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INTEGRAL METHOD SOLUTION FOR FLOW AND HEAT TRANSFER OVER A PERMEABLE SURFACE WITH CONVECTIVE BOUNDARY CONDITION

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Abstract

A classical problem of hydrodynamic and thermal boundary layers over a permeable plate in a uniform laminar flow is considered in this paper. It is well known that the similarity and integral solutions for the flow and energy equations are possible for the boundary condition of constant surface temperature and constant heat flux but only a few similarity solutions has been attempted for convective surface boundary condition. It is implied in this paper that an integral solution is available if the mass transpiration and the convective heat transfer coefficient associated with the hot lower sided fluid vary like $x^{-1/2}$. Where x is the distance from the leading edge of the solid surface. The governing partial differential equations are first described and then transformed into integral form, before being solved numerically. At the end, the effect of some parameters on the flow and thermal fields are examined and discussed.

Key words: Boundary layer, Convective boundary condition, Integral solution, Permeable surface, Heat transfer

INTEGRAL METHOD SOLUTION FOR FLOW AND HEAT TRANSFER OVER A PERMEABLE SURFACE WITH CONVECTIVE BOUNDARY CONDITION

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Abstract

A classical problem of hydrodynamic and thermal boundary layers over a permeable plate in a uniform laminar flow is considered in this paper. It is well known that the similarity and integral solutions for the flow and energy equations are possible for the boundary condition of constant surface temperature and constant heat flux but only a few similarity solutions has been attempted for convective surface boundary condition. It is implied in this paper that an integral solution is available if the mass transpiration and the convective heat transfer coefficient associated with the hot lower sided fluid vary like $x^{-1/2}$. Where x is the distance from the leading edge of the solid surface. The governing partial differential equations are first described and then transformed into integral form, before being solved numerically. At the end, the effect of some parameters on the flow and thermal fields are examined and discussed.

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1. Introduction

Needless to say, the most important aspect and application of viscous fluid theory is the boundary layer theory. In this theory we are usually confronted with two curve boundary value problems, i.e. one set of conditions is given at the surface and the other at infinity. It is also accepted that a boundary layer is the layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant [1].

Many natural and industrial problems can be simplified and solved using the boundary layer concepts. Moreover, use of these concepts allows scientists and engineers to identify the most dominant parameters governing the process. The first serious industrial application of boundary layer theory occurred in the late 1920s when designers began to use the theory's results to predict skin friction drag on airships and airplanes. Prior to that time, they had been limited to using empirical data obtained primarily from wind tunnels. But because of inaccurate results that had being obtained from those wind tunnels, the designers were reluctant to hinge their designs on them. After that time and when the accuracy and value of skin friction formulas, obtained from boundary layer theory, became more appreciated, the results of the theory became a standard tool for the airplane designers [2].

One of the classical problems in boundary layer theory is flow over a flat plate. This problem was considered by many studies and recently, Aziz [3] and Magyari [4] studied the similar

problem, but with convective boundary condition. Also, Ishak [5] extended the work of Aziz [3] by introducing the effects of suction and injection on the surface. The similarity solution is the method used in all these studies.

One approach to solving the boundary layer equations involves using an approximate integral method. The approach was originally proposed by von Karmen and applied by Pohlhausen [6]. It is without the mathematical complications inherent in the exact method; yet it can be used to obtain reasonably accurate results for key boundary layer parameters.

The objective of the present study is to repeat Ishak's [5] work, but by using the integral method instead of the similarity solution. Besides, it was tried to validate this paper's solution by comparing with works of Ishak [5], Aziz [4] and Shokouhmand [7].

2. Governing Equations

Consider a steady two-dimensional laminar boundary layer flow over a static permeable plate immersed in a viscous fluid of temperature T_{∞} . It is assumed that the free stream moves on the top of the solid surface with a constant velocity U_{∞} . The boundary equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-1}{\rho}\frac{dp_{\infty}}{dx} + v\frac{\partial^2 u}{\partial y^2}$$
(2)

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$
(3)

Where *u* and *v* are the velocity components in the *x* and *y* directions, respectively, *T* is the fluid temperature, *v* is the kinematic viscosity, and α is the thermal diffusivity.

The boundary conditions for the flow field are:

$$u = 0 \quad \& \quad v = V_w(x) \quad at \quad y = 0$$
$$u \to u_\infty \quad \& \quad \frac{\partial u}{\partial y} = 0 \qquad as \quad y \to \infty \tag{4}$$

Where $V_w(x)$ is mass transfer velocity at the plate with $V_w(x) > 0$ for injection (blowing), $V_w(x) < 0$ for suction and $V_w(x) = 0$ corresponds to an impermeable plate. Also, for a flat plate we know that $\frac{-1}{\rho} \frac{dp_{\infty}}{dx} = 0$.

