A new method to measure the distance between interval-valued fuzzy numbers

Mohammad Reza Rabiee[1,*], **Naser Reza Arghami**[1,♣] **and Bahram Sadeghpour**[1,2,♠]

[1]Department of Statistics, Ferdowsi University of Mashhad, Mashhad, 91775 Iran,
[2]Department of Statistics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, 47419 Iran,
[*] Corresponding author. Email: rabie1354@yahoo.com
♣ Email: arghami_nr@yahoo.com
♠ Email: sadeghpour@umz.ac.ir

Abstract

This paper gives a new kind of distance between interval-valued fuzzy sets defined on real line R, denoted by $D^{*}_{f,p}$. The applicability of the proposed method is investigated by a numerical data set and this distance is compared with two other distances by an example.

Keywords: Interval-valued fuzzy number, distance, Hausdorff metric.

1 Introduction

Since fuzzy set theory was introduced by Zadeh [18], many new approaches and theories treating imprecision and uncertainty have been proposed. Specially, the intuitionistic fuzzy set theory pioneered by Atanassov [1], and the interval-valued fuzzy set theory suggested by Grozafczany [6] and Turksen [11] are two well-known generalizations of the fuzzy set theory. In fact, it is pointed out that there is a strong connection between Atanassov's intuitionistic fuzzy sets and the interval-valued fuzzy sets [3, 4, 12]. Over the last decades, the theory of interval-valued fuzzy set has been developed in different directions. For the proposes of this article, we briefly review some works on this topic.

Gorzalczany [7] investigated approximate reasoning based on interval-valued fuzzy sets. Wang and Li [13] presented the applications of interval-valued fuzzy numbers and interval-distribution numbers in pseudo-probability metric spaces. Wang and Li [13, 14] introduced the concept of interval-valued fuzzy number and studied some of its properties and presented a method for calculating correlation and information energy of interval-valued fuzzy numbers. In [8], Hong and Lee presented some algebraic properties and a distance measure for interval-valued fuzzy numbers. Przemyslaw [10] studied some distances between interval-valued fuzzy sets based on the Hausdorff metric. Wang et al. [15] investigated the combination and normalization of the interval-valued belief structures. Deschrijver [5] investigated some arithmetic operators in interval-valued fuzzy set theory. Chen and Chen [2] presented a method for handling information filtering problems based on interval-valued fuzzy numbers and presented a similarity measure between interval-valued fuzzy numbers. In [9], Chen presented a method for handling the similarity measure problems of interval-valued fuzzy numbers.

The structure of this paper is as follows. Section 2 shows the preliminaries, it includes notations and basic concepts which will be used in the following section. Section 3 shows the new kind of distance and discusses some Proposition and Theorem. By using a numerical example, we compare this distance with two other distances in Section 4. A brief conclusion is given in the last section.

2 Preliminaries

In this section, we review some elementary definitions and a well-known result of the interval-valued fuzzy sets and interval-valued fuzzy numbers, biased on Wang and Li [13], Hong and Lee [8], and Zhixn and Hongmei [19]. Let $I=[0,1]$ and $[I]=\{[a,b] \mid a \le b, a,b \in I\}$. For any $a \in I$, define $\bar{a}=[a,a]$.

Definition 2.1 If $a_t \in I, t \in T$, then we define $\vee_{t\in T} a_t = \sup\{a_t : t \in T\}$ and $\wedge_{t\in T} a_t = \inf\{a_t : t \in T\}$. We also define for $[a_t, b_t] \in [I], t \in T$,

$$i)\ \vee_{t\in T}[a_t,b_t] = [\vee_{t\in T} a_t, \vee_{t\in T} b_t], \wedge_{t\in T}[a_t,b_t] = [\wedge_{t\in T} a_t, \wedge_{t\in T} b_t] \quad (1)$$

$$ii)\ [a_1,b_1] = [a_2,b_2] \text{ iff } a_1 = a_2, b_1 = b_2, \quad (2)$$

$$[a_1,b_1] \le [a_2,b_2] \text{ iff } a_1 \le a_2, b_1 \le b_2, \quad (3)$$

$$[a_1,b_1] < [a_2,b_2] \text{ iff } [a_1,b_1] \le [a_2,b_2], \text{ but } [a_1,b_1] \ne [a_2,b_2]. \quad (4)$$

