Journal of Advanced Research in Pure Mathematics
Online ISSN: 1943-2380

Vol. 5, Issue. 4, 2013, pp. 91-95
doi: 10.5373/jarpm.1627.112812

A remark on the capability of finite p-groups

Mohsen Parvizi[1,*], Peyman Niroomand[2]

[1] *Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.*
[2] *School of Mathematics and Computer Science, Damghan University, Damghan, Iran.*

Abstract. In this paper, we classify all capable finite p-groups of order p^n with derived subgroup of order p and G/G' elementary abelian of rank $n-1$.

Keywords: Finite p-groups; Capable groups; Schur multiplier.

Mathematics Subject Classification 2010: 20F99.

1 Motivation

The concept of capability was appeared in works of P. Hall in classifying groups into isoclinism classes. It was important for him to decide when for a group G, there exists a group H with $G \cong H/Z(H)$. Hall and Senior [5] called such groups capable. The problem of finding capable groups, or determining necessary and sufficient conditions for a group or a class of groups to be capable, is therefore worthy enough to be considered. Several works had been done on this topic. For instance in [2] capable groups in the class of the direct sums of cyclic groups are completely determined. As a result, all finitely generated abelian groups which are capable were classified. For nonabelian groups the class of nilpotent groups is a suitable candidate to study. The reason is the variety of tools which can be used for nilpotent groups. Ellis in [4] proved that a finite nilpotent group is capable if and only if all of its Sylow p-subgroups are capable. This suggest to study the class of p-groups. In [8] and [9] the capability of 2-generator 2-groups is considered. In this paper we study the capability of p-groups with derived subgroups of order p and elementary abelianizations. It is known that in the case $|G'| = p$, capability of G implies $[G : Z(G)] = p^2$ (see [6]). We will prove this result in a different way in the special case for which G/G' is elementary abelian p-group after Theorem 3.2.

*Correspondence to: Mohsen Parvizi, Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran. Email: `parvizi@math.um.ac.ir`
†Received: 28 November 2012, revised: 22 February 2013, accepted: 6 May 2013.

2 Preliminaries

This section contains definitions, notations and theorems which are used in main results. We assume that the notion of Schur multiplier, which is denoted by $\mathcal{M}(G)$ for a group G is known, also we use the notion of epicenter and exterior center of a group without defining them. Epicenter of a group G which is denoted by $Z^*(G)$, was introduced by Beyl, Felgner, and Schmid in [3]. They showed a necessary and sufficient condition for a group G to be capable is $Z^*(G) = 1$. The following theorems are used in the rest.

Theorem 2.1. (See [7, Theorem 2.5.10]) Let G be a finite group and N be a central subgroup of G. Then $N \subseteq Z^*(G)$ if and only if the natural map $\mathcal{M}(G) \longrightarrow \mathcal{M}(G/N)$ is injective.

Theorem 2.2. (See [7, Theorem 2.2.10]) Let A and B be finite groups then

$$\mathcal{M}(A \times B) \cong \mathcal{M}(A) \times \mathcal{M}(B) \times \frac{A}{A'} \otimes \frac{B}{B'}.$$

Theorem 2.3. (See [7, Theorem 2.5.6 (i)]) Let G be a finite group and N be a central subgroup of G. Then the following sequence is exact

$$\mathcal{M}(G) \longrightarrow \mathcal{M}(\frac{G}{N}) \longrightarrow N \cap G' \longrightarrow 1.$$

The following lemma is a conclusion of Theorem 2.3 and used in the proof of the main theorem.

Lemma 2.4. Let G be a finite p-group and $N \subseteq Z(G) \cap G'$ be a subgroup of order p. If $|\mathcal{M}(G/N)| = p\ |\mathcal{M}(G)|$ then $N \subseteq Z^*(G)$.

Proof. Using Theorems 2.1 and 2.3 it is enough to show that $\mathcal{M}(G) \longrightarrow \mathcal{M}(G/N)$ has trivial kernel. Let α and β denote the homomorphisms $\mathcal{M}(G) \longrightarrow \mathcal{M}(G/N)$ and $\mathcal{M}(G/N) \longrightarrow N \cap G'$, respectively. Since $|N| = p$, we have $|\text{Ker } \beta| = |\mathcal{M}(G/N)|/p$ which is equal to $|\mathcal{M}(G)|$. Now Theorem 2.3 implies Ker $\alpha = 1$ as required. □

The following lemma is a consequence of [7, Corollary 2.5.3], in where $\phi(G)$ denotes the Frattini subgroup.