It is assumed that the bottom surface of the plate is heated by convection from a hot fluid of temperature T_f which provides a heat transfer coefficient h_f . Under this assumption, the boundary conditions for the thermal field may be written as:

$$-k\frac{\partial T}{\partial y} = h_f (T_f - T_w), \qquad y \to 0$$

$$T = T_{\infty}, \qquad \frac{\partial T}{\partial y} = 0, \qquad y \to \infty$$
 (5)

With k and T_w being the thermal conductivity and the uniform temperature over the top of the surface, respectively.

By using Eq. (1), we can rewrite (2) and (3) as [8]:

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = v \frac{\partial^2 u}{\partial y^2}$$
(6)

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$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha \frac{\partial^2 T}{\partial y^2}$$
(7)

By implementation of integral with respect to y for (6) and after some simplifications we obtain:

$$\frac{d}{dx}\int_{0}^{Y}(u^{2})dy + u_{\infty}\nu(y) = -\nu\frac{\partial u}{\partial y}_{y=0}$$
(8)

Also by substituting (4) into integral form of (1) we have:

$$v(y) = -\frac{d}{dx} \int_0^y u \, dy + v(0) \tag{9}$$

Substituting (9) in (8) and using (4), after some calculations, we obtain:

$$\frac{d}{dx}\int_0^Y u(u_\infty - u)dy = \frac{du_\infty}{dx}\int_0^y u\,dy + u_\infty v_w(x) + v\frac{\partial u}{\partial y_{y=0}}$$
(10)

We know that for flow over a flat plate $\frac{du_{\infty}}{dx}$ is equal to zero, so the integral momentum equation can be written as:

$$\frac{d}{dx}\int_{0}^{Y}u(u_{\infty}-u)dy = u_{\infty}v_{w}(x) + v\frac{\partial u}{\partial y_{y=0}}$$
(11)

In order that the similarity solutions of Eqs. (1)-(5) exists, Ishak [5] considered:

$$v_{w}(x) = -\frac{1}{2} \left(\frac{v u_{\infty}}{x}\right)^{\frac{1}{2}} f_{w}$$
(12)

From scale up [8] we have $\delta \sim x^{\frac{1}{2}}$ and from (12) we have $v_w(x) \sim x^{-1/2}$, so we can take:

$$v_w(x) = \frac{B}{\delta} \tag{13}$$

Where *B* is a constant and δ is hydrodynamic boundary layer thickness. Substituting (13) into (11) we have:

$$\frac{d}{dx}\int_{0}^{Y}u(u_{\infty}-u)dy = u_{\infty}\frac{B}{\delta} + v\frac{\partial u}{\partial y_{y=0}}$$
(14)

In order to solve above equation, we introduce the following transformations [8]:

$$f(\eta) = \frac{u}{u_{\infty}} , \quad \eta = \frac{y}{\delta} , \quad dy = \delta d\eta$$

$$IF \quad \eta > 1 \qquad THEN \qquad f(\eta) = 1 \tag{15}$$

Substituting (15) into (2) and using (4), we obtain from (14):

$$\frac{d}{dx}\left(\frac{\delta^2}{2}\right) = \frac{1}{u_{\infty}} \frac{\nu \frac{\partial f}{\partial \eta_{\eta=0}} + B}{\int_0^1 f(1-f)d\eta}$$
(16)

Considering a polynomial function of third degree for f and after some calculations, by implementing (15) into (4) we have:

$$f(\eta) = -2\frac{B+\nu}{B+4\nu}\eta^3 + \frac{3B}{B+4\nu}\eta^2 + \frac{6\nu}{B+4\nu}\eta$$
(17)

Substituting (17) into (16) we obtain:

$$\frac{\delta}{x} = \left(2\frac{C_2}{C_1} + c_3\right)^{\frac{1}{2}} Re_x^{-\frac{1}{2}}$$
(18)

Where in (18) C_1 , C_2 and C_3 are:

$$C_1 = \int_0^1 f(1-f)d\eta = \frac{9B^2 + 1341B\nu + 156\nu^2}{70(B+4\nu)^2}, \quad C_2 = \frac{\partial f}{\partial \eta_{\eta=0}} = \frac{6\nu}{B+4\nu}, \quad C_3 = 2\frac{B}{\nu C_1}$$
(19)

