10}$$

where $\tilde{G}_{ab}$ and $\widetilde{B}_{ab}$ are pull-back of the bulk fields $G_{\mu\nu}$ and $B_{\mu\nu}$ onto the world-volume of the D-brane[1], e.g.,

$$\tilde{G}_{ab}=\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}G_{\mu\nu}\,. \tag{1.11}$$

The gauge field is added to this action by the replacement $B\to B+2\pi\alpha'F$. The transverse scalar fields appear in the static gauge where $X^a=\sigma^a$ and $X^i=2\pi\alpha'\Phi^i$. The DBI action also describes the dynamics of D-branes of superstring theory at low energy. This action is covariant under T-duality transformations [24].

The α' corrections to the D-brane action (1.10) have been studied in [15, 25]. By requiring the consistency of the effective action with α'-order terms of the disk-level scattering amplitude of two gravitons, the following gravity couplings have been found in [15]:

$$S_{D_p}^{(1)}=-\frac{T_p}{2}\int d^{p+1}xe^{-\phi}\sqrt{-\tilde{G}}\Big[\tilde{R}+\perp_{\alpha\beta}(\Omega^\alpha{}_a{}^a\Omega^\beta{}_b{}^b-\Omega^\alpha{}_{ab}\Omega^{\beta\,ab})+\dots\Big], \tag{1.12}$$

where $\tilde{R}$ is the scalar curvature made out of the pull-back metric $\tilde{G}_{ab}$ and Ω is the second fundamental form

$$\Omega^\alpha{}_{ab}=\frac{\partial^2X^\alpha}{\partial\sigma^a\partial\sigma^b}+\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\Gamma^\alpha_{\mu\nu}\,. \tag{1.13}$$

[1] Our index convention is such that the Greek letters are used as space-time indices as usual. The Latin letters $(a,b,c,\dots)$ denote the world-volume indices while $(i,j,k,\dots)$ denote the transverse or normal bundle indices.

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The operator $\perp^{\mu\nu}$ is the projection operator to the transverse space, i.e.,

$$\perp^{\mu\nu} = G^{\mu\nu} - \tilde{G}^{\mu\nu}\,, \qquad \tilde{G}^{\mu\nu} = \frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b}\tilde{G}^{ab}\,, \tag{1.14}$$

where $\tilde{G}^{\mu\nu}$ which projects operators to the world volume space, is the first fundamental form, and the indices of the operator $\perp^{\mu\nu}$ in (1.12) are lowered by the spacetime metric[2].

The dots in (1.12) refer to the dilaton and B-field couplings. The coefficients of these couplings that have been found in [15], however, depend on the dimension of the D_p-brane. We refer the interested readers to [15] for the explicit form of these couplings. Under the T-duality transformations, the graviton transforms into B-field, so in a T-duality invariant action the coefficients of the B-field and dilaton, like the coefficients of the gravity couplings in (1.12), must be independent of p. As a result, the above action is not manifestly invariant under the T-duality.

In this paper, we are going to find an action which is manifestly invariant under the T-duality. We have found that it is impossible to find appropriate dilaton and B-field couplings which make the gravity couplings in (1.12) to be invariant under the standard T-duality transformations. However, when we use the α'-corrected T-duality transformations [16], we will find the following T-duality invariant couplings:

$$\begin{aligned} S^{(1)}_{D_p} = & -\frac{T_p}{2}\int d^{p+1}x e^{-\phi}\sqrt{-\tilde{G}}\Big[\tilde{R} + \perp_{\mu\nu}(\Omega^\mu{}_a{}^a\Omega^\nu{}_b{}^b - \Omega^\mu{}_{ab}\Omega^{\nu\,ab}) + 2\perp_{\mu\nu}\Omega^\mu{}_a{}^a\partial^\nu\phi + \partial_\mu\phi\partial^\mu\phi \\ & -\frac{1}{8}\tilde{H}^2 - \frac{1}{8}\perp^{\mu\nu}H^2_{\mu\nu} + \frac{1}{8}\perp^{\alpha\beta}\perp^{\mu\nu}H_{\alpha\mu\lambda}H_{\beta\nu}{}^\lambda + \frac{1}{24}\perp^{\alpha\beta}\perp^{\mu\nu}\perp^{\lambda\sigma}H_{\alpha\mu\lambda}H_{\beta\nu\sigma}\Big]\,, \end{aligned} \tag{1.15}$$

where $\tilde{H}^2 = H_{abc}H^{abc}$. The above D-brane action is consistent with the bulk action (1.4), i.e., they both are manifestly invariant under the T-duality. Similar calculations have been done in [17, 26] to find the T-dual completion of the gravity couplings on the world volume of D-branes in the superstring theory.

The actions (1.4) and (1.15) may be used to study various physical phenomena like the scattering of external particles from the D-branes in the bosonic string theory. However, if one prefers to use the simplified bulk action (1.9) in which the field redefinition (1.6) has been used, then the same field redefinition must be applied on the DBI action to modify the brane action (1.15). The field redefinition (1.6) produces the following couplings from the DBI action:

$$-\frac{\alpha' T_p}{2}\int d^{p+1}x e^{-\phi}\sqrt{-\tilde{G}}\Big[-2a_1H^2 + b_1\tilde{G}^{ab}H^2_{ab} + b_2\partial^a\phi\partial_a\phi\Big]\lambda_0\,. \tag{1.16}$$

[2]The relation between the second fundamental form K^i_{ab} that has been used in [15] and Ω^μ_{ab} is $K^i_{ab} = \Omega^\mu_{ab}n^i_\mu$ where n^i_μ for $i = p+1, \cdots, D$, are the orthonormal basis of the transverse space, i.e., $\perp_{\mu\nu} = \sum_{i=p+1}^{D} n^i_\mu n^i_\nu$.

