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ON TOPOLOGICAL HOMOTOPY GROUPS OF PRODUCT SPACES

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Abstract. In this talk we present some conditions under which the product map $q \times q$, where $q : \Omega^n(X, x) \to \pi_n(X, x)$, is a quotient map for some spaces. As a consequence, we show that the topological homotopy groups preserve the product of spaces under certain circumstances.

1. Introduction

Endowed with the quotient topology induced by the natural surjective map $q : \Omega^n(X, x) \to \pi_n(X, x)$, where $\Omega^n(X, x)$ is the nth loop space of (X, x) with the compact-open topology, the familiar homotopy group $\pi_n(X, x)$ becomes a quasi-topological group which is called the topological nth homotopy group of the pointed space (X, x), denoted by $\pi_n^{top}(X, x)$ (see [1, 3]).

It was claimed by Biss [1] that $\pi_1^{top}(X, x)$ is a topological group. However, Calcut and McCarthy and Fabel showed that there is a gap

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in the proof of [1, Proposition 3.1]. The misstep in the proof is repeated by Ghane et al. [3] to prove that $\pi_n^{top}(X,x)$ is a topological group [3, Theorem 2.1].

Calcut and McCarthy showed that $\pi_n^{top}(X,x)$ is a homogeneous space. It is an open problem to understand when $\pi_n^{top}(X,x)$ is or is not a topological group.

The crucial misstep in proving that $\pi_n^{top}(X,x)$ is a topological group is that for the natural quotient map $q : \Omega^n(X,x) \to \pi_n(X,x)$, the map $q \times q$ can fail to be a quotient map. In this paper, we intend to obtain some conditions in which the product map $q \times q$ is a quotient map for some spaces. Using this fact, we can show that the topological homotopy groups preserve the product of spaces under certain circumstances.

2. Main results

It is well-known that if $(X,x) = \prod_{i\in I}(X_i,x_i)$, then $\pi_n(X,x) \cong \prod_{i\in I}\pi_n(X_i,x_i)$. In this paper, we intend to obtain some conditions under which topological nth homotopy groups preserve the product of spaces. With the notation and assumption of the previous section, Consider the following commutative diagram:

$$\begin{array}{ccc} \Omega^n(X,x) & \xrightarrow{\phi} & \prod\limits_{\leftarrow} \Omega^n(X_i,x_i) \\ \Big\downarrow q & & \prod q_i \Big\downarrow \\ \pi_n^{top}(X,x) & \xrightarrow{\psi} & \prod\limits_{\leftarrow} \pi_n^{top}(X_i,x_i), \end{array} \tag{2.1}$$

where ϕ is the natural homeomorphism using the fact that loop spaces preserve product, ψ is isomorphisms of groups and $q_i : \Omega^n(X_i,x_i) \to \pi_n^{top}(X_i,x_i)$ is the natural quotient map, for all $i \in I$. Since the map q is a quotient map, if we show that $\prod q_i$ is also a quotient map, then $\psi : \pi_n^{top}(X,x) \to \prod\limits_{\leftarrow} \pi_n^{top}(X_i,x_i)$ will be an isomorphism of quasi-topological groups. We are going to find some conditions in which the product of quotient maps q_i is also a quotient map. Michael [4] introduced a new class of quotient maps, called bi-quotient maps. A map $f : X \to Y$ is called a bi-quotient map if, whenever $y \in Y$ and $\mathcal{U}$ is a covering of $f^1(y)$ by open subsets of X, then finitely many $f(U)$, $U \in \mathcal{U}$, cover some neighborhood of y in Y [4, Definition 1.1]. He showed that any product (finite or infinite) of bi-quotient maps is also a bi-quotient map [4, Theorem 1.2]. Thus it is sufficient to see that in which conditions the quotient map q_i is a bi-quotient map. We need the following interesting result of Michael.

Lemma 2.1. *([4, Corollary 3.5]). Let $f : X \to Y$ be a quotient map, where Y is Hausdorff and X is second countable. Then f is a bi-quotient map if and only if Y is second countable.*

We shall also need the following well-known result.

