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A TOPOLOGICAL APPROACH TO THE NILPOTENT MULTIPLIERS

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Abstract. In this talk, using the topological interpretation of the c-nilpotent multiplier of a group G, $M^{(c)}(G)$, we extend a result of Burns and Ellis (Math. Z. 226 (1997) 405-428) on the second nilpotent multiplier of a free product of two groups to the c-nilpotent multipliers, for all $c \geq 1$.

1. Introduction

Let $G \cong F/R$ be a free presentation of G, then the c-nilpotent multiplier of G, denoted by $M^{(c)}(G)$, is defined to be

$$M^{(c)}(G) \cong \frac{R \bigcap \gamma_{c+1}(F)}{[R,\ {}_cF]}.$$

Definition 1.1. A *simplicial group* $K_.$ is a sequence of groups $K_0, K_1, K_2, \ldots$ together with homomorphisms $d_i : K_n \to K_{n-1}$ (faces) and $s_i : K_n \to K_{n+1}$ (degeneracies), for each $0 \leq i \leq n$, with the following

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conditions:

$$\begin{array}{rcll} d_j d_i & = & d_{i-1} d_j & \text{for } j < i \\ s_j s_i & = & s_{i+1} s_j & \text{for } j \leq i \\ d_j s_i & = & \begin{cases} s_{i-1} d_j & \text{for } j < i; \\ identity & \text{for } j = i, i+1; \\ s_i d_{j-1} & \text{for } j > i+1. \end{cases} \end{array}$$

Definition 1.2. If $K_.$ is a simplicial group, then the *Moore complex* $(NK_., \partial)$ of $K_.$ is the (nonabelian) chain complex defined by $(NK)_n = \cap_{i=0}^{n-1} \ker d_i$ with $\partial_n : NK_n \to \ NK_{n-1}$ which is a restriction of d_n.

A simplicial group $K_.$ is said to be *free* if each K_n is a free group and degeneracy homomorphisms s_i's send the free basis of K_n into the free basis for K_{n+1}.

Definition 1.3. A free simplicial resolution of G consists of a free simplicial group $K_.$ with $\pi_0(K_.) = G$ and $\pi_n(K_.) = 0$, for all $n \geq 1$.

In the following, we recall some results that will be needed in sequel.

Theorem 1.4. [2]

(1) *Every epimorphism between simplicial groups is a fibration.*
(2) *Let $K_.$ be a simplicial group, then $\pi_*(K_.) \cong H_*(NK_.)$.*
(3) *Let $K_.$ and $L_.$ be simplicial abelian groups, then* $H_n\big(N(K_. \otimes L_.)\big) \cong H_n\big(N(K_.) \otimes N(L_.)\big)$.

The following topological interpratation of c-nilpotent multiplier of a group is proved by Burns and Ellis.

Theorem 1.5. [1, Proposition 4.1] *Let $K_.$ be a free simplicial resolution of G, Then the following isomorphisms hold.*

$$\begin{array}{rcl} \pi_1\big(K_./\gamma_c(K_.)\big) & \cong & M^{(c)}(G) \\ \pi_0\big(K_./\gamma_c(K_.)\big) & \cong & G/\gamma_{c+1}(G). \end{array}$$

2. Main results

Proposition 2.1. *Let $F = K * L$ be the free product of two free groups K and L and let $\varphi : F \to K \times L$ be the natural epimorphism. Then for all $c \geq 1$, there exists the following short exact sequence*

$$0 \to ker\bar{\varphi}_c \to \frac{F}{\gamma_{c+1}(F)} \xrightarrow{\bar{\varphi}_c} \frac{K}{\gamma_{c+1}(K)} \times \frac{L}{\gamma_{c+1}(L)} \to 0,$$

where $\ker \bar{\varphi}_c \cong \frac{[K,L]^F}{[K,L,\ _{c-1}F]^F}$ *which satisfies in the following exact sequence*

$$0 \to \frac{[K,L,\ _{c-2}F]^F}{[K,L,\ _{c-1}F]^F} \to \ker \bar{\varphi}_c \to \frac{[K,L]^F}{[K,L,\ _{c-2}F]^F} \to 0.$$