In Eq. (18), for $C_3 > 0$ we have injection, for $C_3 < 0$ suction and for $C_3 = 0$ impermeable surface. For $C_3 = 0$ we obtain from (19) B = 0 and so we have:

$$C_1 = \frac{39}{280}$$
 , $C_2 = \frac{3}{2}$ (20)

Substituting (20) into (18) we obtain:

$$\frac{\delta}{x} = \left(2\frac{\frac{3}{2}}{\frac{39}{280}} + 0\right)^{\frac{1}{2}} Re_x^{-\frac{1}{2}} = (21.54)^{\frac{1}{2}} Re_x^{-\frac{1}{2}} , \quad \frac{\delta}{x} = 4.64 Re_x^{-\frac{1}{2}}$$
(21)

That the above result is relatively a good approximation for Blasius [9] similarity solution [8].

In order to solve thermal field, we can substitute (9) into (7) and then by using (4) and (5) we obtain:

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)dy = \alpha \frac{\partial T}{\partial y}_{y=0} + v_w(x)(T_\infty - T_0) + \frac{dT_\infty}{dx}\int_0^Y u\,dy$$
(22)

We know that the ambient temperature T_{∞} is constant and not dependent on x, so $\frac{dT_{\infty}}{dx} = 0$. In order to solve Eq. (22), we introduce the following transformations too [8]:

$$\theta(\xi) = \frac{T_f - T}{T_f - T_{\infty}} , \qquad \xi = \frac{y}{\delta_T} , \qquad \xi = \frac{\eta}{\Delta} , \qquad \Delta = \frac{\delta_T}{\delta}$$
(23)

Substituting (23) into (21), we obtain:

$$\delta_T = 4.64 \Delta \left(\frac{\nu}{u_{\infty}}\right)^{\frac{1}{2}} x^{\frac{1}{2}}$$
(24)

Substituting (23) into (22), using (15), (23) and (13), we have:

$$\frac{d}{dx}\left(\frac{\delta_T^2}{2}\right) = \frac{\frac{\alpha}{u_\infty}\frac{\partial\theta}{\partial\xi}_{\xi=0} + \frac{B\Delta}{u_\infty}\left(\frac{T_0 - T_\infty}{T_f - T_\infty}\right)}{\int_0^1 f(\Delta\xi) \left(1 - \theta(\xi)\right)d\xi}$$
(25)

Considering a polynomial function of third degree for θ and after some calculations, by using (23) and (5) we have:

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$$\theta(\xi) = a_1 \xi^3 + b_1 \xi^2 + c_1 \xi + d_1 \tag{26}$$

Where in (26), a_1 , b_1 , c_1 and d_1 are:

$$a_{1} = -\frac{2\delta_{T}h_{f}B\Delta + 2\alpha\delta_{T}h_{f}}{\delta_{T}h_{f}B\Delta + 4\alpha\delta_{T}h_{f} + 6\alpha k} , \quad b_{1} = \frac{3\delta_{T}h_{f}B\Delta}{\delta_{T}h_{f}B\Delta + 4\alpha\delta_{T}h_{f} + 6\alpha k}$$

$$c_{1} = \frac{6\alpha\delta_{T}h_{f}}{\delta_{T}h_{f}B\Delta + 4\alpha\delta_{T}h_{f} + 6\alpha k} , \quad d_{1} = \frac{6\alpha k}{\delta_{T}h_{f}B\Delta + 4\alpha\delta_{T}h_{f} + 6\alpha k}$$
(27)

For B = 0 (impermeable plate) we have no injection and suction. We also can obtain from (27):

$$a_{1} = -\frac{\delta_{T}h_{f}}{2\delta_{T}h_{f} + 3k} , \qquad b_{1} = 0$$

$$c_{1} = \frac{3\delta_{T}h_{f}}{2\delta_{T}h_{f} + 3k} , \qquad d_{1} = \frac{3k}{2\delta_{T}h_{f} + 3k}$$
(28)

Considering above simplifications, we have two case for thermal field:

1-
$$\Delta < \mathbf{1}$$

 $\frac{\delta_T}{x} = \left(\frac{280}{14\Delta - \Delta^3}\right)^{\frac{1}{2}} Pr^{-\frac{1}{2}} Re_x^{-\frac{1}{2}}$ and $\Delta = 3.6 \left(\frac{1}{14\Delta - \Delta^3}\right)^{\frac{1}{2}} Pr^{-\frac{1}{2}}$ (29)