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Adding the above couplings for the specific values (1.8) to the action (1.15), one finds

$$
\begin{aligned}
S^{(1)}_{D_p} = & -\frac{T_p}{2}\int d^{p+1}x e^{-\phi}\sqrt{-\tilde{G}}\Big[\tilde{R} + \perp_{\mu\nu}(\Omega^{\mu}{}_{a}{}^{a}\Omega^{\nu}{}_{b}{}^{b} - \Omega^{\mu}{}_{ab}\Omega^{\nu\,ab}) - 3\partial_a\phi\partial^a\phi \\
& + \perp_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + 2\perp_{\mu\nu}\Omega^{\mu}{}_{a}{}^{a}\partial^\nu\phi + \frac{1}{24}\tilde{H}^2 + \frac{1}{8}\perp^{\mu\nu}H^2_{\mu\nu} \\
& - \frac{3}{8}\perp^{\alpha\beta}\perp^{\mu\nu}H_{\alpha\mu\lambda}H_{\beta\nu}{}^{\lambda} + \frac{5}{24}\perp^{\alpha\beta}\perp^{\mu\nu}\perp^{\lambda\sigma}H_{\alpha\mu\lambda}H_{\beta\nu\sigma}\Big]\,,
\end{aligned}
\tag{1.17}
$$

which is the D-brane action corresponding to the bulk action (1.9). Unlike the D-brane action (1.15), the above action is not manifestly invariant under the T-duality transformations. To compare the D-brane actions (1.15) or (1.17) with the string theory S-matrix elements, we have to transform them to the Einstein frame and use some other field redefinitions. Then we compare them with the corresponding disk-level S-matrix elements.

An outline of the paper is as follows: In section 2 we give a brief review of the standard T-duality transformation along with its α' corrections [16], and review the prescription given in [17] for finding the T-dual completion of a gravity coupling in the effective action of D-branes. In section 3, using this method we find the appropriate B field and dilaton couplings which make the graviton couplings in (1.12) to be invariant under the T-duality. In section 4, in order to compare the action (1.17) with the string theory S-matrix elements, we transform the actions (1.9) and (1.17) to the Einstein frame. We find perfect agreement between the B-field couplings that we have found and the B-field couplings that have been found in [15] from S-matrix calculations. The dilaton couplings, however, are not exactly the couplings that have been found in [15]. In appendix A, we reexamine the extraction of the dilaton couplings from the S-matrix element of two gravitons and find an exact agreement with the couplings that we have found from T-duality.

2 T-duality

The full set of nonlinear T-duality transformations for massless fields have been found in [2, 3, 27–29]. When the T-duality transformation acts along the Killing coordinate y, the transformations are

$$
\begin{aligned}
e^{2\tilde{\phi}} &= \frac{e^{2\phi}}{G_{yy}}\,, & \tilde{G}_{yy} &= \frac{1}{G_{yy}}\,, \\
\tilde{G}_{\mu y} &= \frac{B_{\mu y}}{G_{yy}}\,, & \tilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}B_{\nu y}}{G_{yy}}\,, \\
\tilde{B}_{\mu y} &= \frac{G_{\mu y}}{G_{yy}}\,, & \tilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}}\,,
\end{aligned}
\tag{2.1}
$$

where $\mu,\nu \neq y$. In above transformation the metric is given in the string frame. If y is identified on a circle of radius ρ, i.e., $y \sim y + 2\pi\rho$, then after T-duality the radius becomes $\tilde{\rho} = \alpha'/\rho$. The string coupling is also transformed as $\tilde{g} = g\sqrt{\alpha'}/\rho$.

It is known that the T-duality transformations in the superstring theory do not receive α' corrections, however, they receive such corrections in the heterotic and bosonic string

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theories [16]. That is, the T-duality operator has an α' expansion

$$T = \sum_{n=0}^{\infty} (\alpha')^n T^{(n)} \,, \tag{2.2}$$

where $T^{(n)}$ for $n > 0$ are all zero in the superstring theory and are non-zero in other cases. In all theories $T^{(0)}$ is given by the Buscher rules (2.1). The invariance of the effective actions at order $(\alpha')^0$ then means that

$$S^{(0)} \xrightarrow{T^{(0)}} S^{(0)} \,. \tag{2.3}$$

At order α', the action has two terms, i.e., $S = S^{(0)} + \alpha' S^{(1)}$. The invariance then means

$$\begin{aligned} S^{(1)} &\xrightarrow{T^{(0)}} S^{(1)} + \delta S \,, \\ S^{(0)} &\xrightarrow{T^{(1)}} -\delta S \,. \end{aligned} \tag{2.4}$$

At order $(\alpha')^2$, the action has three terms, i.e., $S = S^{(0)} + \alpha' S^{(1)} + (\alpha')^2 S^{(2)}$ and again the invariance means that

$$\begin{aligned} S^{(2)} &\xrightarrow{T^{(0)}} S^{(2)} + \delta S_1 + \delta S_2 \,, \\ S^{(1)} &\xrightarrow{T^{(1)}} -\delta S_1 \,, \\ S^{(0)} &\xrightarrow{T^{(2)}} -\delta S_2 \,. \end{aligned} \tag{2.5}$$

Similarly for the action at higher orders of α'.