Moreover, we have the following isomorphism

$$\frac{[K, L, \ _{c-2}F]^F}{[K, L, \ _{c-1}F]^F} \cong \oplus \sum_{for\ \ some\ \ i+j=c} \underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{i-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{j-times}.$$

Theorem 2.2. *Let G, H be two groups with*

$$G^{ab} \otimes H^{ab} = M^{(1)}(G) \otimes H^{ab} = M^{(1)}(H) \otimes G^{ab} = Tor(G^{ab}, H^{ab}) = 0.$$

Then the following isomorphism holds, for all $c \geq 1$,

$$M^{(c)}(G * H) \cong M^{(c)}(G) \oplus M^{(c)}(H).$$

Proof. For $c = 1$, by a well-known result on Schur multiplier of the free product (see [4, Theorem 2.6.8]).

Let $K_.$ and $L_.$ be free simplicial groups corresponding to G and H, respectively, then $F_. = K_. * L_.$ is a free simplicial resolution of $G * H$. By Proposition 2.1, consider the following short exact sequence of simplicial groups

$$0 \to (\ker \bar{\varphi}_c)_. \to \frac{F_.}{\gamma_{c+1}(F_.)} \overset{\bar{\varphi}_c}{\to} \frac{K_.}{\gamma_{c+1}(K_.)} \times \frac{L_.}{\gamma_{c+1}(L_.)} \to 0.$$

Theorem 1.4 (1) yields the following long exact sequence

$$\begin{aligned} \cdots \quad &\to \pi_2\big((\ker \bar{\varphi}_c)_.\big) \to \pi_2\big(\tfrac{F_.}{\gamma_{c+1}(F_.)}\big) \overset{\pi_2(\bar{\varphi}_c)}{\to} \pi_2\big(\tfrac{K_.}{\gamma_{c+1}(K_.)}\big) \oplus \pi_2\big(\tfrac{L_.}{\gamma_{c+1}(L_.)}\big) \\ &\to \pi_1\big((\ker \bar{\varphi}_c)_.\big) \to \pi_1\big(\tfrac{F_.}{\gamma_{c+1}(F_.)}\big) \overset{\pi_1(\bar{\varphi}_c)}{\to} \pi_1\big(\tfrac{K_.}{\gamma_{c+1}(K_.)}\big) \oplus \pi_1\big(\tfrac{L_.}{\gamma_{c+1}(L_.)}\big). \end{aligned}$$

Since the homomorphisms $\pi_1(\bar{\varphi}_c)$ and $\pi_2(\bar{\varphi}_c)$ are split, we have the following isomorphism

$$\pi_1(\ker \bar{\varphi}_c) \oplus \pi_1\big(K_./\gamma_{c+1}(K_.)\big) \oplus \pi_1\big(L_./\gamma_{c+1}(L_.)\big) \cong \pi_1\big(F_./\gamma_{c+1}(F_.)\big).$$

Proposition 2.1 and Theorem 1.4 (1) yield the following long exact sequence of homotopy groups

$$\cdots \to \pi_1\Big(\frac{[K_., L_., \ _{c-2}F_.]^{F_.}}{[K_., L_., \ _{c-1}F_.]^{F_.}}\Big) \to \pi_1(\ker \bar{\varphi}_c)_. \to \pi_1(\ker \bar{\varphi}_{c-1})_. \to \cdots .$$

By induction on c, we prove that $\pi_1(\ker \bar{\varphi}_c)_. = 0$. For $c = 2$, Burns and Ellis in [1, Lemma 4.2] proved that $(\ker \bar{\varphi}_2)_. \cong K_.^{ab} \otimes L_.^{ab}$. Hence by Theorem 1.4 (3) and (2), we can prove that

$$\pi_1(\ker \bar{\varphi}_2)_. \cong M^{(1)}(G) \otimes H^{ab} \oplus M^{(1)}(H) \otimes G^{ab} \oplus Tor(G^{ab}, H^{ab}) \cong 0$$

and $\pi_0(K_.^{ab} \otimes L_.^{ab}) \cong G^{ab} \otimes H^{ab} \cong 0$.

