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SOLVABILITY OF FREE PRODUCTS, CAYLEY GRAPHS AND COMPLEXES

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Abstract. In this paper, we verify the solvability of free product of finite cyclic groups with topological methods. We use Cayley graphs and Everitt methods to construct suitable 2-complexes corresponding to the presentations of groups and their commutator subgroups. In particular, using these methods, we prove that the commutator subgroup of $\mathbb{Z}_m * \mathbb{Z}_n$ is free of rank $(m-1)(n-1)$ for all $m, n \geq 2$.

1. Introduction

This paper is based on combinatorial algebraic topology. We use topological interpretations of groups [2], whose main objects are combinatorial 2-complexes. In sections 2, 3, 4 and 5 we give some preliminaries from [2], [3], and [4]. First we study the topology of 2-complexes. In section 3 and 4, we define the fundamental groups of these complexes and deduce their presentations using topological figures of their complexes. In section 5, we introduce the coverings of complexes, and we apply them to construct suitable complexes corresponding to several subgroups.

Using these topological preliminaries, we present our main results in section 6. In section 6, we recall some notes about solvable groups and then, using algebraic topology, we prove that the commutator subgroup of free product of two finite cyclic groups is free, specially we compute its rank. From this fact, we conclude that free product

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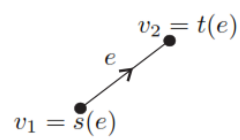
of two finite cyclic groups is solvable, if and only if each of its factors is of order 2. Of course, the fact is proved by group theoretical methods [1].

2. The topology of complexes

Here we give some preliminaries from [2], [3] and [4]. Also, the figures of sections 2 and 3, are from [2]. combinatorial 2-complex K is made up of three sets V_K, E_K and F_K (vertices, edges and faces), together with maps that describe how the pieces fit together. We have $s, t : E_K \longrightarrow V_K$ and ${}^{-1} : E_K \longrightarrow E_K$ so that ${}^{-1}$ assigns each edge to another, called its inverse, and s, t assign start and terminal vertices to e.



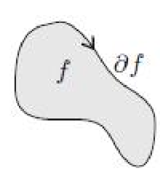
These maps satisfy $e^{-1} \neq e$, $(e^{-1})^{-1} = e$, $s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$. The vertex and edge sets together with these maps form a directed graph called the $1-skeleton$ K^1 of K.(the vertices alone form $0-skeleton$ K^0).

A path w in K is a sequence of edges $e_1^{\varepsilon_1} \cdots e_k^{\varepsilon_k}, \varepsilon_i = \pm 1$ with $t(e_i^{\varepsilon_i}) = s(e_{i+1}^{\varepsilon_{i+1}})$. This path is closed if $s(w) = t(w)$, which $s(w) = s(e_1^{\varepsilon_1})$ and $t(w) = t(e_k^{\varepsilon_k})$.

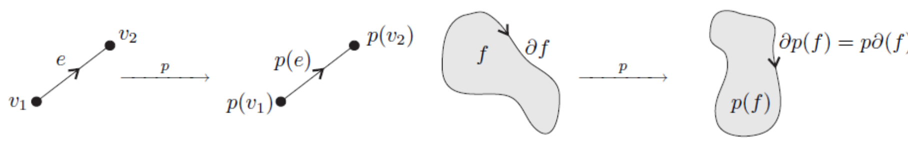
A 2-complex is connected if there is a path between any two of its vertices.

Two paths w_1 and w_2 are cyclic permutations of each other if $w_1 = e_1^{\varepsilon_1} \cdots e_k^{\varepsilon_k}$, then $w_2 = e_j^{\varepsilon_j} \cdots e_k^{\varepsilon_k} e_1^{\varepsilon_1} \cdots e_{j-1}^{\varepsilon_{j-1}}$ for some k. A cycle in the $1-skeleton$ is a set consisting of a path and all of its cyclic permutations.

Finally, to define faces we consider the following maps that say how the faces are glued onto the $1-skeleton$, ${}^{-1} : F_K \longrightarrow F_K$ and $\partial : F_K \longrightarrow cycles$ which must satisfy $f^{-1} \neq f$, $(f^{-1})^{-1} = f$ and $w \in \partial(f)$ if and only if $w^{-1} \in \partial(f^{-1})$.



