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Archive of Applied Mechanics

ISSN 0939-1533
Volume 83
Number 12

Arch Appl Mech (2013) 83:1773-1787
DOI 10.1007/s00419-013-0777-y

Springer

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Received: 29 December 2012 / Accepted: 22 August 2013 / Published online: 4 September 2013

Abstract The solutions to H_∞ and H_2 optimization problems of a variant dynamic vibration absorber (DVA) applied to suppress vibration in beam structures are derived analytically. The H_∞ optimum parameters such as tuning frequency and damping ratios are expressed based on fixed-point theory to minimize the resonant vibration amplitude, as well as, the H_2 optimum parameters to minimize the total vibration energy or the mean square motion of a beam under random force excitation as analytical formulas. The reduction in maximum amplitude responses and mean square motion of a beam using the traditional vibration absorber is compared with the proposed dynamic absorber. Numerical results show the non-traditional DVA under optimum conditions has better vibration suppression performance on beam structures than the traditional design of DVA. Furthermore, comparing H_∞ and H_2 optimization procedures shows that for a beam under random force excitation, use of H_2 optimum parameters resulting in smaller mean square motion than the other optimization.

Keywords Dynamic vibration absorber · Passive vibration control · Harmonic excitation · Random excitation · Beam structures

1 Introduction

Dynamic vibration absorbers (DVA) or tuned mass dampers (TMD) are a well-established passive vibration control devices which when tuned correctly and attached properly could suppress vibration of structures in a good manner. These absorbers have wide application in different fields such as mechanical, civil and aerospace structures due to unique specifications such as easy-to-maintain, uncomplicated design, high reliability and excellent performance.

The first DVA invented by Frahm [1] in 1911 had no damper, and it was useful just in the range of frequencies close to the natural frequency of the DVA. Ormondroyd and Den Hartog [2] pointed out that damping element could widen the frequency band of the DVA's efficient operation and had an optimum value; they used damping element and spring in a parallel manner ;this configuration is known as traditional DVA here, Fig. 1b. Arrangement of mass, spring and damper besides frequency and damping ratios of DVA plays an important role in the performance of the absorbers. As a result, several optimization criteria and various configurations are proposed to have better vibration suppression performance. In the following, briefly review H_2 and H_∞ optimizations which are two most useful approaches in finding optimum parameters.

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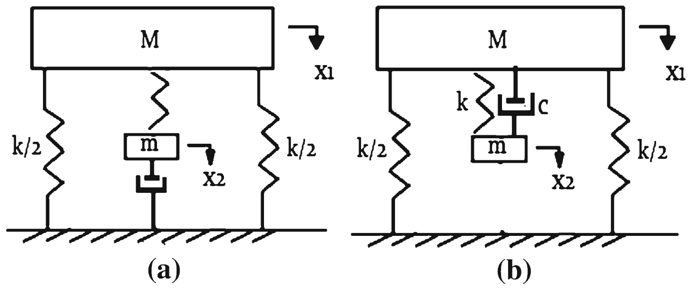

Fig. 1 **a** Non-traditional DVA, **b** traditional DVA

H_∞ optimization:

Ormondroyd and Den Hartog [2] indicated that the damping of the DVA had an optimum value for the minimization of the amplitude response of system in 1928; such optimization criterion is known as H_∞ optimization and is based on fixed-point theory which is the first method in DVA optimization. The objective is to minimize the maximum amplitude response of the primary system.

H_2 optimization:

This optimization criterion was proposed by Crandall and Mark [3] in 1963. The objective is to reduce the total vibration energy of the system over all frequencies. In this optimization criterion, the area under the frequency response curve of the system is minimized.

Some approximations were taken in Den Hartog and Ormondroyd optimization [2], but Nishihara and Asami [4] found the closed-form solution to the exact optimization of dynamic vibration absorbers and showed the both optimum parameters were very close to the exact values; they also derived the analytical solutions to H_∞ and H_2 optimization problems of DVAs attached to damped linear systems [5]. Eliot et al. [6] found the optimum parameters such that either the kinetic energy of the host structure is minimized or the power dissipation within the absorber is maximized, and Tigli [7] studied the optimum design of dynamic vibration absorbers (DVAs) installed on linear damped systems that are subjected to random loads. According to the role of these absorbers in vibration control of sdof systems, substantial research work has been done to derive optimum parameters of DVAs on multi-degree-of-freedom (mdof) or continuous systems. Ozer and Royston [8] extended Den Hartog's vibration absorber technique to multi-degree-of-freedom systems. Rice [9] used SIMPLEX nonlinear optimization to determine the H_∞ optimum parameters of a DVA applied for suppressing the vibration of a beam. Hadi and Arfiadi [10] reported the use of a genetic algorithm to solve optimum tuning for mdof systems numerically. Wong et al. [11] proposed a new dynamic vibration absorber combining a translational-type absorber and a rotational-type absorber and used finite element analysis to evaluate the performance of proposed absorber mounted on beam. Cheung and Wong [12] established a theory for describing the excitation–response relation leading to the H_∞ and H_2 optimum tuning of the DVA attached onto a plate structure. They derived optimum tunings such as tuning frequency and damping ratios of the absorber and also the position of the absorber on the vibrating structure in order to minimize vibrational displacement, velocity and acceleration.

As mentioned before, the configuration of absorbers affect its performance. In the last few years, several non-traditional configurations of dynamic vibration absorbers reported to improve the performance of the absorbers. Some examples of non-traditional DVAs are series TMDs [13], parallel multiple TMDs [14] and multi-degree-of-freedom TMDs [15]. A variant design of the damped dynamic vibration absorber as shown in Fig. 1a was proposed by Ren [16], and Liu [17] recently which provide a feasible substitute for some applications. Sometimes a damper is too massive to be attached like traditional DVA, and this variant design offers a solution; moreover, analytical derivation of the optimum parameters for minimizing the resonant vibration of sdof systems under various excitations such as force [16–18] or ground motion [19] was studied in this case which showed it provides greater vibration suppression than ordinary absorber. Chitba et al. [20] proposed an optimal design for supplementing flexible structures with a set of absorbers and piezoelectric devices for vibration confinement and energy harvesting. They considered two possible configurations for each of the additional piezoelectric devices, either embedded between the structure and the absorbers or between the ground and absorbers; they found that the second configuration yields faster extraction of vibration energy, so using this configuration lead us to achieve vibration suppression and energy harvesting together. It has been shown that the non-traditional absorber which optimized by H_∞ procedure can result in a more reduction in the vibration amplitude of the sdof systems than the ordinary one. In 2011, Cheung and Wong [21] proposed

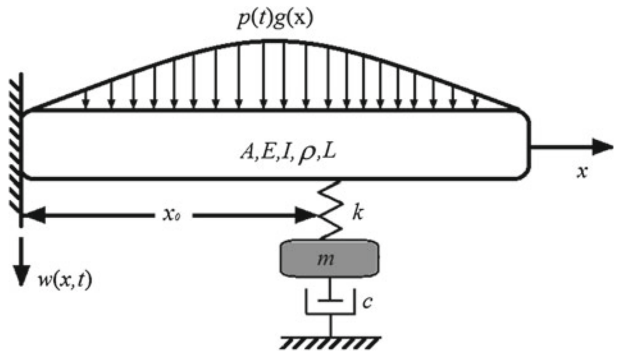

Fig. 2 Cantilever beam with a DVA under external force

H_2 optimization of this kind of DVAs for vibration control of sdof systems. Furthermore, they suggested new procedure for the H_∞ optimization and new optimum parameters derived which resulted in lower maximum amplitude responses [22].