Now let $\pi_1(\ker\bar{\varphi}_{c-1})_. = 0$. We are going to show that $\pi_1(\ker\bar{\varphi}_c)_. = 0$. The result holds, since

$$\frac{[K_., L_.,\ {}_{c-2}F_.]^{F_.}}{[K_., L_.,\ {}_{c-1}F_.]^{F_.}} \cong \oplus \sum_{for\ \ some\ \ i+j=c} \underbrace{K_.^{ab}\otimes ...\otimes K_.^{ab}}_{i-times}\otimes \underbrace{L_.^{ab}\otimes ...L_.^{ab}}_{j-times},$$

and

$$\begin{aligned}
&\pi_1(\underbrace{K_.^{ab}\otimes ...\otimes K_.^{ab}}_{i-times}\otimes \underbrace{L_.^{ab}\otimes ...\otimes L_.^{ab}}_{j-times})\\
&\cong \pi_1(K_.^{ab}\otimes L_.^{ab})\otimes \pi_0(\underbrace{K_.^{ab}\otimes ...\otimes K_.^{ab}}_{(i-1)-times}\otimes \underbrace{L_.^{ab}\otimes ...\otimes L_.^{ab}}_{(j-1)-times})\\
&\oplus\pi_0(K_.^{ab}\otimes L_.^{ab})\otimes \pi_1(\underbrace{K_.^{ab}\otimes ...\otimes K_.^{ab}}_{(i-1)-times}\otimes \underbrace{L_.^{ab}\otimes ...\otimes L_.^{ab}}_{(j-1)-times})\\
&\oplus Tor\big(\pi_0(K_.^{ab}\otimes L_.^{ab}), \pi_0(\underbrace{K_.^{ab}\otimes ...\otimes K_.^{ab}}_{(i-1)-times}\otimes \underbrace{L_.^{ab}\otimes ...\otimes L_.^{ab}}_{(j-1)-times})\big)\cong 0.
\end{aligned}$$

$\square$

Corollary 2.3. *Let G and H be two groups. Then, for all $c\geq 1$, we have the following isomorphism*

$$M^{(c)}(G*H)\cong M^{(c)}(G)\oplus M^{(c)}(H),$$

if one of the following conditions holds:
(i) G and H are two abelian groups with coprime orders.
(ii) G and H are two finite groups with $\big(|G|, |H^{ab}|\big) = \big(|G^{ab}|, |H|\big) = 1$.
(iii) G and H are two finite groups with

$$\big(|G^{ab}|, |H^{ab}|\big) = \big(|M(G)|, |H^{ab}|\big) = \big(|G^{ab}|, |M(H)|\big) = 1.$$

(vi) G and H are two perfect groups.

Note that parts $(i)-(iii)$ of the above corollary are vast generalizations of a result of the second author (see[3, Theorem 2.5]).

References

1. J. Burns and G. Ellis, *On the nilpotent multipliers of a group*, Math. Z. 226 (1997), 405–428.
2. E.B. Curtis, *Simplicial homotopy theory*, Advances in Math. 6 (1971), 107–209.
3. B. Mashayekhy, *On the nilpotent multiplier of a free product*, Bull. Iranian Math. Soc. 28 (2002), 49–56.
4. C. Miller, *The second homology group of a group; relation among commutators*, Proc. Amer. Math. Soc. 3 (1952), 588–595.