Definition 2.2 Let X be an ordinary nonempty set. Then:

- The mapping $A : X \to [I]$ is called an interval-valued fuzzy set (IVFS) on X. The set of all IVFS on X is denoted by IF(X).
- For $A \in IF(X)$, let $A(x) = [A^-(x), A^+(x)]$, for all $x \in X$. Then two fuzzy sets $A^- : X \to I$ and $A^+ : X \to I$ are called lower fuzzy set and upper fuzzy set of A, respectively.
- The value of $\Pi_A(x) = A^+(x) - A^-(x)$ is called the degree of non-determinancy of the element $x \in X$ to the IVFS A.

Definition 2.3 Let $A \in IF(X)$ and $[\lambda_1, \lambda_2] \in [I]$. We call $A_{[\lambda_1,\lambda_2]} = \{x \in X : A^-(x) \ge \lambda_1,\ A^+(x) \ge \lambda_2\}$ and $A_{(\lambda_1,\lambda_2)} = \{x \in X : A^-(x) > \lambda_1, A^+(x) > \lambda_2\}$ the $[\lambda_1,\lambda_2]$-level set of A and the (λ_1,λ_2)-level set of A, respectively.

Definition 2.4 Let $A \in IF(R)$, where R is the real line. Assume the following conditions are satisfied:

- A is normal, i.e., there exists $x_0 \in R$, such that $A(x_0) = \overline{1}$,
- For arbitrary $[\lambda_1,\lambda_2] \in [I]^+ = [I] - \{\overline{0}\}$, $A_{[\lambda_1,\lambda_2]}$ is a closed bounded interval.

Then we call A an interval-valued fuzzy number (IVFN). We denote the set of all IVFNs by $IF^*(R)$.

Definition 2.5 Let $A, B \in IF(R)$ and $\bullet \in \{+,-,\cdot,\div\}$. We define the extended operations by $(A \bullet B)(z) = \vee_{z = x\bullet y}(A(x) \wedge B(y))$. For each $[\lambda_1,\lambda_2] \in [I]^+$, we write $A_{[\lambda_1,\lambda_2]} \bullet B_{[\lambda_1,\lambda_2]} = \{x \bullet y : x \in A_{[\lambda_1,\lambda_2]}, y \in B_{[\lambda_1,\lambda_2]}\}$.

Definition 2.6 A triangular IVFN is represented as $A = [A^-, A^+] = [(a_1^-, a, a_2^-), (a_1^+, a, a_2^+)]$, where A^- and A^+ denote the lower and upper triangular fuzzy numbers of A, $A^- \subset A^+$. Also, A is denoted by $A = [A^-, A^+] = [(a_1^+, a_1^-), a, (a_2^-, a_2^+)]$ where $a_1^+ \le a_1^- \le a \le a_2^- \le a_2^+$ (see Figure 1).

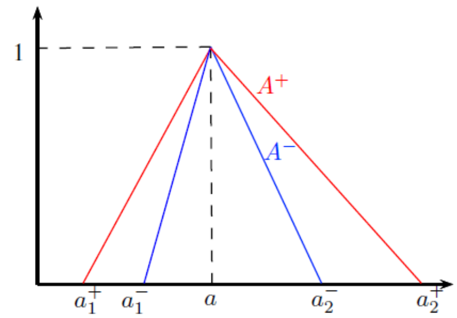

Figure 1: A typical triangular interval-valued fuzzy number

3 A new distance between interval-valued fuzzy numbers

Based on definitions given in [9] and [16], we propose the following definition of distance between IVFNs.

Definition 3.1 Let $A, B \in IF^*(R)$. The $D^*_{p,f}$ distance between A and B is defined as

$$D^*_{p,f}(A,B) = \max\{D_{p,f}(A^-,B^-), D_{p,f}(A^+,B^+)\} \tag{5}$$

where

$$D_{p,f}(A^\bullet,B^\bullet) = \left(\int_0^1 f(\lambda) d^p(A^\bullet_\lambda, B^\bullet_\lambda) d\lambda\right)^{1/p}, \tag{6}$$

where $\bullet \in \{-,+\}$ and

$$d^p(A^\bullet_\lambda, B^\bullet_\lambda) = |a_1(\lambda) - b_1(\lambda)|^p + |a_2(\lambda) - b_2(\lambda)|^p,$$
$$A^\bullet_\lambda = [a_1(\lambda), a_2(\lambda)], \quad B^\bullet_\lambda = [b_1(\lambda), b_2(\lambda)] \tag{7}$$

and $f(\lambda)$ is an increasing function on $[0,1]$ with $f(0) = 0$ and $\int_0^1 f(\lambda) d\lambda = \frac{1}{2}$.