To study the α' corrections to the T-duality transformations, it is convenient to introduce the following new fields:

$$\begin{aligned} &g_{\mu\nu} \equiv G_{\mu\nu} - G^{yy} G_{\mu y} G_{\nu y} \,, \qquad b_{\mu\nu} \equiv B_{\mu\nu} - \frac{1}{2} G^{yy} (B_{\mu y} G_{\nu y} - B_{\nu y} G_{\mu y}) \,, \\ &\bar{\phi} \equiv \phi - \frac{1}{4} \ln G_{yy} \,, \quad V_\mu \equiv \sqrt{G^{yy}} G_{\mu y} \,, \quad W_\mu \equiv \sqrt{G^{yy}} B_{\mu y} \,, \quad \sigma \equiv \frac{1}{2} \ln G_{yy} \,. \end{aligned} \tag{2.6}$$

In terms of these fields, the T-duality operator at leading α'-order $T^{(0)}$ simplifies as

$$\sigma \xrightarrow{T^{(0)}} -\sigma \,, \qquad V_\mu \xrightarrow{T^{(0)}} W_\mu \,, \qquad W_\mu \xrightarrow{T^{(0)}} V_\mu \,. \tag{2.7}$$

The fields $g_{\mu\nu}$, $b_{\mu\nu}$ and $\bar{\phi}$ remain invariant under the T-duality. Up to the order of α', the T-duality operator has been found in [16] to be

$$\begin{aligned} \sigma &\xrightarrow{T} -\sigma - \frac{\alpha' \lambda_0}{2} \left[8(\nabla\sigma)^2 + e^{2\sigma} V_{\mu\nu} V^{\mu\nu} + e^{-2\sigma} W_{\mu\nu} W^{\mu\nu} \right] , \\ V_\mu &\xrightarrow{T} W_\mu - \alpha' \lambda_0 \left[4 W_{\mu\nu} \nabla^\nu \sigma + e^{2\sigma} H_{\mu\nu\lambda} V^{\nu\lambda} \right] , \\ W_\mu &\xrightarrow{T} V_\mu - \alpha' \lambda_0 \left[4 V_{\mu\nu} \nabla^\nu \sigma - e^{-2\sigma} H_{\mu\nu\lambda} W^{\nu\lambda} \right] , \\ H_{\mu\nu\lambda} &\xrightarrow{T} H_{\mu\nu\lambda} - 12\alpha' \lambda_0 \Big[\nabla_{[\mu} (W_\nu{}^\rho V_{\lambda]\rho}) + V_{[\mu\nu} W_{\lambda]\rho} \nabla^2 \sigma + W_{[\mu\nu} V_{\lambda]\rho} \nabla^2 \sigma \,, \\ &\qquad + \frac{1}{4} e^{2\sigma} V^{\rho\chi} V_{[\mu\nu} H_{\lambda]\rho\chi} - \frac{1}{4} e^{-2\sigma} W^{\rho\chi} W_{[\mu\nu} H_{\lambda]\rho\chi} \Big] \,. \end{aligned} \tag{2.8}$$

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The metric $g_{\mu\nu}$ and $\bar{\phi}$ remain invariant. In above transformations, H is the field strength of the b-field, i.e., $H_{\mu\nu\lambda} = \partial_\mu b_{\nu\lambda} + \partial_\lambda b_{\mu\nu} + \partial_\nu b_{\lambda\mu}$, $V_{\mu\nu}$ is the field strength of V_μ, i.e., $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and $W_{\mu\nu}$ is the field strength of W_μ, i.e., $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$.

One may assume that the metric is a small perturbation around the flat space, i.e., $G_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, and also assume that the dilaton and B-field are small perturbations. Then one can find a perturbative expansion for the T-duality transformation of the graviton, dilaton and B-field.

A perturbative method for finding the T-duality invariant world volume couplings is given in [17]. Let us review this method here. A coupling in general has world volume and normal bundle indices. Suppose we are implementing T-duality along a world volume direction y of a D_p-brane. We first separate the world-volume indices along and orthogonal to y, and then apply the T-duality transformations (2.8). The orthogonal indices are the complete world-volume indices of the T-dual D_{p-1}-brane. However, y in the T-dual theory which is a normal bundle index, is not complete. On the other hand, the normal bundle indices of the original theory are not complete in the T-dual D_{p-1}-brane. They do not include the y index. In a T-duality invariant theory, the index y must be combined with the incomplete normal bundle indices to make them complete. This last step can be done by rewriting an incomplete normal bundle index as a complete normal bundle index minus the y-index. Then all the couplings which have the y-index must be canceled. If the world volume couplings with the y-index are not canceled, one should then add new couplings to the original theory to be able to cancel them.

3 T-dual completion of the gravity couplings

In this section we are going to apply the perturbative method outlined in the previous section to extend the brane action (1.12) to be invariant under the T-duality transformations (2.8). Let us first review how the gravity couplings (1.12) have been found in [15]. The authors of [15] consider various gravity couplings at order α' with arbitrary coefficients, i.e.,

$$\frac{\alpha' T_p}{2}\int d^{p+1}x e^{-\phi}\sqrt{-\tilde{G}}\Big[\beta_0\tilde{R}+\beta_1\perp_{\mu\nu}\Omega^\mu{}_a{}^a\Omega^\nu{}_b{}^b+\beta_2\perp_{\mu\nu}\Omega^\mu{}_{ab}\Omega^{\nu ab}+\beta_3\perp^{\mu\nu}R_{\mu\nu}+\beta_4\perp^{\mu\nu}\perp^{\alpha\beta}R_{\mu\alpha\nu\beta}\Big].$$

There is another gravity coupling at this order which is given by the bulk scalar curvature R evaluated on the D-brane. However, it is not independent according to the Gauss identity (see e.g., [31])

$$\tilde{R} - R + 2\perp^{\mu\nu}R_{\mu\nu} - \perp^{\mu\nu}\perp^{\alpha\beta}R_{\mu\alpha\nu\beta} - \perp_{\mu\nu}(\Omega^\mu{}_a{}^a\Omega^\nu{}_b{}^b - \Omega^\mu{}_{ab}\Omega^{\nu ab}) = 0\,. \tag{3.1}$$

Using the fact that the α' terms should not change the propagator of the transverse scalars, the relation $\beta_1 = -\beta_2$ has been found. Then using these D-brane couplings and the Gauss-Bonnet couplings in the bulk action, the massless poles and the contact terms of the scattering amplitude of two gravitons from D-brane have been calculated. By equating these terms with the corresponding terms in the disk-level S-matrix element of two gravitons, one finds the gravity couplings in (1.12) uniquely.