Lemma 2.5. Let G be a finite p-group then

$$|\mathcal{M}(\frac{G}{\phi(G)})| \leq |\mathcal{M}(G)||G'|$$

3 Main results

Let G be a group of order p^n and G', the derived subgroup of G is of order p, and G/G' is elementary abelian. It is easy to see that in this case $exp(G) \leq p^2$. Although the capability of all extra special p-groups was considered in [3], so all the groups we consider are not extra special. By [10, Lemma 2.1] we have $G = H \cdot Z(G)$ (the central product of H and $Z(G)$) in which H is an extra-special p-group. We know that $G' \subseteq Z(G)$ and G' is cyclic of order p, so G' may be a direct summand of $Z(G)$. The following theorem gives the structure of G depending on the way G' is embedded in $Z(G)$. We consider only groups with noncyclic centers, the groups with cyclic centers are considered in Theorem 3.4.

Theorem 3.1. Let $|G| = p^n$ and $Z(G)$ is not cyclic then

(i) if for some K, $Z(G) = G' \times K$ then $G = H \times K$;

(ii) if G' is not a direct summand of $Z(G)$ then $G = \big(H \cdot \mathbb{Z}_{p^2}\big) \times K$ in which $Z(G) = \mathbb{Z}_{p^2} \times K$ and $G' \subseteq \mathbb{Z}_{p^2}$.

Proof. (i) Since $G = H \cdot Z(G)$ and $H \cap Z(G) = G'$, so $G = H \times K$.
(ii) The proof is similar to the pervious part. □

We state the main theorem of this paper as follows:

Theorem 3.2. Let $G = H \cdot Z(G)$ be a p-group of order p^n with derived subgroup of order p and G/G' elementary abelian of order p^{n-1}, then G is capable if and only if H is capable and G' is a direct summand of $Z(G)$.

Remark 3.3. The above theorem shows that if G is a finite p-group with G' of order p and elementary abelian abelianization, The capability of G implies $[G : Z(G)] = p^2$. So in this case the result of Beyl and Tappe which was mentioned by Isaacs [6] can be proved in a quite different way.

The proof of the Main Theorem is partitioned into some cases as follows. In the rest we assume that G is of order p^n and G/G' is elementary abelian of order p^{n-1}.

Theorem 3.4. Let G be a p-group of order p^n with derived subgroup of order p and G/G' elementary abelian of order p^{n-1}. If $Z(G)$ is cyclic, then G is not capable.

Proof. Since G/G' is an elementary abelian p-group, we have $\phi(G) = G'$. Now using Lemma 2.5 and Main Theorems of [10, 11], we have

$$p^{\frac{1}{2}(n-1)(n-2)-1} \leq |\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}.$$

Again using Main Theorems of [10, 11], we deduce that

$$|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)-1},$$

so the following sequence is exact

$$1 \longrightarrow \mathcal{M}(G) \longrightarrow \mathcal{M}(\frac{G}{G'}) \longrightarrow G' \longrightarrow 1$$

which implies $G' \subseteq Z^*(G)$. □

In the following theorem we consider the groups satisfying in the condition in part (i) of Theorem 3.1.

Theorem 3.5. Let $G = H \times K$ where K is the complement of G' in $Z(G)$. Then G is capable if and only if H is capable.

Proof. We have $Z^*(G) \subseteq G' = H'$.

If H is capable, for $p \neq 2$, H is the extra-special p-group of order p^3 and exponent p and that $|\mathcal{M}(H)| = p^2$. We do the job by proving $H' \nsubseteq Z^*(G)$. To do this we use Theorems 2.1 and 2.3. The sequence

$$\mathcal{M}(G) \longrightarrow \mathcal{M}(\frac{G}{H'}) \longrightarrow H' \longrightarrow 1$$

is exact, but by Theorem 2.2 we have $|\mathcal{M}(G)| = p\ |\mathcal{M}(G/H')|$ so

$$1 \neq |\text{Ker}\ (\mathcal{M}(G) \longrightarrow \mathcal{M}(G/H'))|$$

and the result holds. For $p = 2$, H is isomorphic to the dihedral group of order 8, and a similar technique shows the result.