Specially, for $p = 2$, we have:

$$d^2(A^\bullet_\lambda, B^\bullet_\lambda) = (a_1(\lambda) - b_1(\lambda))^2 + (a_2(\lambda) - b_2(\lambda))^2, \tag{8}$$

Note. [16] Clearly, $d^p(A_\lambda, B_\lambda)$ is a distance of the λ-level set of fuzzy numbers A and B. It reflects the degree of closeness between A_λ and B_λ. Function $f(\lambda)$ can be understood as the weight of $d^2(A_\lambda, B_\lambda)$, and the property of monotone increasingness of $f(\lambda)$ means that the higher the membership of the level set, the more important it is in determining the distance between A and B. The conditions $f(0) = 0$ and $\int_0^1 f(\lambda) d\lambda = \frac{1}{2}$ ensure that the distance defined here is the extension of ordinary distance in R defined by an absolute value. That is, this distance becomes an ordinary one in R when the fuzzy numbers become decadent to crisp. In actual applications, function $f(\lambda)$ can be chosen according to the actual situation. In the following, we put $f(\lambda) = \lambda$ and we denote $D_{p,f}$ and $D^*_{p,f}$ by D_p and D^*_p, respectively.

In the following, we prove that $D^*_{p,f}$ is a metric on the space of IVFNs. At first, we need to express the following lemma.

Lemma 3.1 If a,b,c and d are real numbers, then

$$\max\{a+b,c+d\} \le \max\{a,c\} + \max\{b,d\} \tag{9}$$

Proof: We have 24 possible permutations of a,b,c and d. We prove (9) for two cases.

1) Let $a \le b \le c \le d$. Then $a+b \le c+d$, and therefore $\max\{a+b,c+d\} = c+d$, $\max\{a,c\} = c$ and $\max\{b,d\} = d$. Hence, relation (9) is satisfied.

2) Let $b \le c \le d \le a$. Then $\max\{a,c\} = a$ and $\max\{b,d\} = d$. If $\max\{a+b,c+d\} = c+d$, then

$$c \le a \Rightarrow c+d \le a+d \quad i.e. \max\{a+b,c+d\} \le \max\{a,c\} + \max\{b,d\}$$

and so, relation (9) is held. And if $\max\{a+b,c+d\} = a+b$, then

$$b \le d \Rightarrow a+b \le a+d \quad i.e. \max\{a+b,c+d\} \le \max\{a,c\} + \max\{b,d\}$$

and hence, relation (9) is satisfied.

Similarly, the remained 22 other cases can be proved. ■

Theorem 3.2 $D^*_{p,f}$ is a metric on $IF^*(R)$.

Proof: Suppose that $A,B,C \in IF^*(R)$.

- $D^*_{p,f}(A,B) \ge 0$ is obviously held.
- If $A = B$, then $D^*_{p,f}(A,B) = 0$. Conversely, if $D^*_{p,f}(A,B) = 0$, then $D_{p,f}(A^-,B^-) =$. $D_{p,f}(A^+,B^+) = 0$ Therefore, $\forall x \in R, A^-(x) = B^-(x)$ and $A^+(x) = B^+(x)$, and so we conclude $A = B$.
- Symmetry property i.e. $D^*_{p,f}(A,B) = D^*_{p,f}(B,A)$ is clearly held.
- Triangular inequality: Since $D_{p,f}(A^-,B^-)$ and $D_{p,f}(A^+,B^+)$ are metrics on the space of $F(R)$ [17, 16], if A^-,B^-,C^-,A^+,B^+,C^+ are fuzzy numbers, then

$$D_{p,f}(A^-,B^-) \le D_{p,f}(A^-,C^-) + D_{p,f}(C^-,B^-),$$
$$D_{p,f}(A^+,B^+) \le D_{p,f}(A^+,C^+) + D_{p,f}(C^+,B^+).$$