In this article, the H_∞ optimum parameters such as tuning frequency and damping ratios of proposed absorber and also the position of the absorber for minimizing the maximum amplitude response of the beam have been derived analytically, as well as, the H_2 optimum parameters for minimizing mean square motion. According to the author's knowledge, no research has been reported on this topic. It is proven by numerical simulations that these optimized non-traditional DVAs have better operation in vibration control of beam structures than the ordinary one, Furthermore, the effect of mass ratio, DVA position, kind of DVA and optimization procedure is studied which improves our approach in selecting appropriate absorber and optimization procedure in different situations.

2 Frequency response function of the beam with non-traditional DVA

The beam is considered as an Euler–Bernoulli beam, and the dynamic response of the beam is due to the dominant mode only, i.e., single-mode response only, and the responses of other modes may be ignored. The modes can also be well separated. The equation of motion due to external distributed force $p(t)g(x)$ where $p(t)$ is a function of time, $g(x)$ is a deterministic function of x and a point force $f(t)$ generated by non-traditional DVA located at $x = x_0$ as shown in Fig. 2 may be written as

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = p(t)g(x) + f(t)\delta(x - x_0) \tag{1}$$

the length of the beam is L, mass per unit length ρ A, bending stiffness *EI*. The boundary conditions are any combination of pinned, clamped or free supports.

The methodology followed here is based on reference [12] but with a different model. The solution to Eq. (1) can be expanded in a Fourier series written as

$$w(x, t) = \sum_{i=1}^{\infty} q_i(t)\varphi_i(x) \tag{2}$$

where $\varphi_i(x)$ is the eigenfunction of the beam without absorber.

The spatial part of the forcing function $g(x)$ and the Dirac delta function $\delta(x - x_0)$ can be expanded similarly as, respectively

$$g(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x) \tag{3}$$

and

$$\delta(x - x_0) = \sum_{i=1}^{\infty} b_i \varphi_i(x) \tag{4}$$

where the Fourier coefficients are

$$a_i = \frac{1}{L}\int_0^L g(x)\varphi_i(x)\mathrm{d}x \text{ and } b_i = \frac{\varphi_i(x_0)}{L} \tag{5}$$

Substituting Fourier series expansion of forcing, Dirac delta functions and Eq. (2) into Eq. (1) and taking Laplace transformation on the resulting equation with respect to time, an algebraic equation achieved as

$$\rho A s^2 Q_i(s) + EI\beta_i^4 Q_i(s) = a_i P(s) + b_i F(s) \tag{6}$$

where β_i is the eigenvalues of the beam.

By putting Eq. 6 in Laplace transformation of Eq. 2 with respect to time, the s-domain motion of any point on the beam could be written as

$$W(x,s) = \sum_{i=1}^{\infty} \frac{a_i P(s) + b_i F(s)}{\rho A s^2 + EI\beta_i^4}\varphi_i(x) \tag{7}$$

The force due to DVA applied to beam at the attachment point could be written as

$$F(s) = -\frac{k\left(cs + ms^2\right)}{ms^2 + cs + k} W(x_0, s) \tag{8}$$

Eliminating function $F(s)$ in Eq. (7) by use of Eq. (8) result in

$$W(x,s) = \sum_{i=1}^{\infty} \frac{a_i P(s) - b_i \frac{k(cs+ms^2)}{ms^2+cs+k} W(x_0,s)}{\rho A s^2 + EI\beta_i^4}\varphi_i(x) \tag{9}$$

Putting $x = x_0$ in Eq. (7) result in the s-domain motion of the beam at the DVA attachment point

$$W(x_0,s) = \frac{\sum_{i=1}^{\infty} \frac{a_i \varphi_i(x_0) P(s)}{\rho A s^2 + EI\beta_i^4}}{1 + \frac{k\left(cs+ms^2\right)}{ms^2+cs+k}\sum_{i=1}^{\infty}\frac{b_i\varphi_i(x_0)}{\rho A s^2 + EI\beta_i^4}} \tag{10}$$

Substituting the s-domain motion of the beam at the DVA attachment point in Eq. (9), the s-domain motion of any point of the beam could be written as

$$\frac{W(x,s)}{P(s)} = \sum_{i=1}^{\infty} \frac{a_i - b_i \frac{\sum_{i=1}^{\infty}\frac{a_i\varphi_i(x_0)}{\rho A s^2 + EI\beta_i^4}}{\frac{k\left(cs+ms^2\right)}{ms^2+cs+k} + \sum_{i=1}^{\infty}\frac{b_i\varphi_i(x_0)}{\rho A s^2 + EI\beta_i^4}}}{\rho A s^2 + EI\beta_i^4}\varphi_i(x) \tag{11}$$

This equation could be simplified by representing mass, damping, stiffness and natural frequencies of the non-traditional DVA and the beam parameters in non-dimensional forms

$$\mu = \frac{m}{M},\ \zeta_a = \frac{c}{2\sqrt{mk}},\ \omega_a = \sqrt{k/m},\ \gamma = \frac{\omega_a}{\omega_n},\ \gamma_i = \frac{\omega_i}{\omega_n},\ \lambda = \frac{\omega}{\omega_n} \tag{12}$$

where μ is the mass ratio between the absorber mass and the beam mass, ζ_a is damping ratio of the absorber, ω_a is the natural frequency of the absorber, γ is the ratio between the absorber frequency and a reference natural frequency of the beam, γ_i is the non-dimensional natural frequency of the beam referred to ω_n and λ is the normalized frequency.

Using the above non-dimensional parameters in Eq. (11) and replacing s by $j\omega$, the frequency response function of the beam can be obtained as

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{1}{\rho A\omega_n^2}\sum_{i=1}^{\infty} \frac{a_i - b_i \frac{\mu L \sum_{i=1}^{\infty}\frac{a_i\varphi_i(x_0)}{\gamma_i^2-\lambda^2}}{-\frac{2\zeta_a j\gamma\lambda + \gamma^2 - \lambda^2}{\gamma^2(2\zeta_a j\gamma\lambda - \lambda^2)} + \mu L\sum_{i=1}^{\infty}\frac{b_i\varphi_i(x_0)}{\gamma_i^2-\lambda^2}}}{\gamma_i^2 - \lambda^2}\varphi_i(x) \tag{13}$$

The frequency and damping ratio of the absorber besides the mass ratio and the position of the absorber are parameters which affected the performance of DVA in minimizing the vibration of the primary system. In the following, H_∞ and H_2 optimizations are used to find the optimum parameters analytically.