If H is not capable, it can be either an extra-special p-group of order p^3 and exponent p^2, or an extra-special p-group of order p^{2m+1} with $m > 1$ which multipliers are trivial and of order p^{2m^2-m-1}, respectively. For H of order p^3 a similar argument to that of the first case shows that

$$\mathcal{M}(G) \longrightarrow \mathcal{M}(\frac{G}{H'})$$

is injective and so $H' \subseteq Z^*(G)$. On the other hand if H is of order p^{2m+1} for $m > 1$, using Theorem 2.2 the following sequence is exact.

$$1 \longrightarrow \mathcal{M}(G) \longrightarrow \mathcal{M}(\frac{G}{H'}) \longrightarrow H' \longrightarrow 1$$

Therefore G is not capable.

□

Now the only case which needs to be discussed is the groups satisfying in the condition in part (ii) of Theorem 3.1.

Theorem 3.6. Let G be a p-group of order p^n with derived subgroup of order p and G/G' elementary abelian of order p^{n-1} with noncyclic $Z(G)$ and G' is not a direct summand of $Z(G)$, then G is not capable.

Proof. In this case we have $G = H \cdot \mathbb{Z}_{p^2} \times K$ where H is an extra special p-group of order p^{2m+1} and K is elementary abelian of order p^{n-2m-2}. By Theorem 2.4, it is enough to show that $|\mathcal{M}(G/(H \cdot \mathbb{Z}_{p^2})')| = p\ |\mathcal{M}(G)|$. Using Theorem 2.2 we have $\mathcal{M}(G) = \mathcal{M}(H \cdot \mathbb{Z}_{p^2}) \times \mathcal{M}(K) \times (H \cdot \mathbb{Z}_{p^2} \otimes K)$. But $|\mathcal{M}(H \cdot \mathbb{Z}_{p^2})| = p^{\frac{1}{2}(2m)(2m+1)}$ due to Theorem 3.4, $|\mathcal{M}(K)| = p^{\frac{1}{2}(n-2m-2)(n-2m-3)}$ and $|H \cdot \mathbb{Z}_{p^2} \otimes K| = p^{(2m+1)(n-2m-2)}$. After some computations we have $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)-1}$. On the other hand $|\mathcal{M}(G/(H \cdot \mathbb{Z}_{p^2})')| = p^{\frac{1}{2}(n-1)(n-2)}$, so the result follows. □

References

[1] M.R. Bacon, L.-C. Kappe. On capable p-groups of nilpotency class two. Ill. J. Math., 2003, 47: 49 - 62.

[2] R. Baer. Groups with preassigned central and central quotient group. Trans. Amer. Math. Soc., 1938, 44: 387 - 412, doi: 10.1090/S0002-9947-1938-1501973-3.

[3] F.R. Beyl, U. Felgner, and P. Schmid. On groups occurring as center factor groups. J. Algebra, 1979, 61: 161 - 177, doi: 10.1016/0021-8693(79)90311-9.

[4] G. Ellis. Tensor products and q-cross modules. J. Lond. Math. Soc., 1995, 51(2): 241 - 258, doi: 10.1112/jlms/51.2.243.

[5] M. Hall, Jr., J.K. Senior. The Groups of Order 2^n ($n \leq 6$). Macmillan Co., New York, 1964.

[6] M.I. Isaacs. Derived subgroups and centers of capable groups. Proc. Amer. Math. Soc., 2001, 129: 2853 - 2859.

[7] G. Karpilovsky. The Schur multiplier. London Math. Soc. Monogr., (N.S.) 2, 1987.

[8] A. Magidin. Capable 2-generator 2-groups of class two. Comm. Algebra, 2006, 34: 2183 - 2193, doi: 10.1080/00927870600549717.

[9] A. Magidin, R.F. Morse. Certain homological functors of 2-generator p-group of class 2. Contemp. Math., 2010, 511: 127 - 166.

[10] P. Niroomand. On the order of Schur multiplier of non-abelian p-groups. J. Algebra, 2009, 322: 4479 - 4482, doi: 10.1016/j.jalgebra.2009.09.030.

[11] P. Niroomand. A note on the Schur multiplier of groups of prime power order. Ric. Mat., 2012, 61(2): 341 - 346, doi: 10.1007/s11587-012-0134-4.

[12] P. Niroomand, R. Rezaei, F.G. Russo. Commuting powers and exterior degree of finite groups. J. Korean Math. Soc., 2012, 49: 855 - 865, doi: 10.4134/JKMS.2012.49.4.855.

[13] P. Niroomand, F.G. Russo. An improvement of a bound of Green. J. Algebra Appl., 2012, 11(6): 1250116 (11 pages), doi: 10.1142/S0219498812501162.