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Now to extend the gravity couplings (1.12) to be invariant under the T-duality, one has to again consider all the B-field and dilaton couplings at order α' with arbitrary coefficients, i.e.,

$$\frac{\alpha' T_p}{2}\int d^{p+1}x e^{-\phi}\sqrt{-\tilde{G}}\Big[\alpha_1 H_{abc}H^{abc}+\alpha_2\perp^{\alpha\beta}H_{\alpha\mu\nu}H_\beta{}^{\mu\nu}+\alpha_3\perp^{\alpha\beta}\perp^{\mu\nu}H_{\alpha\mu\lambda}H_{\beta\nu}{}^{\lambda}$$
$$+\alpha_4\perp^{\alpha\beta}\perp^{\mu\nu}\perp^{\lambda\sigma}H_{\alpha\mu\lambda}H_{\beta\nu\sigma}+\alpha_5 B_{ab}\nabla_\mu H^{\mu ab}+\sigma_1 G^{\mu\nu}\nabla_\mu\partial_\nu\phi$$
$$+\sigma_2\perp^{\mu\nu}\nabla_\mu\partial_\nu\phi+\sigma_3\tilde{G}^{ab}\partial_a\phi\partial_b\phi+\sigma_4 G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big]. \tag{3.2}$$

Note that the coupling H^2 is not independent of the other B-field couplings that we have considered, i.e.,

$$H^2=H_{abc}H^{abc}+3\perp^{\alpha\beta}H_{\alpha\mu\nu}H_\beta{}^{\mu\nu}-3\perp^{\alpha\beta}\perp^{\mu\nu}H_{\alpha\mu\lambda}H_{\beta\nu}{}^{\lambda}+\perp^{\alpha\beta}\perp^{\mu\nu}\perp^{\lambda\sigma}H_{\alpha\mu\lambda}H_{\beta\nu\sigma}\,. \tag{3.3}$$

Neither the dilaton coupling $\perp^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ nor the coupling $\perp^{\mu\nu}\Omega^\mu{}_a{}^a\partial^\nu\phi$ are independent, i.e.,

$$\begin{aligned}\perp^{\mu\nu}\partial_\mu\phi\partial_\nu\phi &= G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\tilde{G}^{ab}\partial_a\phi\partial_b\phi\,,\\ \tilde{G}^{\mu\nu}\nabla_\mu\partial_\nu\phi &= \tilde{\nabla}^a\partial_a\phi-\perp_{\mu\nu}\Omega^\mu{}_a{}^a\partial^\nu\phi\,.\end{aligned} \tag{3.4}$$

Moreover, up to a total derivative term, one has the identity $\tilde{\nabla}^a\partial_a\phi=\partial^a\phi\partial_a\phi$ in the string frame action. So the terms in (3.2) are all independent B-field and dilaton couplings at order α'.

After adding the couplings (3.2) to the gravity couplings (1.12), we use the perturbation $G_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ and keep the terms with linear- and quadratic-order fields. Following the method outlined in the previous section, we separate the world volume indices along and orthogonal to the Killing index y. Then we use the T-duality transformations (2.7). In the T-dual theory we also complete the transverse indices. Doing all these steps, we have found that it is impossible to cancel the couplings which have the y-index. This confirms the observation made in [16] that the T-duality in the bosonic and the heterotic string theories must receive α' corrections (2.8).

The α'-corrected T-duality transformations (2.8) must be applied on the DBI action (1.10). Since we keep the perturbative fields up to the quadratic order, we have to take into account only the α' corrections in the first line of (2.8). The α' terms in (2.8) start at quadratic order terms, so we have to consider the linear order terms in the DBI action which are

$$S^{(0)}_{D_p}=-T_p\int d^{p+1}x\left(1-\phi+\frac{1}{2}\eta^{ab}h_{ab}+\cdots\right), \tag{3.5}$$

where dots refer to the higher-order terms. Now we have to separate the world volume indices along and orthogonal to y, i.e.,

$$S^{(0)}_{D_p}=-T_p\int d^{p+1}x\left(1-\phi+\frac{1}{2}h_{yy}+\frac{1}{2}\eta^{\tilde{a}\tilde{b}}h_{\tilde{a}\tilde{b}}+\cdots\right), \tag{3.6}$$

where the world volume indices $\tilde{a}$, $\tilde{b}$ are orthogonal to y. Under the T-duality (2.8), $-\phi+\frac{1}{4}h_{yy}$ is invariant, and the α'-corrected T-duality of $h_{\tilde{a}\tilde{b}}$ has no quadratic order term. So

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the T-duality of above action is given by the T-duality transformation of $\frac{1}{4}h_{yy}$ which is

$$\frac{1}{4}h_{yy} \xrightarrow{T} -\frac{1}{4}h_{yy}+\frac{\alpha'}{8}\Big[\partial_\mu h_{yy}\partial_\mu h_{yy}+\partial_\mu h_{\nu y}(\partial_\mu h_{\nu y}-\partial_\nu h_{\mu y})+\partial_\mu B_{\nu y}(\partial_\mu B_{\nu y}-\partial_\nu B_{\mu y})\Big]. \quad (3.7)$$

Taking into account the above y-dependent terms at order α' and ignoring some total derivative terms, one is able to find the T-dual completion of the gravity couplings. With the assistance of the computer algebra system, "Cadabra", [32, 33], we have found the result in (1.15).