3 H_∞ optimization: minimizing the vibration at a point on the beam

As mentioned before, the natural frequency of the beam considered being well separated, so in the vicinity of the nth natural frequency, the modal displacement response could approximate by $i = n$ and ignoring other modes in Eq. (13)

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{1}{\rho A\omega_n^2} \frac{a_n - b_n \frac{\mu L \frac{a_n\varphi_n(x_0)}{1-\lambda^2}}{-\frac{2j\zeta_a\gamma\lambda+\gamma^2-\lambda^2}{\gamma^2(2j\zeta_a\gamma\lambda-\lambda^2)}+\mu L\frac{b_n\varphi_n(x_0)}{1-\lambda^2}}}{1-\lambda^2}\varphi_n(x) \tag{14}$$

Frequency response function could be simplified by defining an equivalent mass ratio ε,

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{a_n\varphi_n(x)}{\rho A\omega_n^2} \frac{\left(\gamma^2-\lambda^2\right)+(2j\zeta_a\gamma\lambda)}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\varepsilon\gamma^2\lambda^2\right]+\left[2j\zeta_a\gamma\lambda\left(1-\lambda^2+\varepsilon\gamma^2\right)\right]}, \varepsilon = \mu\varphi_n^2(x_0) \tag{15}$$

It may be rewritten as

$$\frac{W(x,\lambda)}{P(\lambda)} = \frac{a_n\varphi_n(x)}{\rho A\omega_n^2} G(\lambda) \tag{16}$$

where

$$G(\lambda) = \frac{\left(\gamma^2-\lambda^2\right)+(2j\zeta_a\gamma\lambda)}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\varepsilon\gamma^2\lambda^2\right]+\left[2j\zeta_a\gamma\lambda\left(1-\lambda^2+\varepsilon\gamma^2\right)\right]} \tag{17}$$

In considering H_∞ optimization for specific point, the objective is to minimize the maximum vibration amplitude response of the primary system at point x.

$$\max\left(\left|\frac{W\left(x,\lambda,\gamma_{H_\infty},\zeta_{H_\infty}\right)}{P(\lambda)}\right|\right) = \min\left(\max_{\gamma,\zeta_a}\left|\frac{W(x,\lambda)}{P(\lambda)}\right|\right) \tag{18}$$

The modal displacement consists of a constant term, $\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}$ which is independent of optimum parameters, so the objective function is changed to Eq. (19) and we just need to optimized $G(\lambda)$ which is equivalent to the amplitude ratio as derived and optimized by Ren [16] in the sdof system attached to a non-traditional DVA if the term ε is replaced by the mass ratio μ.

$$\max\left(\left|\frac{W\left(x,\lambda,\gamma_{H_\infty},\zeta_{H_\infty}\right)}{P(\lambda)}\right|\right) = \frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\max\left(\left|G(\lambda,\gamma_{H_\infty},\zeta_{H_\infty})\right|\right) \tag{19a}$$

H_∞ optimization can be derived based on the fixed-point theory that expresses in the frequency response spectrum there are three points that their amplitudes are independent of damping ratio. $G(\lambda)$is calculated according to Eq. (17) with three damping ratios, and the results are shown in Fig. 3. It can be observed that there are intersecting points O, a and b which are independent of the damping of the absorber.

At any damping ratio, the frequency response must include these three fixed points O, a and b. So the H_∞ optimum condition of the DVA may be expressed as

$$\max\left(\left|G\left(\lambda,\gamma_{H_\infty},\zeta_{H_\infty}\right)\right|\right) = \min\left(\max_{\gamma,\zeta_a}\left(\left|G(\lambda_a)\right|,\left|G(\lambda_b)\right|,\left|G(\lambda_0)\right|\right)\right) \tag{19b}$$

In the other words, regardless of damping, we tune the frequency ratio such a way that the fixed points have minimum amplitude. In the next step, we determine the absorber damping such a way that fixed point becomes the peaks of the response. In order to find the tuning frequency, the heights of the fixed points are calculated at different values of γ and the results are plotted in Fig. 4.

By comparing the height of the fixed points at different values of γ Fig. 5 may be found which consist of local and global minimums that are shown by points A and C, respectively.

Point A is the local minimum of the graph and that is the case which two fixed points have the same amplitude; Ren [16] found the local optimum tuning frequency of the absorber by using this equalization, and

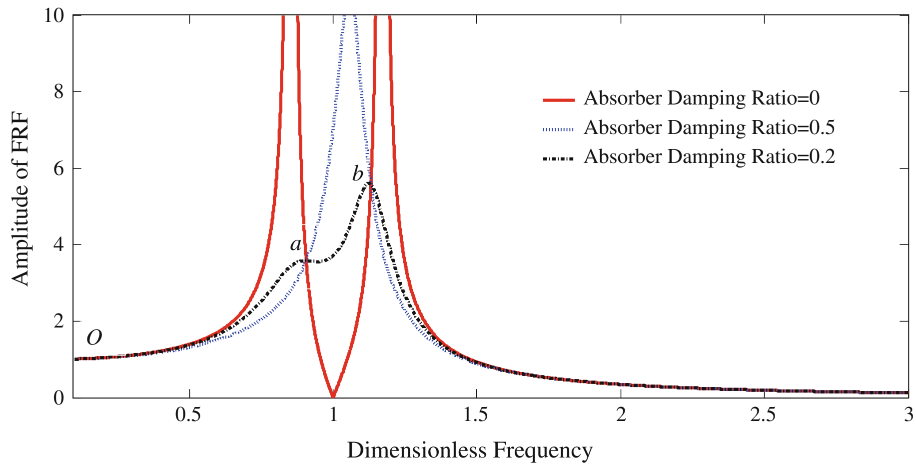

Fig. 3 Amplitude of $G(\lambda)$ at $\varepsilon = 0.15$ and $\gamma = 1$

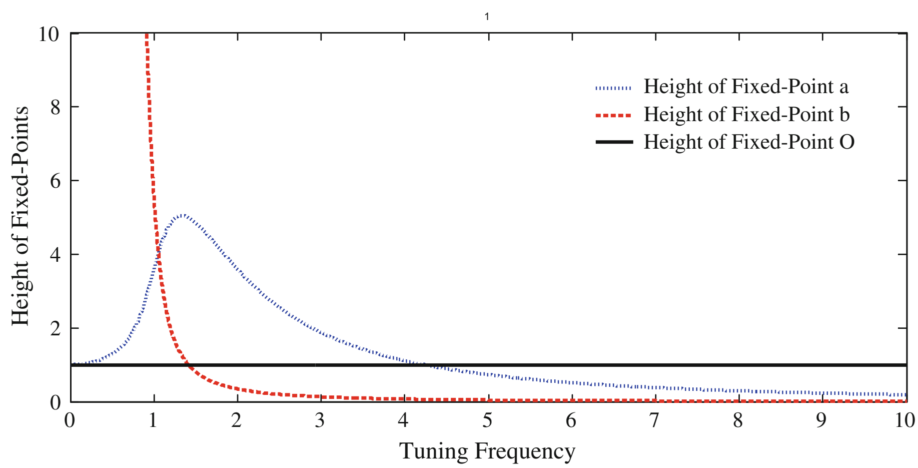

Fig. 4 The height of the fixed points versus tuning frequency γ at $\varepsilon = 0.15$