2-
$$\Delta > 1$$

$$\frac{\delta_T}{x} = \left(\frac{280}{-35\Delta^{-1} + 14\Delta^{-2} - \Delta^{-4} + 35}\right)^{\frac{1}{2}} Pr^{-\frac{1}{2}} Re_x^{-\frac{1}{2}} \quad \text{and} \quad (30)$$

$$\Delta = 3.6 \left(\frac{1}{-35\Delta^{-1} + 14\Delta^{-2} - \Delta^{-4} + 35}\right)^{\frac{1}{2}} Pr^{-\frac{1}{2}}$$

As we saw in (29) and (30), $\delta_T \sim x^{\frac{1}{2}}$. In order that Eq. (26) not be dependent on *x*, we easily find in (27) and (28) that h_f must be proportional to $x^{-\frac{1}{2}}$. This is the vital condition for the similarity solution that the solution must be independent of *x* [8]. So we can assume that [3]:

$$h_f = c \, x^{-\frac{1}{2}} \tag{31}$$

In order to simplify equations and have comparability with Ishak's [5], Aziz's [3] and Shokouhmand's [4] results, Eq. (28) was rewritten and a constant a, was introduced:

$$a = \frac{c}{k} \left(\frac{\nu}{u_{\infty}}\right)^{\frac{1}{2}}$$
(32)

$$a_1 = -\frac{4.64\Delta a}{9.28\Delta a + 3}$$
, $c_1 = \frac{13.92\Delta a}{9.28\Delta a + 3}$, $d_1 = \frac{3}{9.28\Delta a + 3}$ (33)

3. Results and Discussion

The focus in this section will be on the solving of the flow and the thermal fields. In order to approach the goal, the FORTRAN software was used.

In order to compare our results with those of Ishak [5], we took the assumption that was introduced by him:

$$v_w(x) = \frac{1}{2} \left(\frac{\nu u_\infty}{x}\right)^{\frac{1}{2}} f_w \tag{34}$$

Using iterative method to find *B*, we tried to obtain δ from Eq. (18) and $f(\eta)$ from Eq. (17) for various values of f_w .



Figure 1: Velocity profiles for various values of f_w Figure 2: Hydrodynamic boundary layer thickness for various values of f_w

Fig. 1 shows the velocity profiles for different values of suction/injection parameter f_w . As it was seen in Eqs. (17) and (18), parameters *a* and *pr* give no influence to the flow field. It was observed that the velocity gradient was increased by suction and decreased by injection. It means that the skin friction, correspondingly, increased as f_w increased and decreased as f_w decreased. Thus the surface shear stress is higher for suction ($f_w < 0$) compared to injection ($f_w > 0$).

Fig. 2 shows the hydrodynamic boundary layer thickness δ , for different values of f_w . It was observed that δ increased as f_w increased and decreased as f_w decreased. This observation can easily justify our results in Fig. 1.

We also tried to compare our results with aziz's [3], Ishak's [5] and Shokouhmand;s [7] for impermeable surface of the plate. In order to that, we took new variable γ , and new dimensionless temperature function that were introduced by Aziz [3]:

$$\vartheta(\xi) = \frac{T - T_{\infty}}{T_f - T_{\infty}} = 1 - \frac{T_f - T}{T_f - T_{\infty}} = 1 - \theta(\xi)$$
(35)

$$\gamma = \gamma \left(\frac{u_{\infty}}{\nu x}\right)^{\frac{1}{2}} \xrightarrow{Eq. (24)} \gamma = 4.64 \frac{\gamma}{\delta_T} \Delta \left(\frac{\nu}{u_{\infty}}\right)^{\frac{1}{2}} \left(\frac{u_{\infty}}{\nu}\right)^{\frac{1}{2}} \rightarrow \xi = \frac{\gamma}{4.64 \Delta}$$
(36)

With this new variable we can obtain from Eqs. (26), (32), (34) and (35):

$$\vartheta'(0) = \frac{3a}{9.28\,\Delta\,a + 3}\tag{37}$$

Needless to say, we obtained Eqs. (34)-(37) just to compare our results with those of Aziz [3], Ishak [5] and Shokouhmand [7]. Now we can calculate $\vartheta(0)$ and $\vartheta'(0)$ from Eqs. (34) and (37), respectively, with iterative method to obtain Δ .