Therefore, we have

$$\max\{D_{p,f}(A^-,B^-), D_{p,f}(A^+,B^+)\} \le$$
$$\max\{D_{p,f}(A^-,C^-) + D_{p,f}(C^-,B^-), D_{p,f}(A^+,C^+) + D_{p,f}(C^+,B^+)\}. \tag{10}$$

By using relation (10) and Lemma 3.1, we have

$$D^*_{p,f}(A,B) = \max\{D_{p,f}(A^-,B^-), D_{p,f}(A^+,B^+)\}$$
$$\le \max\{D_{p,f}(A^-,C^-) + D_{p,f}(C^-,B^-), D_{p,f}(A^+,C^+) + D_{p,f}(C^+,B^+)\}$$
$$\le \max\{D_{p,f}(A^-,C^-), D_{p,f}(A^+,C^+)\} + \max\{D_{p,f}(C^-,B^-), D_{p,f}(C^+,B^+)\}$$
$$= D^*_{p,f}(A,C) + D^*_{p,f}(C,B). \quad ■$$

In the following, the $D^*_{p,f}$ distance will be used in triangular interval-valued fuzzy numbers and with a numerical example, we compare the distance with some other distances.

Proposition 3.3 Let $A = (a_1, a, a_2)$ and $B = (b_1, b, b_2)$ be two triangular fuzzy numbers. Then

$$D_2^2(A,B) = \frac{(a-b)^2}{2} + \frac{1}{12}[(a_2-b_2)^2 + (a_1-b_1)^2] + \frac{1}{6}(a-b)[(a_2-b_2)+(a_1-b_1)]. \tag{11}$$

Proof: The level sets of triangular fuzzy numbers A and B can be expressed as

$$A_\lambda = [a_1 + \lambda(a - a_1), a_2 - \lambda(a_2 - a)], \quad B_\lambda = [b_1 + \lambda(b - b_1), b_2 - \lambda(b_2 - b)] \tag{12}$$

According to Eq. (6), we have

$$D_2^2(A,B) = \int_0^1 \lambda[(a_1 - b_1) + \lambda((a - a_1) - (b - b_1))]^2 d\lambda$$
$$+ \int_0^1 \lambda[(a_2 - b_2) - \lambda((a_2 - a) - (b_2 - b))]^2 d\lambda$$
$$= \frac{(a-b)^2}{2} + \frac{1}{12}[(a_2 - b_2)^2 + (a_1 - b_1)^2] + \frac{1}{6}(a-b)[(a_2 - b_2) + (a_1 - b_1)],$$

and the proof is complete. ■

Theorem 3.4 Let $A = ((a_1^+, a_1^-), a, (a_2^-, a_2^+))$ and $B = ((b_1^+, b_1^-), b, (b_2^-, b_2^+))$ be two triangular IVFNs. Then

$$D_2^{*2}(A,B) = \frac{(a-b)^2}{2} + \max\{\frac{1}{12}[(a_2^- - b_2^-)^2 + (a_1^- - b_1^-)^2] + \frac{1}{6}(a-b)[(a_2^- - b_2^-) + (a_1^- - b_1^-)],$$
$$\frac{1}{12}[(a_2^+ - b_2^+)^2 + (a_1^+ - b_1^+)^2] + \frac{1}{6}(a-b)[(a_2^+ - b_2^+) + (a_1^+ - b_1^+)]\}. \tag{13}$$

Proof: The proof is straightforward in view of Eq. (5).■

Definition 3.2 The mean distance between A_i and $B_i, i = 1, \cdots, m$ is defined by

$$MD_{f,p}^* = \frac{1}{m}\sum_{i=1}^m D_{f,p}^*(A_i, B_i). \tag{14}$$

4 Comparison with two other distances

In the following, we introduce two distances between interval-valued fuzzy numbers based on Hausdorff metric for evaluating the goodness of fit of an IVF regression model. Let $u = [u_1, u_2]$ and $v = [v_1, v_2]$ be two closed intervals. The Hausdorff metric between u and v is defined by [9]

$$d_H(u,v) = \max\{|u_1 - v_1|, |u_2 - v_2|\}. \tag{15}$$

Definition 4.1 [9] Let $A, B \in IF^*(R)$. The D_p^* distance between A and B is defined as

$$D_p^*(A,B) = \max\{D_p(A^-, B^-), D_p(A^+, B^+)\} \tag{16}$$

where

$$D_p(A^\bullet, B^\bullet) = \left(\int_0^1 d_H^p(A_\lambda^\bullet, B_\lambda^\bullet) d\lambda\right)^{1/p}. \tag{17}$$