4 Couplings in the Einstein frame

To compare the actions (1.15) or (1.17) with the S-matrix elements in string theory, we have to transform them to the Einstein frame $G^s_{\mu\nu} = e^{\gamma\phi}G^E_{\mu\nu}$ where $\gamma = 4/(D-2)$. Since the S-matrix elements are independent of field redefinitions, we use the simplified actions in which the field redefinitions have been used. We first transform the bulk actions to the Einstein frame.

For those terms which have no derivative of the metric, the transformation gives only an overall dilaton factor. In other cases, there are some extra terms involving the derivative of the dilaton, e.g., the transformation of scalar curvature is

$$R \Longrightarrow e^{-\gamma\phi}\Big[R-\gamma(D-1)\nabla^2\phi-\frac{\gamma^2}{4}(D-1)(D-2)(\partial\phi)^2\Big], \quad (4.1)$$

where on the right hand side the metric is in the Einstein frame. This transforms the string frame action (1.3) to the following action in the Einstein frame:

$$S^{(0)} = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}\Big[R-\gamma(\partial\phi)^2-\frac{1}{12}e^{-2\gamma\phi}H^2\Big], \quad (4.2)$$

where the total derivative term has been dropped.

To transform the α'-order terms in (1.9), one needs the following relations between the two frames:

$$\nabla_\mu\partial_\nu\phi \Longrightarrow \nabla_\mu\partial_\nu\phi-\frac{\gamma}{2}\Big(2\partial_\mu\phi\partial_\nu\phi-G_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi\Big), \quad (4.3)$$

$$R_{\mu\nu\alpha\beta} \Longrightarrow e^{\gamma\phi}R_{\mu\nu\alpha\beta}+2\gamma e^{\gamma\phi}\Big[G_{[\mu[\beta}\nabla_{\nu]}\partial_{\alpha]}\phi+\frac{\gamma}{2}G_{[\mu[\alpha}\partial_{\nu]}\phi\partial_{\beta]}\phi+\frac{\gamma}{4}G_{[\mu[\beta}G_{\nu]\alpha]}\partial_\lambda\phi\partial^\lambda\phi\Big].$$

Using these transformations, one finds the transformation of various terms in the ac-

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tion (1.9), i.e.,

$$
\begin{aligned}
R_{\mu\nu\lambda\rho}H^{\mu\nu\alpha}H^{\lambda\rho}{}_{\alpha} &\Longrightarrow e^{-4\gamma\phi}\Big[R_{\mu\nu\lambda\rho}H^{\mu\nu\alpha}H^{\lambda\rho}{}_{\alpha}-2\gamma\nabla^{\mu}\partial^{\nu}\phi H^2_{\mu\nu}+\gamma^2\partial^{\mu}\phi\partial^{\nu}\phi H^2_{\mu\nu}-\frac{1}{2}\gamma^2(\partial\phi)^2H^2\Big],\\
\nabla^2\phi &\Longrightarrow e^{-\gamma\phi}\Big[\nabla^2\phi+2(\partial\phi)^2\Big],\\
R^2_{GB} &\Longrightarrow e^{-2\gamma\phi}\Big[R^2_{GB}+4\gamma(D-3)\Big(R_{\mu\nu}-\frac{1}{2}G_{\mu\nu}R\Big)\nabla^{\mu}\partial^{\nu}\phi\\
&\quad +4\gamma(D-3)\Big((\nabla^2\phi)^2-\nabla_{\mu}\partial_{\nu}\phi\nabla^{\mu}\partial^{\nu}\phi\Big)\\
&\quad -\frac{\gamma^2}{2}(D-3)\Big((D-4)(\partial\phi)^2R+4\partial_{\mu}\phi\partial_{\nu}\phi R^{\mu\nu}\Big)\\
&\quad +2\gamma^2(D-3)\Big((D-3)\nabla^2\phi(\partial\phi)^2+2\nabla_{\mu}\partial_{\nu}\phi\partial^{\mu}\phi\partial^{\nu}\phi\Big)\\
&\quad +\frac{\gamma^3}{4}(D-1)(D-3)(D-4)(\partial\phi)^4\Big].
\end{aligned}
\tag{4.4}
$$

In the string frame, the Gauss-Bonnet term does not change the graviton propagator, so one expects that it does not change the graviton and dilaton propagators when transforming it to the Einstein frame. In fact one can easily show that the quadratic order terms in the Gauss-Bonnet can be written as cubic order terms. To this end, consider the following identities:

$$
\begin{aligned}
\Big(R_{\mu\nu}-\frac{1}{2}G_{\mu\nu}R\Big)\nabla^{\mu}\partial^{\nu}\phi &= \nabla^{\mu}[\Big(R_{\mu\nu}-\frac{1}{2}G_{\mu\nu}R\Big)\partial^{\nu}\phi],\\
(\nabla^2\phi)^2-\nabla_{\mu}\partial_{\nu}\phi\nabla^{\mu}\partial^{\nu}\phi &= \nabla_{\mu}[\partial^{\mu}\phi\nabla^2\phi]-\nabla_{\mu}[\partial_{\nu}\phi\nabla^{\mu}\partial^{\nu}\phi]+\partial_{\mu}\phi\partial_{\nu}\phi R^{\mu\nu},
\end{aligned}
\tag{4.5}
$$