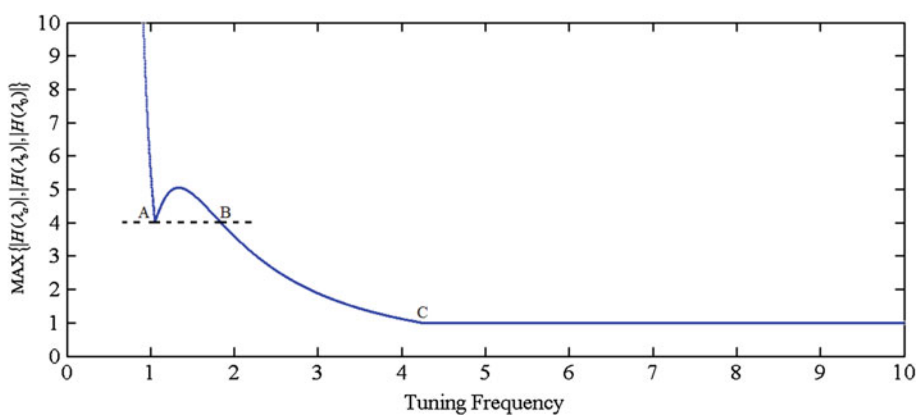

Fig. 5 Max $\{|G(\lambda_a)|, |G(\lambda_b)|, |G(\lambda_0)|\}$ versus γ at $\varepsilon = 0.15$

then, the local optimum absorber damping was determined such a way that the fixed point become the peaks of the response. These optimum parameters set known as the *first set* in the following. Recently, Cheung and Wong [22] used point C which is the global minimum of the graph; they presented a set of global optimum parameters by using the same procedure. In other words, at first they put the height of the fixed point equal to one, they

Table 1 The H_∞ optimum tuning at a point x in the beam

	First set	Second set	Third set
Tuning ratio	$\frac{1}{\sqrt{1-\varepsilon}}$	$\sqrt{\frac{\left(\sqrt{2}+\sqrt{\varepsilon}\right)+\left(1-\sqrt{2\varepsilon}\right)}{\sqrt{\varepsilon}(1-\varepsilon)}}$	$\sqrt{\frac{2(1-\varepsilon)}{\varepsilon}}$
Damping ratio	$\sqrt{\frac{3\varepsilon}{8(1-0.5\varepsilon)}}$	$\sqrt{\frac{1-2\gamma^2(1-\varepsilon)+\gamma^4(1+\varepsilon+\varepsilon^2)-(1-\gamma^2(1-\varepsilon))\sqrt{1-2(1-\varepsilon)\gamma^2+(1+\varepsilon^2)\gamma^4}}{4\gamma^2(1+\gamma^2+\varepsilon\gamma^2-\sqrt{1-2(1-\varepsilon)\gamma^2+(1+\varepsilon^2)\gamma^4})}}$	
Height of fixed points	$\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}(1-\varepsilon)\sqrt{\frac{2}{\varepsilon}}$	$\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\frac{2}{1-\gamma^2+\varepsilon\gamma^2+\sqrt{1-2(1-\varepsilon)\gamma^2+(1+\varepsilon^2)\gamma^4}}$	

found the optimum tuning frequency, and then, zero derivation assumption was used which makes the fixed point peak of the response. In this way, the global optimum damping ratio was derived. These global optimum parameters known as the *Third set* in the following. However, γ_C may be too high to be applied in practice, by assuming the practical constraints, and they may consider a practical range of the optimum tuning frequency parameter of non-traditional DVA such as point B [22]. The optimum parameters derived by using point B is known as the *second set* in this article, and these parameters can be found by using the above procedure.

The optimum parameters of the non-traditional DVA attached to the beam are derived and listed in Table 1. is the damping ratio, and γ_{H_∞} is tuning frequency of H_∞ optimization related to non-traditional DVA attached on the beam.

The optimum frequency and damping ratio of the non-traditional DVA on the beam structure are derived analytically. Now that is time to find the optimum mass ratio and position of the absorber on beam structures. The amplitude of $G(\lambda)$ is calculated for different equivalent mass ratios or ε in Fig. 6.

As mentioned before in Eq. (15), the equivalent mass ratio consists of two terms, the mass ratio and the eigenfunction of the beams at the attachment point. As you can see in the Fig. 6, by increasing the equivalent mass ratio, the absorber becomes more effective, so the optimum location for DVA is the location which makes the eigenfunction maximum or equal to one, and the mass ratio has to be considered as high as possible.

4 H_∞ optimization–minimizing the root mean square motion over whole domain of the beam

The root mean square motion over the whole domain of the beam structure may be written as

$$\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 \mathrm{d}x = \int_0^L \left(\frac{1}{\rho A\omega_n^2}\right)^2 \left(\sum_{i=1}^{\infty} \frac{a_i - b_i \frac{\mu L \sum_{i=1}^{\infty} \frac{a_i\varphi_i(x_0)}{\gamma_i^2-\lambda^2}}{-\frac{2\zeta_a j\gamma\lambda+\gamma^2-\lambda^2}{\gamma^2(2\zeta_a j\gamma\lambda-\lambda^2)}+\mu L\sum_{i=1}^{\infty}\frac{b_i\varphi_i(x_0)}{\gamma_i^2-\lambda^2}}}{\gamma_i^2-\lambda^2}\varphi_i(x)\right)^2 \mathrm{d}x \tag{20}$$

Only the eigenfunction of the beam depends on x, so Eq. (20) could be simplified as

$$\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 \mathrm{d}x = \left(\frac{1}{\rho A\omega_n^2}\right)^2 \int_0^L \left(\sum_{i=1}^{\infty} \frac{a_i - b_i \frac{\mu L \sum_{i=1}^{\infty} \frac{a_i\varphi_i(x_0)P(s)}{\gamma_i^2-\lambda^2}}{-\frac{2\zeta_a j\gamma\lambda+\gamma^2-\lambda^2}{\gamma^2(2\zeta_a j\gamma\lambda-\lambda^2)}+\mu L\sum_{i=1}^{\infty}\frac{b_i\varphi_i(x_0)}{\gamma_i^2-\lambda^2}}}{\gamma_i^2-\lambda^2}\varphi_i(x)\right)^2 \mathrm{d}x \tag{21}$$

According to orthogonality relations of the eigenfunctions and well-separated natural frequencies assumption, the mean square motion is simplified as

$$\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 \mathrm{d}x = \left(\frac{\sqrt{L}}{\rho A\omega_n^2}\frac{a_n - b_n\frac{\mu L\frac{a_n\varphi_n(x_0)}{1-\lambda^2}}{-\frac{2j\zeta_a\gamma\lambda+\gamma^2-\lambda^2}{\gamma^2(2j\zeta_a\gamma\lambda-\lambda^2)}+\mu L\frac{b_n\varphi_n(x_0)}{1-\lambda^2}}}{1-\lambda^2}\right)^2 \tag{22}$$

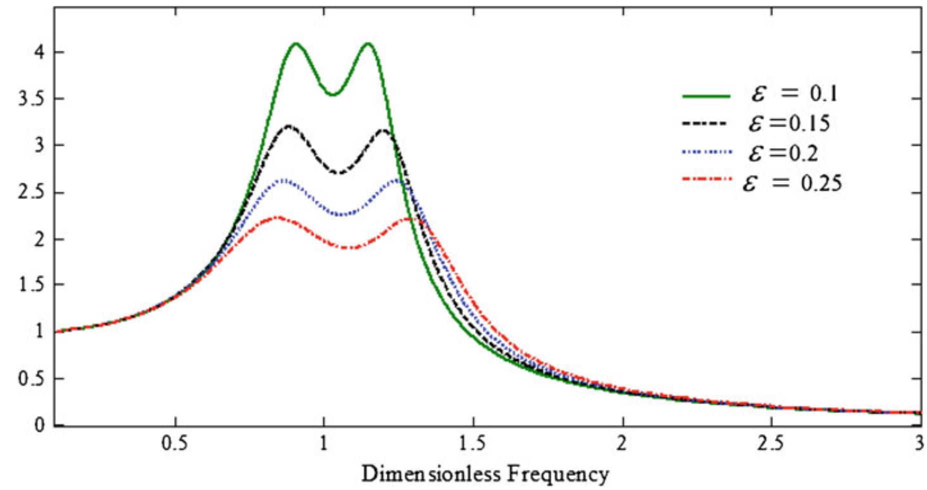

Fig. 6 Amplitude of $G(\lambda)$ at different equivalent mass ratios

Consequently,

$$\sqrt{\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 \mathrm{d}x} = \frac{a_n\sqrt{L}}{\rho A\omega_n^2} \frac{\left(\gamma^2-\lambda^2\right)+(2j\zeta_a\gamma\lambda)}{\left[\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-\varepsilon\gamma^2\lambda^2\right]+\left[2j\zeta_a\gamma\lambda\left(1-\lambda^2+\varepsilon\gamma^2\right)\right]} \tag{23}$$