A map $p : K_1 \longrightarrow K_2$ between 2-complexes assigns to each vertex of K_1 a vertex of K_2, to each edge of K_1 an edge or vertex of K_2, and to each face of K_1 a face, path or vertex of K_2 with $p(e^{-1}) = p(e)^{-1}, p(f^{-1}) = p(f)^{-1}$.



This map is dimension preserving if $p : V_{K_1} \longrightarrow V_{K_2}$, $p : E_{K_1} \longrightarrow E_{K_2}$ and $p : F_{K_1} \longrightarrow F_{K_2}$.

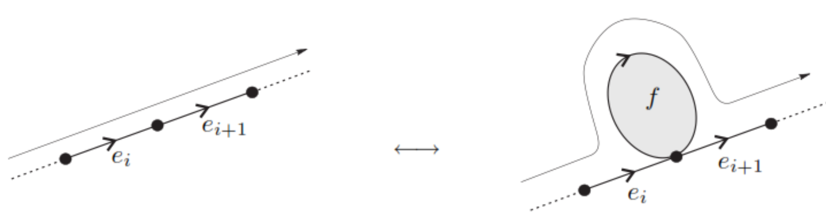
A map is an isomorphism if it preserves dimension and is bijection on the vertex, edge and face sets.

3. Fundamental groups of complexes

Two paths w_1 and w_2 are homotopic ($w_1 \sim_h w_2$) if and only if there is a finite sequence of these two moves taking one path to the other; the first inserts or deletes a spur: an edge /inverse edge pair of the form ee^{-1} or $e^{-1}e$.

$e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k}$ $\qquad$ $e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} (ee^{-1}) e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k}$

The second inserts or deletes the boundary of a face: a $w \in \partial(f)$ for some face f of K with $s(w) = t(w) = t(e_i^{\varepsilon_i}) = s(e_{i+1}^{\varepsilon_{i+1}})$.



$e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k}$ $\qquad$ $e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} \partial f e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k}$

If w_1 and w_2 are paths in a 2-complex K with $t(w_1) = s(w_2)$, then let $w_1 w_2$ be the path obtained by juxtaposing these two, by traversing the edges of w_1 and then the edges of w_2.

In particular, if $w_1 \sim_h w_1'$ and $w_2 \sim_h w_2'$ then $w_1 w_2 \sim_h w_1' w_2'$. So we can extend a product on the homotopy classes of paths in K, if $[w_1]_h$ and $[w_2]_h$ are two such, where $t(w_1) = s(w_2)$, then $[w_1]_h [w_2]_h := [w_1 w_2]_h$.

To get a group from homotopy classes of paths, we need to ensure we can always multiply paths. Let K be a 2-complex and fix a vertex v. Let $\pi_1(K, v)$ be the set of all homotopy classes of closed paths with start and terminal vertex v. $\pi_1(K, v)$ together with product $[w_1]_h [w_2]_h = [w_1 w_2]_h$ forms a group with identity $[v]_h$ and inverse element $[w]_h^{-1} = [w^{-1}]_h$.

A 2-complex K is called a tree if and only if the face set of K is empty and $\pi_1(K)$ is the trivial group.

4. Complexes and presentations

First we obtain a group presentation for the fundamental group $\pi_1(K)$ of a complex K. Let K be a connected 2-complex and v a vertex of K. Let T be a connected tree that contains all the vertices of K. Choose an edge e_α from each edge or its inverse in $K^1 \backslash T$. Then there are unique paths $w_\alpha, \overline{w_\alpha}$ without spurs in T, such that w_α connects v to the start vertex of e_α and $\overline{w_\alpha}$ connects v to the terminal vertex. Let $x_\alpha = w_\alpha e_\alpha \overline{w_\alpha}^{-1}$, a loop based at v, and $X = \{x_\alpha \mid e_\alpha \in K^1 \setminus T\}$. Choose f_β from each face or its inverse in K. Let $\partial(f_\beta) = e_{\alpha_1}^{\varepsilon_1} \cdots e_{\alpha_k}^{\varepsilon_k}$ be the boundary label after the edges that are contained in the tree T have been removed. Take $w_\beta = x_{\alpha_1}^{\varepsilon_1} \cdots x_{\alpha_k}^{\varepsilon_k}$, a word in $X \cup X^{-1}$ and $R = \{w_\beta \mid f_\beta \text{ is a face}\}$.