where in the first equation we have used the Bianchi identity $\nabla^{\mu}R_{\mu\nu}-\frac{1}{2}\nabla_{\nu}R=0$, and in the second equation we have used the (non)commutative property of the covariant derivative, i.e., $[\nabla_{\mu},\nabla_{\nu}]A^{\alpha}=R^{\alpha}{}_{\beta\mu\nu}A^{\beta}$ and $[\nabla_{\mu},\nabla_{\nu}]\phi=0$. The above total derivative terms, however, can not be dropped because of the overall dilaton factor $e^{-\gamma\phi}$ in the Einstein frame. Using the integration by part, one can write the Gauss-Bonnet term as

$$
\begin{aligned}
e^{-2\phi}\sqrt{-G}R^2_{GB} \Longrightarrow e^{-\gamma\phi}\sqrt{-G}\Big[&R^2_{GB}+\gamma^2D(D-3)\partial_{\mu}\phi\partial_{\nu}\phi\Big(R^{\mu\nu}-\frac{1}{2}G^{\mu\nu}R\Big)\\
&+2\gamma^2(D-1)(D-3)\nabla^2\phi(\partial\phi)^2\\
&+\frac{\gamma^3}{4}(D-1)(D-3)(D-4)(\partial\phi)^4\Big].
\end{aligned}
\tag{4.6}
$$

It is interesting to note that the dilaton terms vanish in three dimensions. This is consistent with the fact that the Gauss-Bonnet term is zero in three dimensions.

Using the above transformations, one finds that the action (1.9) transforms to the following action in the Einstein frame:

$$
\begin{aligned}
S^{(1)}=\frac{\lambda_0}{2\kappa^2}\int d^Dx\, e^{-\gamma\phi}\sqrt{-G}\Big[&-R^2_{GB}+e^{-2\gamma\phi}\Big(\frac{1}{2}R_{\mu\nu\lambda\rho}H^{\mu\nu\alpha}H^{\lambda\rho}{}_{\alpha}-\gamma\nabla^{\mu}\partial^{\nu}\phi H^2_{\mu\nu}\Big)\\
&-\gamma^2D(D-3)\partial_{\mu}\phi\partial_{\nu}\phi\Big(R^{\mu\nu}-\frac{1}{2}G^{\mu\nu}R\Big)-\Big(2\gamma^2(D-1)(D-3)-32\Big)\nabla^2\phi(\partial\phi)^2+\cdots\Big],
\end{aligned}
\tag{4.7}
$$

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where dots represent the quartic order terms. The above action can be simplified by using field redefinition in the Einstein frame. Since we are interested in actions which can be compared with the S-matrix elements, we are not going to use the field redefinitions which change the Gauss-Bonnet combination of the curvature squared. Moreover, the Riemann curvature in the second term above can not be changed under the field redefinition. All other cubic terms in the above action can be converted to some quartic order terms by an appropriate field redefinition.

The dilaton and the metric variations of the action (4.2) are

$$\frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}\Big[\delta\phi\Big(2\gamma\nabla^2\phi+\frac{\gamma}{6}e^{-2\gamma\phi}H^2\Big)-\delta G_{\mu\nu}\Big(R^{\mu\nu}-\gamma\partial^\mu\phi\partial^\nu\phi \\ -\frac{1}{4}e^{-2\gamma\phi}H^{2\,\mu\nu}-\frac{1}{2}G^{\mu\nu}[R-\gamma(\partial\phi)^2-\frac{1}{12}e^{-2\gamma\phi}H^2]\Big)\Big]\,.$$

Using the following field redefinitions:

$$\begin{aligned}\delta\phi &= \alpha'\lambda_0 e^{-\gamma\phi}\Big(\gamma(\partial\phi)^2+\frac{1}{12}e^{-2\gamma\phi}H^2\Big)\,,\\ \delta G_{\mu\nu} &= -\alpha'\lambda_0 e^{-\gamma\phi}\Big(\gamma^2[D(D-3)+4]\partial_\mu\phi\partial_\nu\phi-4\gamma\nabla_\mu\partial_\nu\phi\Big)\,,\end{aligned} \tag{4.8}$$

one finds that the Einstein frame action (4.7) converts to the following standard form of the bulk action at order α':

$$S^{(1)}=\frac{\lambda_0}{2\kappa^2}\int d^Dx\, e^{-\gamma\phi}\sqrt{-G}\Big[-R^2_{GB}+\frac{1}{2}e^{-2\gamma\phi}R_{\mu\nu\lambda\rho}H^{\mu\nu\alpha}H^{\lambda\rho}{}_\alpha-\frac{\gamma^2(D-4)}{(D-2)}(\partial\phi)^4 \\ -e^{-2\gamma\phi}\Big(\gamma\partial^\mu\phi\partial^\nu\phi H^2_{\mu\nu}-\frac{\gamma}{6}(\partial\phi)^2H^2\Big)-e^{-4\gamma\phi}\Big(\frac{1}{8}H^2_{\mu\nu}H^{2\,\mu\nu} \\ +\frac{1}{24}H_{\mu\nu\lambda}H^\nu{}_{\rho\alpha}H^{\rho\sigma\lambda}H_\sigma{}^{\mu\alpha}-\frac{(D+6)}{144(D-2)}(H^2)^2\Big)\Big]\,. \tag{4.9}$$

These terms are exactly the couplings that have been found in [12] from the corresponding sphere-level S-matrix elements.