In the previous section, we optimized dynamic vibration absorber to minimize the vibration of a point, and now we are going to minimize the root mean square motion over the whole domain of the beam. By considering H_∞ optimization, the objective function may be defined as

$$\max\left(\left|\sqrt{\int_0^L\left(\frac{W\left(x,\lambda,\gamma_{H_\infty},\zeta_{H_\infty}\right)}{P(\lambda)}\right)^2\mathrm{d}x}\right|\right) = \min\left(\max_{\gamma,\zeta_a}\left|\sqrt{\int_0^L\left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2\mathrm{d}x}\right|\right) \tag{24a}$$

It is noted that only the function $G(\lambda)$ which is defined in Eq. (17) is needed to be considered in the optimization because of the constant term, so the objective function can be rewritten as

$$\max\left(\left|\sqrt{\int_0^L\left(\frac{W\left(x,\lambda,\gamma_{H_\infty},\zeta_{H_\infty}\right)}{P(\lambda)}\right)^2\mathrm{d}x}\right|\right) = \frac{a_n\sqrt{L}}{\rho A\omega_n^2}\min\left(\max_{\gamma,\zeta_a}|G(\lambda)|\right) \tag{24b}$$

Therefore, applying the fixed-point theory, the optimum parameters can be found in the same way as in the case of sdof. The outcomes for tuning ratio and damping ratio are the same as those achieved for a point, but $\sqrt{L}$ is replaced by $\varphi_n(x)$ in the height of fixed points.

5 H$_2$ optimization: minimizing the area under the frequency response curve of the beam

In considering H_2 optimization, the objective is to minimize the total vibration energy of the system over all frequencies and the performance index can be defined as

$$\min_{\gamma.\zeta_a}\left(E\left[w^2(x,t)\right]\right) \tag{25}$$

The mean square motion, $E\left[w^2(x,t)\right]$, of the stationary response can be obtained when the spectral density, $S_w(\omega)$, of the response is known, according to the following formulae

$$S_w(\omega) = \left|\frac{W(x,\omega)}{P(\omega)}\right|^2 S_P(\omega) \tag{26}$$

So, the mean square motion can be written in terms of the input mean square spectral density as

$$E\left[w^2(x,t)\right] = \int_{-\infty}^{+\infty} \left|\frac{W(x,\omega)}{P(\omega)}\right|^2 S_P(\omega)\mathrm{d}\omega \tag{27}$$

If the input spectrum assumed to be ideally white, i.e., $S_P(\omega) = S_0$, a constant for all frequencies, the integral of Eq. (27) can be reduced to

$$E\left[w^2(x,t)\right] = S_0 \int_{-\infty}^{+\infty} \left|\frac{W(x,\omega)}{P(\omega)}\right|^2 \mathrm{d}\omega \tag{28}$$

Consequently, the non-dimensional mean square motion can be defined as

$$E\left[w^2(x,t)\right] = \frac{\omega_n S_0}{2\pi} \int_{-\infty}^{+\infty} \left|\frac{a_n\varphi_n(x)}{\rho A\omega_n^2} G(\lambda)\right|^2 \mathrm{d}\lambda \tag{29}$$

As $\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}$ is a constant term, only $\frac{\omega_n S_0}{2\pi}\int_{-\infty}^{+\infty}|G(\lambda)|^2\,\mathrm{d}\lambda$, need to be considered in optimization, which is equal to mean square motion of variant design of DVA attached to sdof system [14], Therefore, the optimum DVA parameters can be found in the same way as in the case of the sdof system if the term ε replaced by the mass ratio μ [21]. By using the formula of Gradshteyn and Ryzhik [21], the mean square motion of the beam may be defined as

$$E\left[w^2(x,t)\right] = \frac{\omega_n S_0}{4\varepsilon\zeta\gamma}\left(\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\right)^2\left[1 + \frac{\varepsilon + 4\zeta^2 - 2}{\gamma^2} + \frac{1}{\gamma^4}\right] \tag{30}$$

If $\frac{\partial}{\partial\gamma}E\left[w^2(x,t)\right] = \frac{\partial}{\partial\zeta_a}E\left[w^2(x,t)\right] = 0$, the system has optimum tuning conditions. In the other words, the vanishing of the both derivatives results in optimum conditions. The derivatives of Eq. (30) are

$$\frac{\partial}{\partial\gamma}E\left[w^2(x,t)\right] = \frac{\omega_n S_0}{4\mu\zeta_a\gamma^6}\left(\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\right)^2\left[\gamma^4 + \left(3\varepsilon + 12\zeta_a^2 - 6\right)\gamma^2 + 5\right] = 0 \tag{31}$$

$$\frac{\partial}{\partial\zeta_a}E\left[w^2(x,t)\right] = -\frac{\omega_n S_0}{4\mu\zeta_a^2\gamma^5}\left(\frac{a_n\varphi_n(x)}{\rho A\omega_n^2}\right)^2\left[\gamma^4 + \left(\varepsilon - 4\zeta_a^2 - 2\right)\gamma^2 + 1\right] = 0 \tag{32}$$

Solving Eqs. (31) and (32) leads to the optimum damping ratio written as

$$\zeta_{H_2\,\text{optimization}} = \sqrt{\frac{\gamma^4 + (\varepsilon - 2)\gamma^2 + 1}{4\gamma^2}} \tag{33}$$

and the local minimum of tuning ratios written as

$$\gamma_{H_2\text{-local optimization}} = \frac{1}{2}\sqrt{6 - 3\varepsilon - \sqrt{(6 - 3\varepsilon)^2 - 32}} \tag{34}$$

No global optimum tuning frequency exists in the proposed absorber, and it is advised that a high tuning frequency ratio be used. Furthermore, using of equivalent mass ratio, less than 0.1144, the frequency ratio more than $0.5\sqrt{6 - 3\varepsilon - \sqrt{(6 - 3\varepsilon)^2 - 32}}$ needed; otherwise, contrary results will be achieved. The best value of damping ratio after selecting the tuning frequency ratio may be found by using Eq. (33).

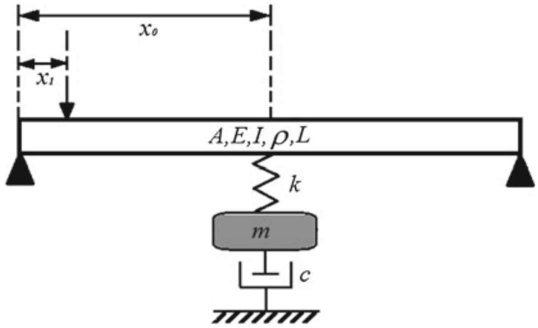

Fig. 7 Schematic model which used in numerical test

6 Simulation results and discussion

To show the efficiency of this optimization procedure of non-traditional DVA on beam structures in vibration suppression, the numerical case studies are presented in this section. At first, vibration suppression performance of variant design of DVA under harmonic force excitation is studied in comparison with traditional one, and then, the effect of the non-traditional DVA in suppressing random vibration is considered.