Theorem 4.1. [2] *$\langle X; R\rangle$ is a presentation for the fundamental group of K.*

Corollary 4.2. [2] *Suppose that K is a graph (F_K is empty), then we can find such a presentation for $\pi_1(K)$ with $|E_K| - |V_K| + 1 = 1 - \chi(K)$ generators and no relation.*

Now we want to obtain a 2-complex from a group presentation. Let $\langle X; R\rangle$ be a presentation for a group G. Define a 2-complex $K = K\langle X; R\rangle$ with a single vertex v. For each $x \in X$ take an $e_x^{\pm 1} \in E_K$ and for each $w \in R$ an $f_w^{\pm 1} \in F_K$. We have $s(e_x^{\pm 1}) = t(e_x^{\pm 1}) = v$, $\partial(f_w)$ is cyclic permutation of $e_{x_{\alpha_1}}^{\varepsilon_1} \cdots e_{x_{\alpha_k}}^{\varepsilon_k}$, if $w = x_{\alpha_1}^{\varepsilon_1} \cdots x_{\alpha_k}^{\varepsilon_k}$.

Theorem 4.3. [2] *$\langle X; R\rangle$ is a presentation for $\pi_1(K\langle X; R\rangle, v)$.*

Corollary 4.4. [2] *A group is free if and only if it is the fundamental group of a graph.*

5. Covering of complexes and Cayley graphs

A map $p : \tilde{K} \longrightarrow K$ of 2-complexes is a covering if and only if
1. p preserves dimension;
2. If $p : \tilde{v} \longrightarrow v$ then p is a bijection from the set of edges in $\tilde{K}$ with initial vertex $\tilde{v}$ to the set of edges in K with initial vertex v;
3. For any face f and any vertex v of K, we introduce $m(f, v)$ to be the number of times that v appears in the boundary of f. Then for any $\tilde{v}$ with $p(\tilde{v}) = v$, we have $\sum_{p(\tilde{f})=f} m(\tilde{f}, \tilde{v}) = m(f, v)$.

Theorem 5.1. [2] *(Subgroup Theorem)*
1. Let $p : \tilde{K} \longrightarrow K$ be a covering, then $p_* : \pi_1(\tilde{K}, \tilde{v}) \longrightarrow \pi_1(K, v)$ is injective.
2. Let K be a 2-complex and H a subgroup of $\pi_1(K, v)$, then there is a connected 2-complex $\tilde{K}$ and a covering $p : \tilde{K} \longrightarrow K$ with $H \cong \pi_1(\tilde{K}, \tilde{v})$, where $p(\tilde{v}) = v$.

For a complex K, a covering $p : \tilde{K} \longrightarrow K$ is universal if and only if for any other covering $p' : K' \longrightarrow K$ there is a covering $q : \tilde{K} \longrightarrow K'$ such that $p'q = p$. In particular, given $G = \langle X, R\rangle$, the universal cover of the presentation 2-complex $K\langle X; R\rangle$ is called Cayley complex of G with respect to $\langle X, R\rangle$.

The 1-skeleton of the Cayley complex is the Cayley graph for G with respect to the generators X. A 1-complex K is the Cayley graph of G with respect to $\langle X\rangle$ if and only if there is a covering $K \longrightarrow K\langle X\rangle$ and a bijection $f : K_0 \longrightarrow G$ such that if $e \in C$ is an edge with initial vertex v and terminal vertex u, and $p(e) = x_i$, then $f(u) = f(v)x_i$ in G.

