



## ON THE DYNAMICS OF BAIRE AND WEAKLY BAIRE SPACES

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ABSTRACT. This is a survey paper of some known results on the  
 preservation of Baire and weakly Baire spaces under images and  
 preimages of some special functions.

### 1. INTRODUCTION

A subset  $A$  of a topological space  $X$  is said to be of the second category if it is contained in the union of countable collection of closed subsets of  $X$  with empty interior in  $X$ ; otherwise  $A$  is said to be of the second category in  $X$ . A topological  $X$  is said to be a Baire space if every non-empty open subset of  $X$  is of the second category. In section 2, we discuss the preservation of Baire category under image and preimage of functions.

Following G. Beer and L. Villar [1], a topological space  $X$  is said to be a weakly Baire space if no non-empty open dense in itself subset is countable. In section 3, we study basic properties of weakly Baire

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spaces. We give an example to illustrate that the class of weakly Baire spaces is strictly larger than the class of Baire spaces. A few other results on weakly Baire spaces are presented.

## 2. IMAGES AND INVERSE IMAGES OF BAIRE SPACES

In this section, the preservation of Baire category under image and preimage of some special functions is studied. Hereafter, we will assume that all spaces are Hausdorff topological spaces.

**Definition 2.1.** A function  $f$  from  $X$  to  $Y$  is called

- (a) feebly open if for every open subset  $U$  of  $X$ ,  $\text{int}(f(U))$  is nonempty.
- (b) quasi-open if for every open set  $V \subseteq X$ ,  $f(U) \subseteq \overline{\text{int} f(U)}$ .
- (c) feebly continuous if for every open subset  $W$  of  $Y$ ,  $\text{int}(f^{-1}(W))$  is nonempty.
- (d) quasi-continuous at  $x_0 \in X$  if for every neighborhood  $U$  of  $x_0$  and every neighborhood  $W$  of  $f(x_0)$ , there is a nonempty open subset  $U'$  of  $U$  such that  $f(U') \subseteq W$ .

The following results may be found in [3] and [4].

**Lemma 2.2. (Frolik's preservation Lemma).** *Let  $f : X \rightarrow Y$  be a quasi-continuous and feebly open surjection. If  $V$  is a dense open subset of  $Y$ , then  $f^{-1}(V)$  is a dense subset of  $X$ .*

**Theorem 2.3. (Frolik's preservation Theorem).** *Let  $f : X \rightarrow Y$  be a quasi-continuous and feebly open surjection. If  $X$  is Baire then so is  $Y$ .*

But what if; we interchange the condition upon  $f$ . I mean if we assume that  $f : X \rightarrow Y$  is feebly continuous and quasi-open. More precisely, one may ask the following questions:

1. Can we have an analogue of Frolik's preservation Lemma?
2. Can we have an analogue of Frolik's preservation Theorem?

The following result gives a positive answer to the first question.

**Lemma 2.4. (Preservation Lemma).** *Let  $f : X \rightarrow Y$  be a feebly continuous and quasi-open function from  $X$  onto  $Y$ . Let  $U$  be an open dense subset of  $X$ . Then  $\text{int} f(U)$  is a dense open subset of  $Y$ .*

*Proof.* Since  $f$  is feebly continuous,  $f(U)$  is dense in  $Y$ . As  $U$  is open and  $f$  is quasi-open, we have

$$\text{int} f(U) \subseteq f(U) \subseteq \overline{\text{int} f(U)}.$$

So that  $Y = \overline{f(U)} \subseteq \overline{\text{int} f(U)}$ . □

Now, let us start “producing” new anti preservation theorems.

**Method.** Let  $\{U_i\}$  be a countable family of open dense subsets of  $X$ . By our new preservation lemma, if  $f$  is a feebly continuous and quasi-open function from  $X$  onto  $Y$ , then  $\text{int}f(U_i)$  is open and dense in  $Y$  for each  $i$ . If  $Y$  is Baire, then  $\bigcap_{i=1}^{\infty} \text{int}f(U_i)$  is dense and feebly openness implies that  $f^{-1}(\bigcap_{i=1}^{\infty} \text{int}f(U_i))$  is dense. Now, we have to show that

$$f^{-1}\left(\bigcap_{i=1}^{\infty} \text{int}f(U_i)\right) \subseteq \bigcap_{i=1}^{\infty} U_i. \quad (2.1)$$

Recall that in the Baire preservation Theorem we used the fact that  $f(f^{-1}(C)) = C$ . But here we have  $A \subseteq f^{-1}(f(A))$  for each  $A \subseteq X$ . The latter formula holds only for injections. This is why the method fails and we cannot obtain an analogue of Theorem 2.3.