Since $A^\bullet$ and $B^\bullet$ are fuzzy numbers, so for each $\lambda \in (0,1]$, $A_\lambda^\bullet$ and $B_\lambda^\bullet$ are bounded closed intervals, i.e. $A_\lambda^\bullet = [a_1(\lambda), a_2(\lambda)], B_\lambda^\bullet = [b_1(\lambda), b_2(\lambda)]$. Therefore, from Eq. (15), we have

$$d_H(A_\lambda^\bullet, B_\lambda^\bullet) = \max\{|a_1(\lambda) - b_1(\lambda)|, |a_2(\lambda) - b_2(\lambda)|\}, \tag{18}$$

where $\bullet \in \{-, +\}$.

Theorem 4.1 [9] D_p^* is a metric on $IF^*(R)$.

Proposition 4.2 Let $A = ((a_1^+, a_1^-), a, (a_2^-, a_2^+))$ and $B = ((b_1^+, b_1^-), b, (b_2^-, b_2^+))$ be two triangular IVF numbers. Then, by Eq. (12) and Eq. (18), $D_p^*(A,B)$ is obtained as

$$D_p^*(A,B)=\max\{D_p(A^-,B^-),D_p(A^+,B^+)\},\tag{19}$$

where

$$D_p^p(A^-,B^-)=\int_0^1 \max\{|(1-\lambda)(a_1^- - b_1^-)+\lambda(a-b)|^p, |(1-\lambda)(a_2^- - b_2^-)+\lambda(a-b)|^p\}d\lambda,$$
$$D_p^p(A^+,B^+)=\int_0^1 \max\{|(1-\lambda)(a_1^+ - b_1^+)+\lambda(a-b)|^p, |(1-\lambda)(a_2^+ - b_2^+)+\lambda(a-b)|^p\}d\lambda.$$

Definition 4.2 The mean distance between A_i and $B_i, i=1,\cdots,m$ is defined by

$$MD_p^*=\frac{1}{m}\sum_{i=1}^m D_p^*(A_i,B_i).\tag{20}$$

Definition 4.3 [9] Let $A,B\in IF^*(R)$. The D_∞^* distance between A and B is defined as

$$D_\infty^*(A,B)=\max\{D_\infty(A^-,B^-),D_\infty(A^+,B^+)\}\tag{21}$$

where

$$D_\infty(A^\bullet,B^\bullet)=\sup_{\lambda\in[0,1]} d_H(A_\lambda^\bullet,B_\lambda^\bullet),\tag{22}$$

for $\bullet\in\{-,+\}$ and $d_H(A_\lambda^\bullet,B_\lambda^\bullet)$ can be obtained by Eq. (18).

Theorem 4.3 [9] D_∞^* is a metric on $IF^*(R)$.

Proposition 4.4 Let $A=((a_1^+,a_1^-),a,(a_2^-,a_2^+))$ and $B=((b_1^+,b_1^-),b,(b_2^-,b_2^+))$ be two triangular IVF numbers. Then, by Eq. (12) and Eq. (18), $D_\infty^*(A,B)$ is obtained as

$$D_\infty^*(A,B)=\max\{D_\infty(A^-,B^-),D_\infty(A^+,B^+)\},\tag{23}$$

where

$$D_\infty(A^-,B^-)=\sup_{\lambda\in[0,1]}\max\{|(1-\lambda)(a_1^- - b_1^-)+\lambda(a-b)|, |(1-\lambda)(a_2^- - b_2^-)+\lambda(a-b)|\},$$
$$D_\infty(A^+,B^+)=\sup_{\lambda\in[0,1]}\max\{|(1-\lambda)(a_1^+ - b_1^+)+\lambda(a-b)|, |(1-\lambda)(a_2^+ - b_2^+)+\lambda(a-b)|\}.$$