Theorem 2.2. *([2, Theorem 12.5.2]). If X and Y are second countable, then the function space X^Y is second countable. In particular, if X is second countable, then $\Omega^n(X, x)$ is also second countable, for all $x \in X$.*

The following theorem is one of the main results of this section.

Theorem 2.3. *Let $(X, x) = \prod_{i\in I}(X_i, x_i)$, where X_i's are second countable, and let $\pi_n^{top}(X_i, x_i)$ be Hausdorff and second countable, for all $i \in I$. Then the isomorphism $\pi_n^{top}(X, x) \cong \prod_{i\in I} \pi_n^{top}(X_i, x_i)$ holds in topological groups.*

Proof. Using diagram (2.1), it is enough to show that the map $\prod q_i$ is a quotient map. Since X_i is second countable, by Theorem 2.2, $\Omega^n(X_i, x_i)$ is second countable, for all $i \in I$. Since $\pi_n^{top}(X_i, x_i)$ is Hausdorff and second countable, by Lemma 2.1 the quotient map $q_i : \Omega^n(X_i, x_i) \to \pi_n^{top}(X_i, x_i)$ is a bi-quotient map, for all $i \in I$. Hence by [4, Theorem 1.2] the product map $\prod q_i$ is a bi-quotient map which yields the result. □

As a consequence of the above theorem, we can present a class of spaces whose topological homotopy groups are topological groups.

Corollary 2.4. *Let $(X, x) = \prod_{i\in I}(X_i, x_i)$, where X_i's are second countable, locally $(n-1)$-connected and semilocally n-connected at x_i whose nth homotopy groups are countable. Then the isomorphism $\pi_n^{top}(X, x) \cong \prod_{i\in I} \pi_n^{top}(X_i, x_i)$ holds in topological groups.*

Proof. Since X_i is locally $(n-1)$-connected and semilocally n-connected at x_i, by [5, Theorem 6.7] $\pi_n^{top}(X_i, x_i)$ is discrete. Since $\pi_n^{top}(X_i, x_i)$ is countable, it is Hausdorff and second countable. Hence Theorem 2.4 implies the result. □

The following example shows that $\pi_n^{top}(-)$ does not preserve the product of spaces in general. Note that Biss [1, Proposition 5.2] claimed that $\pi_n^{top}(-)$ does preserve the product of spaces, but his proof has the mentioned misstep.

Example 2.5. Let HA be the Harmonic Archipelago space introduced in [1]. We show that there is no quasi-topological isomorphism between $\pi_1^{top}(HA \times HA)$ and $\pi_1^{top}(HA) \times \pi_1^{top}(HA)$. Note that the space $HA \times HA$ has one small point. Hence by the proof of [5, Theorem 6.9], we can show that the topology of $\pi_1^{top}(HA \times HA)$ is indiscrete, while it is easy to see that the topology of $\pi_1^{top}(HA) \times \pi_1^{top}(HA)$ is not indiscrete. In fact, we can say that $\pi_1^{top}(-)$ does not preserve the product of small loop spaces.

Note that HA is not semilocally 1-connected, also $\pi_1^{top}(HA)$ is not Hausdorff and it is uncountable. Therefore, the hypotheses of Corollary 2.4 are essential.

Now, we list some conditions of quotient map which imply that their cartesian product is also quotient map.
Let X_i and Y_i be Hausdorff spaces and $f_i : X_i \to Y_i$ be quotient maps, for $i = 1, 2$. If one of the following conditions holds, then $f_1 \times f_2$ is a quotient map [4].

(a) If X_1 and $Y_1 \times Y_2$ are both k-spaces.
(b) If X_1 is a k-space and Y_2 is locally compact.
(c) If X_1, Y_1 and Y_2 are first countable.
(d) If X_1 and X_2 or X_1 and Y_2 are both k_ω-spaces.
(e) If X_1 and X_2 are lindelof and Y_1 and Y_2 are q-spaces.

The precise definitions can be found in [4].

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