We now transform the brane actions to the Einstein frame. The string frame DBI action (1.10) transforms to the following action:

$$S^{(0)}_{D_p}=-T_p\int d^{p+1}x e^{-\phi\left[1-\gamma(p+1)/2\right]}\sqrt{-det(\tilde{G}_{ab}+e^{-\gamma\phi}\widetilde{B}_{ab})}\,, \tag{4.10}$$

To transform the D-brane action (1.17) to the Einstein frame, one needs the following relations between the two frames:

$$\begin{aligned}\tilde{R} &\Longrightarrow e^{-\gamma\phi}\Big[\tilde{R}-\gamma p\tilde{\nabla}^2\phi-\frac{\gamma^2}{4}p(p-1)\partial_a\phi\partial^a\phi\Big]\,,\\ \Omega^\mu{}_{ab} &\Longrightarrow \Omega^\mu{}_{ab}+\frac{\gamma}{2}\Big(\partial_a\phi\partial_bX^\mu+\partial_b\phi\partial_aX^\mu-\tilde{G}_{ab}\partial^\mu\phi\Big)\,,\\ \Omega^\mu{}_a{}^a &\Longrightarrow e^{-\gamma\phi}\Big[\Omega^\mu{}_a{}^a+\frac{\gamma}{2}\Big(2\partial_a\phi\partial^aX^\mu-(p+1)\partial^\mu\phi\Big)\Big]\,,\end{aligned} \tag{4.11}$$

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where we have also used the relation $\tilde{G}_{ab}\tilde{G}^{ba} = p+1$. Using the above transformations, one can find the brane action corresponding to the bulk action (4.7). However, to find the D-brane action which is corresponding to the bulk action (4.9), we have to add to the D-brane action those terms which are coming from the field redefinition (4.8). Using the relation $\partial_a X^\mu \perp_{\mu\nu} = 0$, one finds

$$\begin{aligned} S^{(1)}_{D_p} = & -\frac{T_p}{2}\int d^{p+1}x e^{-\phi\left[1-\gamma(p-1)/2\right]}\sqrt{-\tilde{G}}\Big[\tilde{R} + \perp_{\mu\nu}\Big(\Omega^\mu{}_a{}^a\Omega^\nu{}_b{}^b - \Omega^\mu{}_{ab}\Omega^{\nu\,ab}\Big) \\ & + e^{-2\gamma\phi}\Big(\frac{1}{24}H_{abc}H^{abc} + \frac{1}{8}\perp^{\alpha\beta}H_{\alpha\mu\nu}H_\beta{}^{\mu\nu} - \frac{3}{8}\perp^{\alpha\beta}\perp^{\mu\nu}H_{\alpha\mu\lambda}H_{\beta\nu}{}^\lambda \\ & + \frac{5}{24}\perp^{\alpha\beta}\perp^{\mu\nu}\perp^{\lambda\sigma}H_{\alpha\mu\lambda}H_{\beta\nu\sigma}\Big) + \Big(\gamma(p+1)\Big(\frac{\gamma p}{4}-1\Big)+1\Big)\perp_{\mu\nu}\partial^\mu\phi\partial^\nu\phi \\ & - (\gamma p - 2)\perp_{\mu\nu}\Omega^\mu{}_a{}^a\partial^\nu\phi - \Big(3 + \frac{\gamma^2}{4}p(p-1)\Big)\partial_a\phi\partial^a\phi - \gamma p\tilde{\nabla}^2\phi + \Delta\Big], \end{aligned} \tag{4.12}$$

where Δ is coming from the field redefinition (4.8) which is

$$\begin{aligned} \Delta = & \frac{1}{2}\Big(1 - \frac{\gamma}{2}(p+1)\Big)\Big(\gamma(\perp_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + \partial_a\phi\partial^a\phi) + \frac{1}{12}H^2 e^{-2\gamma\phi}\Big) \\ & + \frac{\gamma^2}{4}\Big(D(D-3)+4\Big)\partial_a\phi\partial^a\phi - \gamma\Big(\tilde{\nabla}^2\phi - \perp_{\mu\nu}\Omega^\mu{}_a{}^a\partial^\nu\phi\Big). \end{aligned}$$

Using the identity (3.3), one can easily observe that the B-field couplings in the above action are exactly the couplings that have been found in [15] from the corresponding disk-level S-matrix elements.

To check the dilaton couplings with the S-matrix elements, one may use once more field redefinition. Since we have already used the field redefinition on the bulk fields to convert the bulk action to the standard form (4.9), we are not allowed anymore to use the field redefinition on the bulk fields. However, we can still use field redefinition on the brane fields. The variation of the DBI action (4.10) under $\perp_{\mu\nu}\delta X^\nu$ is

$$-T_p\int d^{p+1}x\, e^{-\phi\left[1-\gamma(p+1)/2\right]}\sqrt{-\tilde{G}}\Big[\perp_{\mu\nu}\delta X^\mu\Big(-\left[1-\gamma(p+1)/2\right]\partial^\nu\phi - \Omega^\nu{}_a{}^a\Big)\Big]. \tag{4.13}$$

To convert the coupling with structure $\perp_{\mu\nu}\Omega^\mu{}_a{}^a\partial^\nu\phi$ to the coupling $\perp_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$, one can use the following field redefinition:

$$\delta X^\mu = -\frac{\alpha'}{2}(2+\gamma-\gamma p)e^{-\gamma\phi}\partial^\mu\phi\,. \tag{4.14}$$