Model introduction

We consider an Euler beam with simply support boundary condition and attached with non-traditional DVA, and it is shown in Fig. 7. The material of beam supposed to be aluminum of Young's modulus and density of 207 Gpa and 7,870 kg/m^3, respectively, and other properties are $A = 2.24 * 10^{-4}m^2$, $I = 6.31 * 10^{-10}m^4$, $L = 1m$. Absorber attached in $x_0 = 0.5L$, the force applied at $x_1 = 0.1L$ and the mass ratio μ is 0.2.

The eigenfunction and natural frequencies of beam considered as

$$\varphi_i(x) = \sin\left(\frac{i\pi x}{L}\right), \omega_i^2 = \left(\frac{i\pi}{L}\right)^4 \frac{EI}{\rho A}, i = 1, 2, 3, \ldots. \tag{35}$$

and Fourier coefficients are

$$a_i = \frac{1}{L}\sin\left(\frac{i\pi x_1}{L}\right), b_i = \frac{1}{L}\sin\left(\frac{i\pi x_0}{L}\right), i = 1, 2, 3, \ldots \tag{36}$$

Case 1: Vibration suppression performance under harmonic force excitation

We employed the model which introduced in the previous section to show that non-traditional DVA has better outcomes in minimizing vibration of a point and kinetic energy of a beam under harmonic force excitation than use of traditional DVA, while one of the best optimization procedures which followed by Cheung and Wong [12] applied on it.

Case 1-a: Minimizing the vibration at a point on the beam

Performance of proposed DVA in suppressing vibration of a point under harmonic force excitation is compared by traditional DVA, while three optimum parameters which introduced in H_∞ optimization and optimum parameters that presented by Cheung and Wong [12] are applied on non-traditional and traditional DVAs, respectively. Percentage reduction in frequency response of a point by use of non-traditional DVA relative to traditional one was 13.3, 27.0, 53.9 % for *first, second and third set*, respectively. Results in Fig. 8 show that *third set* has the best performance among others, but there are some constraints in real application at times which guide us to use other sets, i.e., small mass ratios cause large frequency, and damping ratio in *third set* or sometimes a damping element could not be so massive hence the use of *the first set* is the best solution.

Case 1-b: Minimizing the kinetic energy of whole beam

The kinetic energy amplitude of whole beam calculated at different excitation frequency according to Eq. (37), while the first mode considered as target mode

$$\frac{\rho h\omega_1^2}{2}\int_0^L \left(\frac{W(x,\lambda)}{P(\lambda)}\right)^2 \mathrm{d}x = \frac{\rho h\omega_1^2}{2}\sum_{i=1}^{10}\left(\frac{L\lambda}{\rho h\omega_1^2}\right)\frac{\left(a_i - b_i\frac{\mu L\sum_{i=1}^{\infty}\frac{a_i\varphi_i(x_0)}{\gamma_i^2-\lambda^2}}{-\frac{2\zeta_a j\gamma\lambda+\gamma^2-\lambda^2}{\lambda^2(2\zeta_a j\gamma\lambda+\gamma^2)}+\mu L\sum_{i=1}^{\infty}\frac{b_i\varphi_i(x_0)}{\gamma_i^2-\lambda^2}}\right)^2}{\left(\gamma_i^2-\lambda^2\right)^2} \tag{37}$$

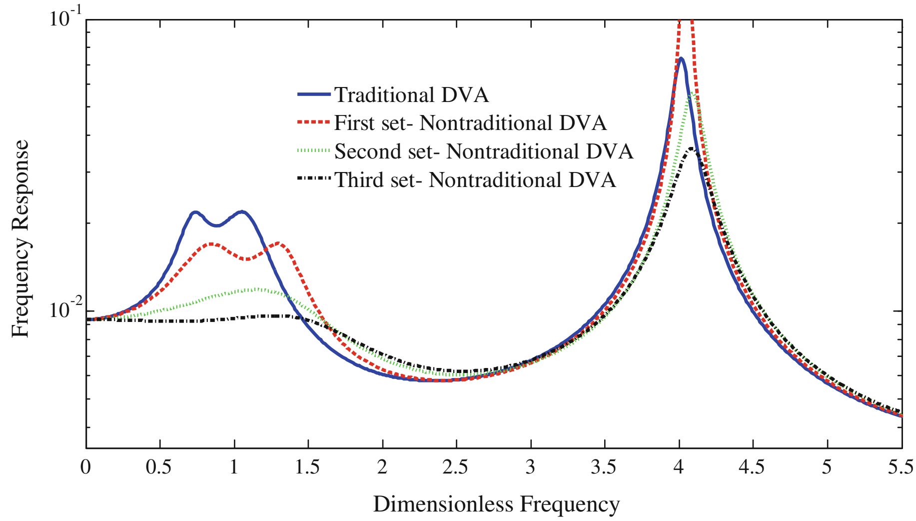

Fig. 8 Minimizing vibration of a point by use of traditional and non-traditional DVA

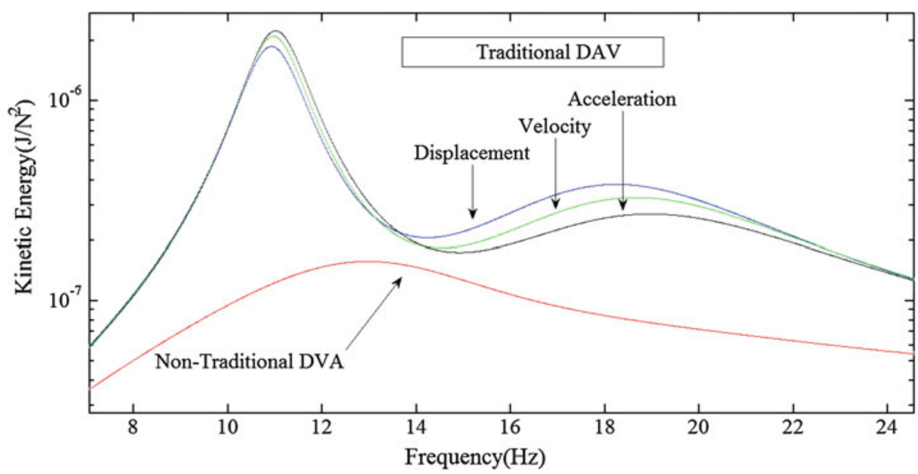

Fig. 9 Minimizing kinetic energy by using non-traditional and traditional DVA with different transfer functions

To calculate these kinetic energy amplitudes, a Matlab program is written, and the results were plotted in Fig. 9. Ten vibration modes ($i_{\max} = 10$) of the beam were used in the calculation. As shown in Fig. 9, the amplitude of the kinetic energy of the first resonance of the beam was suppressed after adding the vibration absorber. By use of traditional dynamic vibration absorber, acceleration transfer function has better results in minimizing kinetic energy at the first resonance of the primary system than other transfer functions. While traditional DVA has a good effect in minimizing kinetic energy, better results achieved by use of non-traditional DVA due to eliminating the pick around objective mode. Consequently, the maximum response of the system can be reduced by more than 35 % in this case while the *first set* of optimum parameters is used.