The following example shows that the Baire anti-preservation is not true in general.

**Example 2.5.** Let  $X = \bigcup_{i=1}^{\infty} [2i, 2i+1] \cup (\mathbb{Q} \cap [0, 1])$  and  $Y = \mathbb{N}$ . Let  $\{q_1, q_2, \dots\} = \mathbb{Q} \cap [0, 1]$ . Define  $f : X \rightarrow Y$  by:

$$f(x) = i \text{ if } x \in [2i, 2i+1] \cup \{q_i\}, \quad i = 1, 2, \dots$$

Then  $f$  is clearly open and feebly continuous.  $Y$  is Baire but  $X$  is not. We can construct a sequence of dense open sets  $\{U_i\}$  of  $X$  such that  $f^{-1}(\bigcap_{i=1}^{\infty} \text{int}f(U_i))$  is dense but  $\bigcap_{i=1}^{\infty} U_i$  is not dense. In fact, if  $U_i = X \setminus \{q_i\}$  for  $i = 1, 2, \dots$  then  $U_i$  is dense and open in  $X$  and  $f(U_i) = Y$  for every  $i = 1, 2, \dots$ . So  $\text{int}(f(U_i)) = Y$ . We have  $X = f^{-1}(Y) = f^{-1}(\bigcap_{i=1}^{\infty} \text{int}f(U_i))$ . But

$$\bigcap_{i=1}^{\infty} U_i = \bigcap_{i=1}^{\infty} (X \setminus \{q_i\}) = \bigcup_{i=1}^{\infty} [2i, 2i+1]$$

and the latter set is not dense in  $(0, 1) \cap \mathbb{Q}$  which is an open subspace of  $X$  in the relative topology.

### 3. WEAKLY BAIRE SPACES

Let  $X$  be a  $T_1$ -space Baire space. Since singletons are closed in  $X$ , each countable dense-in-itself subset is necessarily meager. Hence Baire spaces belong to the class of weakly Baire spaces. The following example shows that the converse is not true in general.

**Example 3.1.** [1] and [5]. Let  $X = ([0, 1] \cap \mathbb{Q}) \times [0, 1]$ , as a subspace of  $\mathbb{R}^2$  with the usual topology. Every non-empty open subset of  $X$  is clearly uncountable; so  $X$  is weakly Baire. However, the space  $X$  is

itself meager, hence it is not Baire. Let  $p : X \rightarrow [0, 1] \cap \mathbb{Q}$  be the projection on the first variable. Then  $f$  is continuous and open surjective onto a countable meager space. This shows that unlike Baireness, weak Baireness is not preserved by continuous open surjections.

But still some basic properties are shared by both classes. Every non-empty open subspace of a Baire space is Baire. The same is true for weakly Baire spaces, that is, every non-empty open subset  $V$  of a weakly Baire space  $X$  is weakly Baire.

In 1979, W. Fleissner and K. Kunen [2] proved that there is a metric Baire space whose Cartesian square is of the first category. However, for weakly Baire spaces, the situation is different:

**Theorem 3.2.** [1, Theorem 1] *The product of an arbitrary family of weakly Baire spaces is weakly Baire.*

In [5], the authors gave the following equivalent definition of a weakly Baire space. Let  $(X, \tau)$  be a  $T_1$ -space. Let  $M(\tau)$  and  $C(X)$  denote meager and countable subsets of  $X$ , respectively. Define  $X_c = \bigcup(C(X) \cap \tau) = \bigcup\{U : U \in C(X) \cap \tau\}$  and  $\tau_c = \tau|_{X_c}$ . Then  $(X, \tau)$  is called weakly Baire if  $M(\tau) \cap C(X) \cap \tau = \emptyset$ . The authors used this definition to prove the following results.

**Theorem 3.3.** [5, Theorem 1] *The space  $(X, \tau)$  is weakly Baire if and only if  $(X_c, \tau_c)$  is Baire.*

The above link between Baire and weakly Baire category explained above enable us to prove the following.

**Theorem 3.4.** [5, Theorem 4] *If  $f : X \rightarrow Y$  is a quasi-continuous feebly open surjection with countable fibers over  $V_c$ , i.e.  $f^{-1}(y) \in C(X)$  for each  $y \in Y_c$ , then  $Y$  is weakly Baire if  $X$  is weakly Baire.*

*Remark 3.5.* Pertaining to the preimages of Baire spaces, a much stronger result related to the continuity and openness can be derived by using W.Fleissner and K.Kunen example mentioned in part 3.

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