Figure 3: A comparison between presented results for $\vartheta'(0)$, a) pr = 0.1, b) pr = 10

а	Similarity	Integral method	а	Similarity	Integral method
0.05	0.04192336	0.0417384	0.8	0.19596121	0.191985
0.1	0.07218628	0.0716398	1	0.20605591	0.201665
0.2	0.11295543	0.111623	5	0.24672765	0.240458
0.4	0.1574047	0.154829	10	0.98431	0.986633
0.6	0.1811687	0.177765	20	0.99209	0.993271

Table 1: Values of $\vartheta'(0)$ for pr = 0.5 and various values of a. (The similarity solution was reported by [7])

Fig. 3 shows a comparison between the results that presented by Aziz [3], Ishak [5] and the present study (integral method). It is obvious that the numerical results for pr = 0.1 are relatively not enough accurate, owing to the small boundary layer thickness set in all the computations. It is well known that the prandtl number pr is a ratio of viscous to conduction effects, so the lower prandtl number, the thicker thermal boundary layer [8].

Table 1 shows Values of $\vartheta'(0)$ for various values of a. The similarity solution results in this table were reported by H.Shokouhmand [7]. The difference between our solution and results reported by [7] in the worst situation was about 0.2%, so it is evident that our solution results are relatively accurate.



Figure 4: Temperature profiles for various values of pr when $f_w = 0$ and a = 1

Figure 5: Temperature profiles for various values of *a* when $f_w = 1$ and pr = 0.72

Fig. 4 shows the temperature profiles for different values of pr when the other parameters are fixed. It is evident from this figure that the temperature gradient at the surface increases as pr increases, which implies an increase in the heat transfer rate at the surface. This is because a higher Prandtl number fluid has a relatively lower thermal conductivity, which reduces conduction, and thereby the thermal boundary layer thickness, and as a consequence the heat transfer rate at the surface increases [8].

Fig. 5 shows the temperature distribution for a fixed Prandtl number of 0.72 and for various values of the parameter a. It is seen in Fig. 5 that the surface temperature $\theta(0)$ increases as a increases. The parameter a at any location x is directly proportional to the heat transfer coefficient associated with the hot fluid h_f . The thermal resistance on the hot fluid side is inversely proportional to h_f . Thus as a increases, the hot fluid side convection resistance decreases and consequently, the surface temperature $\theta(0)$ increases [3]. As it is evident in Eqs. (31) and (32), $h_f \to \infty$ as $a \to \infty$, so the solution approaches the classical solution for the constant surface temperature that was considered by Pahlhausen [6].

3. Conclusions

The problem of steady laminar boundary layer flow and heat transfer over a stationary permeable flat plate with convective boundary condition was considered. Integral method solution was used to solve this problem. The solution was compared to the similarity solution results that were reported by some other studies. It was seen that the difference between results was negligible. It was observed that integral method solution exists if the convective heat transfer from the lower surface and the mass transpiration rate at the surface are proportional to $x^{-1/2}$, where x is the distance from the leading edge of the solid surface. It was also found that suction decreases both hydrodynamic and thermal boundary layers. Thus, suction increases surface shear stress and as a consequence increases the heat transfer rate at the surface and increases velocity and temperature gradient while injection acts vice versa.

References

[1] H. Schlichting, Klaus Gersten, Boundary Layer Theory, 8th ed. Springer-Verlag 2004.

[2] S. Saedodin, M.S. Motaghedi, Unexpected Results on the Integral form of the Boundary Layer Momentum Equation, World applied Sciences Journal 18 (2):221-225 (2012)

[3] A. Aziz, A similarity solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1064–1068

[4] E. Magyari, Comment on "A similarity solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition" by A. Aziz, Commun. Nonlinear Sci. Numer. Simul. 2009;14:1064–8, Commun. Nonlinear Sci. Numer. Simul. (2010), doi:10.1016/j.cnsns.2010.03.020.

[5] A. Ishak, Similarity solutions for flow and heat transfer over a permeable surface with convective boundary condition, Appl. Math. Comput. (2010), doi:10.1016/j.amc.2010.06.026

[6] E. Pohlhausen, Der wärmeaustausch zwischen festen körpern und flüssigkeiten mit kleiner reibung und kleiner wärmeleitung, Z. Angew. Math. Mech. 1 (1921) 115–121.

[7] H. shokouhmand, M.Fakoor, M kooshkbaghi, Commun Nonlinear Sci Numer Simulat 15 (2010) 3965–3973

[8] A. Bejan, Convection Heat Transfer, third ed., John Wiley & Son, New York, 2004.

[9] H. Blasius, Grenzschichten in flüssigkeiten mit kleiner reibung, Z. Math. Phys. 56 (1908) 1–37.