Using also the integration by part to convert the coupling $\tilde{\nabla}^2\phi$ to the coupling with structure $\partial_a\phi\partial^a\phi$, one finds the following result:

$$\begin{aligned} S^{(1)}_{D_p} = & -\frac{T_p}{2}\int d^{p+1}x e^{-\phi\left[1-\gamma(p-1)/2\right]}\sqrt{-\tilde{G}}\Big[\tilde{R} + \perp_{\mu\nu}\Big(\Omega^\mu{}_a{}^a\Omega^\nu{}_b{}^b - \Omega^\mu{}_{ab}\Omega^{\nu\,ab}\Big) \\ & + e^{-2\gamma\phi}\Big(\frac{1}{24}H_{abc}H^{abc} + \frac{1}{8}\perp^{\alpha\beta}H_{\alpha\mu\nu}H_\beta{}^{\mu\nu} - \frac{3}{8}\perp^{\alpha\beta}\perp^{\mu\nu}H_{\alpha\mu\lambda}H_{\beta\nu}{}^\lambda \\ & + \frac{5}{24}\perp^{\alpha\beta}\perp^{\mu\nu}\perp^{\lambda\sigma}H_{\alpha\mu\lambda}H_{\beta\nu\sigma} + \frac{1}{24}\Big[1 - \frac{\gamma}{2}\Big(p+1\Big)\Big]H^2\Big) \\ & - \frac{\gamma^2}{16}\left[(D-2p-2)^2 + 2D - 8\right]\Big(\partial_a\phi\partial^a\phi - \perp_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\Big)\Big]. \end{aligned} \tag{4.15}$$

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This is the D-brane action which is corresponding to the bulk action (4.9). In fact, the action (4.9) has been used in [15] to calculate the massless t-channel pole of the scattering amplitude of two B-fields from D-brane. By subtracting this pole from the corresponding disk-level S-matrix elements at order α', the B-field couplings have been found in [15] which are exactly the same as the B-field couplings in above action. The couplings of one graviton and one dilaton that have been found in [15] are also consistent with the above action. However, the couplings of two dilatons in above action are not exactly the same as the couplings that have been found in [15]. Using the S-matrix element calculated in [15], we have reexamined the calculation of the couplings of two dilatons at α'-order and we have found an exact agreement with the above action. The details of this calculation appear in the appendix A. This completes our illustration of perfect agreement between the manifestly T-duality invariant D-brane action (1.15) that we have found in this paper and the S-matrix calculations.

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A S-matrix element of two dilatons

In this appendix we are going to show that the world volume couplings of two dilatons in the Einstein frame that we have found in (4.15) are reproduced by corresponding string theory S-matrix element. The disk-level scattering amplitude of two closed strings off a D-brane in bosonic string theory is given by the following expression [15]:

$$\begin{aligned} A \sim\ & d_1 B(-t/2, 1+2s) + d_2 B(-t/2, 2s) - d_3 B(1-t/2, 2s) + d_4 B(1-t/2, 1+2s) \\ & + d_5 B(-1-t/2, 1+2s) + d_6 B(1-t/2, -1+2s) + d_7 B(-1-t/2, -1+2s) \\ & - d_8 B(-t/2, -1+2s) - d_9 B(2-t/2, -1+2s) + d_{10} B(3-t/2, -1+2s)\,, \end{aligned} \tag{A.1}$$

where $t = -\alpha' k_{1\mu} k_2^\mu$ and $s = -\alpha' k_{1a} k_2^a/2$. In above amplitude, $d_1, \cdots, d_{10}$ are some kinematic factors that depend on momenta and the polarization of the external states. We refer the interested readers to [15] for the explicit form of these factors.

The dilaton amplitude is given by replacing the polarization tensors in d's with the following expression:

$$\varepsilon_{\mu\nu} = \frac{1}{\sqrt{D-2}} (\eta_{\mu\nu} - k_\mu \ell_\nu - k_\nu \ell_\mu)\,, \tag{A.2}$$

where the auxiliary vector ℓ_μ satisfying $\ell \cdot k = 1$, must be canceled in the whole amplitude.

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We have done this replacement and found the following result:

$$
\begin{aligned}
A \sim\ & (D-t-4)B\Big(-\frac{t}{2}-1,1+2s\Big)+\frac{\Gamma(2s-1)\Gamma(1-\frac{t}{2})}{4t\Gamma(2s-\frac{t}{2}+2)}\Big[\\
& +16s^2\Big[t\left(D-4t\mathrm{Tr}V+t(t+7)+(\mathrm{Tr}V)^2-4\right)-4\Big] \\
& -8st\Big[D+t^2(2-\mathrm{Tr}V)+t(1-\mathrm{Tr}V)^2+\mathrm{Tr}V(4-\mathrm{Tr}V)-8\Big] \\
& +256s^4(t+1)-128s^3t(t-\mathrm{Tr}V+2)+(t-2)t^2(2-\mathrm{Tr}V)^2\Big]\,. \qquad \text{(A.3)}
\end{aligned}
$$

The auxiliary vector has been canceled, as expected. In above amplitude the matrix $V_{\mu\nu} = -\mathrm{diag}(-1,1,\cdots,1,-1,-1,\cdots,-1)$.

To study the above amplitude at low energy, we have to expend it at low energy, i.e., $\alpha' \to 0$. The result is

$$
\begin{aligned}
A \sim\ & \frac{(2-\mathrm{Tr}V)^2t^2-4st(2-\mathrm{Tr}V)^2+16s^2(D-2)}{4st} \\
& +\frac{1}{2}(-4s+t)[(\mathrm{Tr}V)^2+2D-8]+O(\alpha'^2)\,.
\end{aligned}
$$

The terms in the first line are at order α'^0 which are reproduced by the couplings in the DBI action (4.10) and bulk action (4.2), [15]. The terms in the second line are only contact terms which must be reproduced by the effective action at order α'. It is easy to verify that these terms are exactly reproduced by the dilaton couplings in the last line of (4.15).

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