The effect of different optimum parameters of non-traditional dynamic vibration absorbers in minimizing the kinetic energy is studied in Fig. 10. The percentage reduction in the maximum amplitude of the kinetic energy of the whole beam around the first natural frequency of the beam by use of proposed DVA toward traditional one was 26.9, 40.3, 59.7 % for *first, second and third set*, respectively. It is noted that the acceleration optimum parameters of traditional DVA are used in this comparison.

Case 2: Vibration suppression performance under a random force excitation

For non-traditional DVA, it is recommended that a high tuning frequency ratio be used if it is possible in H_2 optimization, but in this section, the tuning frequency equals to H_∞ optimization of *first set* optimum parameters is considered here for comparison. At first, the spectral density of the vibration amplitude at point x on the beam is calculated for both traditional and non-traditional DVAs and is plotted for H_∞ and H_2

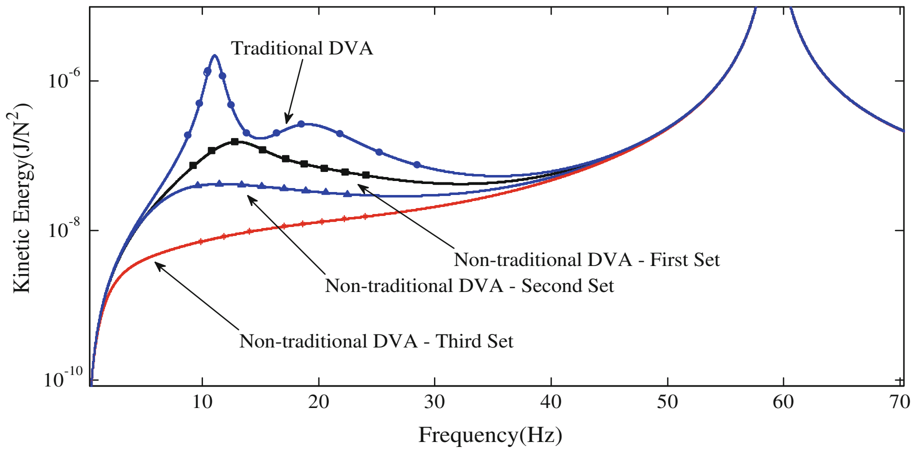

Fig. 10 Using three sets of optimum parameters of non- traditional DVA and optimum parameters of acceleration transfer function of traditional to minimizing kinetic energy

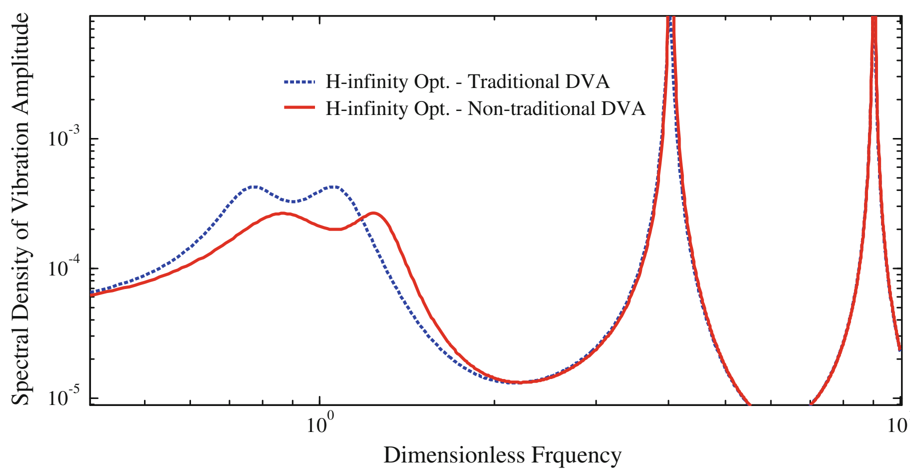

Fig. 11 The spectral density of the vibration amplitude at point x on beam for both DVAs and H_∞ optimization

optimization procedures in Fig. 11 and Fig. 12, respectively. It is obvious that for both optimization procedures, non-traditional DVAs minimize the spectral density of the vibration amplitude at point x better than traditional one about 8.2 % for H_∞ and 11 % for H_2 optimization.

The spectral density of the vibration amplitude is calculated for both optimizations in order to define better procedures for non-traditional DVA. As shown in Fig. 13, H_2 optimization is more effective than the other one in suppressing vibration of the beam under a random force excitation due to 11.4 % reduction in spectral density.

According to the above analysis, using tuning frequency as high as possible in H_2 optimization results in better mean square motion minimization, so the percentage reduction in the mean square motion of the proposed absorber in different tuning frequencies relative to the traditional absorber at different equivalent mass ratios by use of H_2 optimization in both cases is calculated and plotted in Fig. 14. Outcomes show, for a constant equivalent mass ratio, the percentage reduction in the mean square motion raises by increasing tuning frequency ratio. Furthermore, for the fixed value of frequency ratio, increasing equivalent mass ratio results in more percentage reduction in the mean square, although this growth is not significant for frequency ratios less than 1.2. As mentioned before for the equivalent mass ratio less than 1.1, the appropriate frequency ratio needed to choose, if it is not possible, using other optimization procedure is suggested.

In the following, effect of two optimizations, different dynamic vibration absorbers and equivalent mass ratios, ε, in minimizing mean square motion is studied. Mean square motion is calculated for non-traditional and traditional DVA according to Eq. (30) and which proposed by Cheung and Wong [12], respectively. In this calculation, optimum parameters of non-traditional DVA considered as *first set* for H_∞ and Eq. (33), $\gamma = 1.5$

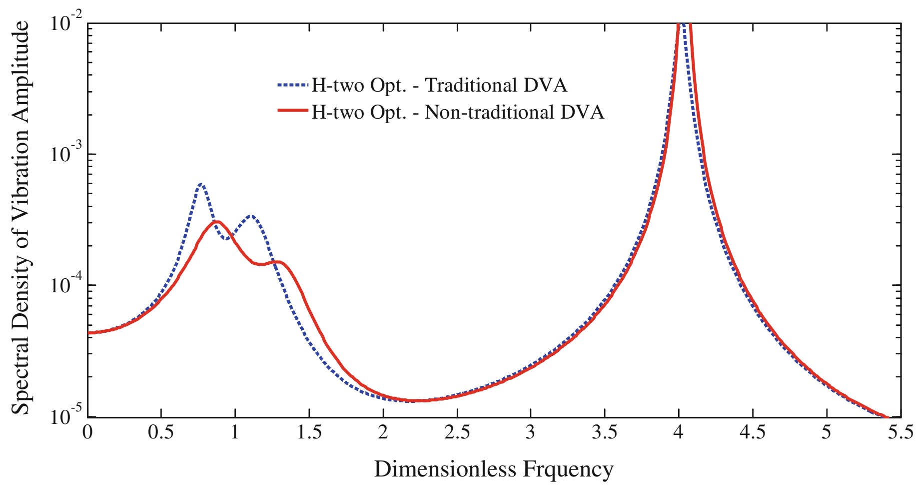

Fig. 12 The spectral density of the vibration amplitude at point x on beam for both DVAs and H_2 optimization