Theorem 5.2. [2] *Let $p : \tilde{K} \longrightarrow K\langle X; R\rangle$ be a regular covering of the presentation 2-complex for $\langle X; R\rangle$, $G = \langle X, R\rangle$ and $H = p_*(\pi_1(\tilde{K}, \tilde{v}))$. Then the 1-skeleton of $\tilde{K}$ is the Cayley graph for G/H with respect to the generators $\langle Hx\rangle_{x\in X}$.*

6. Main results

Recall that a group G is solvable if and only if there is a sequence of subgroups $\{1\} = N_0 \lhd ... \lhd N_k = G$, whose all factor groups N_i/N_{i-1} are abelian for (i=1,2,...,k) [5].

The n-th derived subgroup of G is defined to be $G^{(n)} = (G^{(n-1)})'$, where $G^{(1)}$ is equal to the commutator subgroup of G, $G' = [G, G]$.

In particular, there is a theorem asserting that a group G is solvable, if and only if $G^{(n)} = 1$, for some n [5].

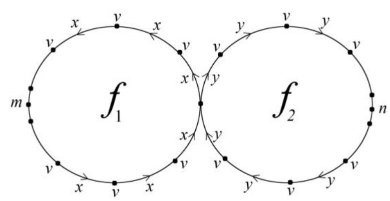
Here is an application of the above notes, specially Theorem 4.2.

Theorem 6.1. [2] *The free product $\mathbb{Z}_2 * \mathbb{Z}_2$ is solvable.*

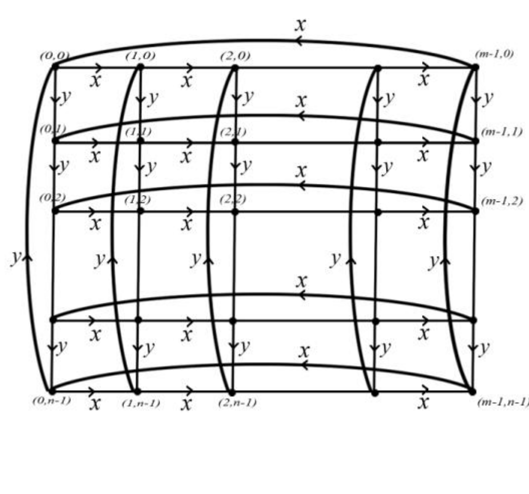
Also, there is a question from [2], asking about the solvability of $\mathbb{Z}_m * \mathbb{Z}_n$, in general. To answer this problem, another question arises as whether we can extend the methods of [2] to compute the commutator subgroup of $\mathbb{Z}_m * \mathbb{Z}_n$ $(m, n \geq 2)$. From group theory, we know that the commutator subgroup of $\mathbb{Z}_m * \mathbb{Z}_n$ is free for all $(m, n \geq 2)$. But we want to prove this fact by topological tools. In particular, we compute the rank of this free group and finally answer the question.

Theorem 6.2. *The commutator subgroup of $\mathbb{Z}_m * \mathbb{Z}_n$ is free of rank $(m-1)(n-1)$.*

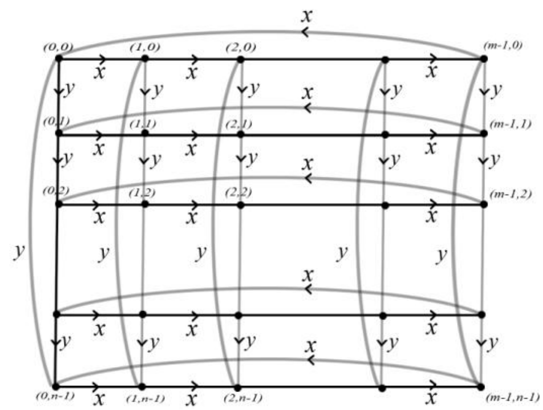
Proof. First, we consider the 2-complex K corresponding to $G = \mathbb{Z}_m * \mathbb{Z}_n \cong \langle x, y; x^m, y^n \rangle$.