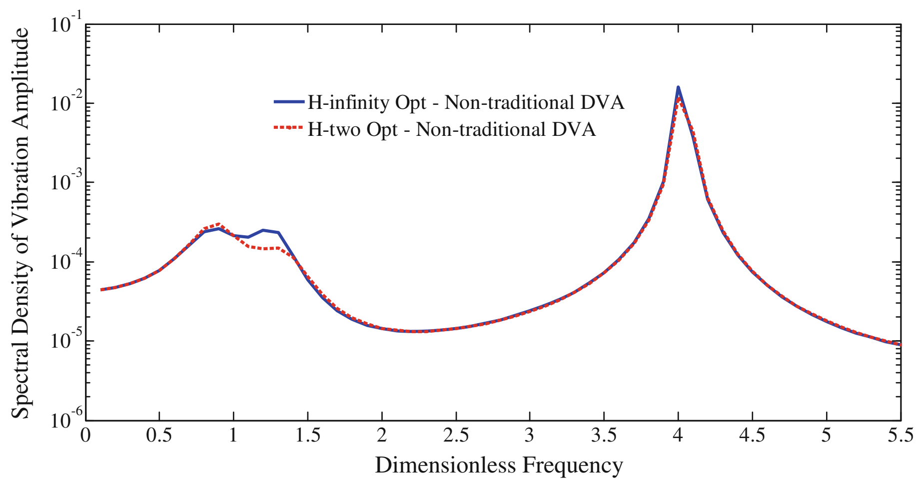

Fig. 13 The spectral density of the vibration amplitude at point x on beam for variant DVA and both optimizations

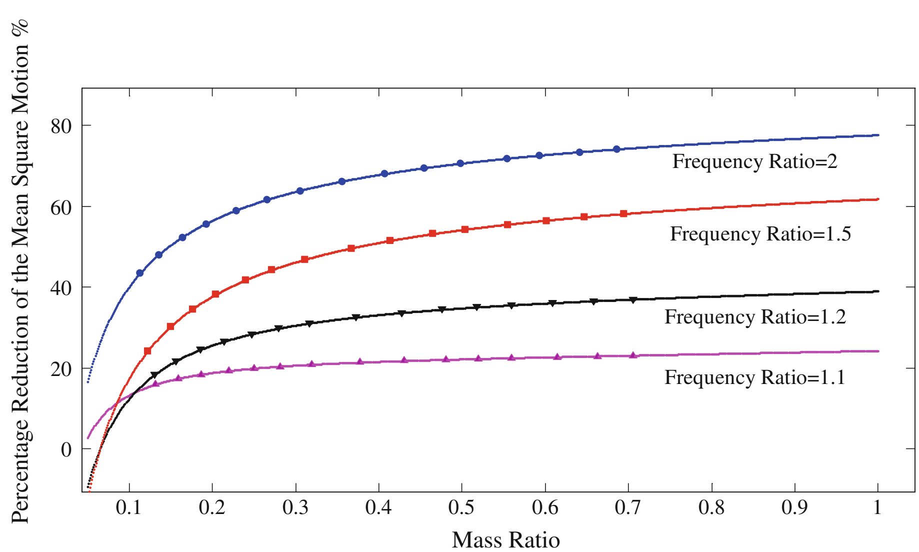

Fig. 14 Percentage reduction in the mean square motion of the proposed absorber relative to the traditional absorber

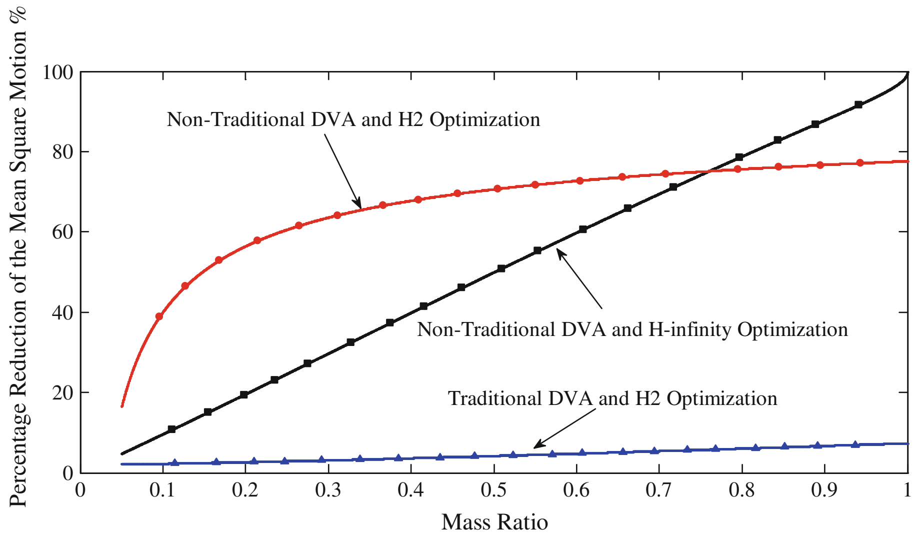

Fig. 15 Percentage reduction in the mean square motion of absorbers relative to the traditional absorber and H_∞ Opt

for H_2 optimization. For tuning frequency and damping ratio of traditional DVA, the formulae which obtained by Cheung and Wong [12] are used. Outcomes show that using H_∞ optimization and traditional DVA has the worst result in minimizing root mean square motion among others, so its percentage reduction in proposed absorber and two optimization procedures relative to the traditional DVA and H_∞ optimization is calculated for different ε and plotted in Fig. 15. It is obvious, in a particular range of equivalent mass ratio, H_2 optimization and non-traditional DVA have the best result in minimizing mean square motion.

7 Conclusion

H_2 and H_∞ optimization procedures of a variant design of dynamic vibration absorbers for suppressing vibration in beam structures under a random and harmonic force excitation were derived. Three sets of optimum parameters proposed for H_∞ optimization, while no global optimum tuning condition exists in the proposed absorber when H_2 optimization is applied, but it is recommended that a high tuning frequency ratio be used if it is possible. The best value of damping ratio after selecting the tuning frequency ratio is derived too. These parameters depend on the value of ε. That means, it depends on both the mass ratio and the modal response of the beam at the attachment point of the DVA. The best location for DVA is the place that makes the eigenfunction of the beam maximum, and the mass ratio has to be chosen as high as possible.

To show the usefulness of non-traditional DVA, numerical results are presented in minimizing the vibration at a point on the beam and kinetic energy of a beam under harmonic force excitation as well as minimizing the mean square motion and the spectral density of the vibration amplitude under a random force excitation which supposed to be ideally white. It has been shown that the performance of the variant DVA can be better than the ordinary one if the frequency and damping ratios of the DVA are chosen properly, i.e., for a practical range of equivalent mass ratio, $\varepsilon = 0.2$, the proposed DVA can provide 13.3 and 26.9 % or more reduction in vibration at a point and kinetic energy of the beam under harmonic excitation, respectively, as well as, 18 and 33 % or more reduction in mean square motion of beam at a point by applying H_∞ optimization and use of H_2 optimization with $\gamma = 1.5$, respectively. Comparison of two optimization procedure for non-traditional DVAs showed the mean square motion at a point of the beam with the proposed absorber and H_2 optimization was 15 % smaller than H_∞ optimization.

Changing the places of the absorber elements is an interesting issue which will be reported elsewhere because the derivation of the H_2 and H_∞ optimal parameters of this variant DVA to control the structural vibration is spacious and the comparison result of the effectiveness of the absorbers is found to be quite different from the result of the present case.

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