To compute the commutator subgroup $[G, G]$, we need to obtain the 2-complex corresponding to $[G, G]$. For this, we consider the Cayley graph for $G/[G, G] \cong \mathbb{Z}_m \times \mathbb{Z}_n \cong \langle x, y; x^m, y^n, [x, y] \rangle$, constructed as follows:



By theorem 4.2, this graph is the 1-skeleton of regular cover $\tilde{K}$ of K, corresponding to $[G,G]$ ($\tilde{K}$ is a 2-complex with $\pi_1(\tilde{K}) \cong [G,G]$). As we see, there are mn vertices in this graph and so in $\tilde{K}$, so the fiber of each face of K must contain mn faces in $\tilde{K}$. Hence, for any closed path with label x^m (or with label y^n) in $\tilde{K}$, consider m faces (or n faces) in $\tilde{K}$, with boundaries x^m (or y^n). Therefore we are finished in constructing the 2-complex $\tilde{K}$. To obtain a presentation for $\pi_1(\tilde{K}) \cong [G,G]$, fix the vertex $v = (0,0)$ and consider he maximal tree T of $\tilde{K}$,



Using the first paragraph of section 4 (theorem 4.1), we obtain $2mn - (mn-1) = mn+1$ elements corresponding to edges of $K^1 \setminus T$, generate $\pi_1(\tilde{K})$. these elements are categorized as follows:

1) The elements corresponding to edges x (from $(m-1,j)$ to $(0,j)$, $0 \le j \le n-1$), which are of the form $x_j = y^j x^{m-1} x y^{-j} = 1$ ($w_\alpha = y^j x^{m-1}$, $\overline{w}_\alpha = y_j$).

2) The elements corresponding to edges y (from $(i,j-1)$ to (i,j), $1 \le i \le m-1$, $1 \le j \le n-1$), which are of the form $y_{ij} = y^{j-1}x^i y x^{-i} y^{-j}$ ($w_\alpha = y^{j-1}x^i$, $\overline{w}_\alpha = y^j x^i$).

3) The elements corresponding to edges y (from $(i,n-1)$ to $(i,0)$, $0 \le i \le m-1$) which are also of the form $y_i = y^{n-1}x^i y x^{-i}$ ($w_\alpha = y^{n-1}x^i$, $\overline{w}_\alpha = x^i$). Using the relation $y^n = 1$ or equivalently $y^{-1} = y^{n-1}$, we have $y_i = y^{n-1}x^i y^{-(n-1)} x^{-i} = y_{i1} y_{i2} \ldots y_{i(n-1)}$.

Finally, as we see, the elements of type (1) are all trivial, and the elements of type (3) are obtained by the elements of type (2). So the presentation of $\pi_1(\tilde{K})$ has $(m-1)(n-1)$ generators $\{y_{ij}\}$ which are all the elements of type (2). In particular, there is no relation between these elements. In fact, as we see, non of the subsets of y_{ij}'s is boundary of any face of the complex $\tilde{K}$. So we have $\pi_1(\tilde{K}) = \langle y_{ij} (1 \le i \le m-1, 1 \le j \le n-1); \varnothing \rangle$. It says that $\pi_1(\tilde{K}) \cong [G,G]$ is free of rank $(m-1)(n-1)$. □

Corollary 6.3. *For any $m, n \ge 2$, $\mathbb{Z}_m * \mathbb{Z}_n$ is solvable if and only if $m = n = 2$.*

Proof. If $m = n = 2$, by Theorem 5.2, the commutator subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$ is free of rank 1. So we have the sequence $1 \lhd \mathbb{Z} \lhd \mathbb{Z}_2 * \mathbb{Z}_2$ with factors $\mathbb{Z}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, $\mathbb{Z}_2 * \mathbb{Z}_2$ is solvable. If m and n are given so that $m + n \geq 5$, by previous theorem, the commutator subgroup of $\mathbb{Z}_m * \mathbb{Z}_n$ is free of rank $(m-1)(n-1) \geq 2$. Thus, by the fact that the commutator subgroup of any free group of rank $k \geq 2$ is free with infinite rank [2], $G^i \neq 1$ for all $i \geq 1$. Therefore in this case, $\mathbb{Z}_m * \mathbb{Z}_n$ is not solvable. $\square$

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