Definition 4.4 The mean distance between A_i and $B_i, i=1,\cdots,m$ is defined by

$$MD_\infty^*=\frac{1}{m}\sum_{i=1}^m D_\infty^*(A_i,B_i).\tag{24}$$

Example 4.1 Table 1 shows the triangular IVF values of A and B and their distances between them. The indices $MD_{f,p}^*, MD_p^*$ and MD_∞^* between A and B values are shown in this table. As we see, the MD_p^* and MD_∞^* are 3.00 and 3.23, respectively, which are very close to 2.78, i.e. the $MD_{f,p}^*$.

Table 1: Triangular interval-valued fuzzy values of A and B and their distances

No.	A					B					$D^*_{f,p}$	D_p	D^*_∞
	a_1^+	a_1^-	a	a_2^-	a_2^+	b_1^+	b_1^-	b	b_2^-	b_2^+			
1	2.75	2.88	3.08	3.51	3.65	5.20	5.57	6.19	6.82	7.17	3.08	3.31	3.52
2	2.31	2.34	2.86	3.01	3.18	4.39	4.72	5.23	5.79	6.07	2.44	2.64	2.90
3	5.13	5.81	6.25	6.61	6.79	4.27	4.60	5.09	5.64	5.92	1.14	1.18	1.21
4	3.80	3.89	4.11	4.33	4.45	3.52	3.80	4.20	4.69	4.90	0.18	0.29	0.45
5	0.93	0.94	1.04	1.15	1.18	7.06	7.52	8.38	9.18	9.66	7.34	7.92	8.49
6	2.42	2.43	2.71	3.04	3.22	4.21	4.54	5.02	5.57	5.84	2.32	2.47	2.62
7	3.94	4.03	4.45	5.21	5.32	4.97	5.33	5.91	6.53	6.85	1.41	1.50	1.53
8	5.55	6.31	6.92	7.32	8.25	7.35	7.82	8.72	9.55	10.05	1.83	2.02	2.23
9	6.13	6.26	7.41	8.30	8.86	6.95	7.39	8.24	9.04	9.51	0.87	0.99	1.14
10	7.60	8.22	9.08	10.01	10.59	2.88	3.14	3.45	3.87	4.05	5.65	6.09	6.54
11	6.04	6.16	6.56	6.95	7.24	4.21	4.54	5.02	5.57	5.84	1.56	1.68	1.82
12	4.41	4.43	5.05	5.59	6.05	6.37	6.79	7.56	8.30	8.73	2.52	2.61	2.71
13	4.79	4.94	5.23	5.50	6.22	4.80	5.14	5.71	6.31	6.62	0.50	0.65	0.81
14	4.70	4.82	5.16	5.94	6.02	3.57	3.87	4.27	4.76	4.98	0.96	1.04	1.18
15	9.20	9.34	11.10	11.80	11.88	5.14	5.51	6.12	6.75	7.09	4.81	5.01	5.05
16	3.74	3.83	4.47	4.81	4.97	6.42	6.85	7.63	8.37	8.81	3.20	3.50	3.84
17	24.32	25.01	28.84	31.07	32.68	21.37	22.49	25.23	27.33	28.86	3.54	3.72	3.82
18	7.59	7.99	9.43	10.65	11.03	5.67	6.06	6.74	7.41	7.79	2.67	2.97	3.24
19	3.73	4.10	4.50	4.85	5.31	4.21	4.54	5.02	5.57	5.84	0.55	0.63	0.73
20	8.68	8.75	9.30	10.80	10.83	4.39	4.72	5.23	5.79	6.07	4.23	4.54	5.00
21	7.60	7.69	9.48	10.73	11.24	4.80	5.14	5.71	6.31	6.62	3.77	4.20	4.62
22	3.10	3.19	3.65	3.91	4.10	4.21	4.54	5.02	5.57	5.84	1.42	1.56	1.74
23	8.42	9.58	10.14	11.67	11.76	4.33	4.66	5.16	5.72	6.00	5.14	5.47	5.95
23	2.59	2.78	3.00	3.32	3.55	7.24	7.70	8.59	9.41	9.90	5.57	5.97	6.35
Mean of distances											2.78	3.00	3.23

5 Conclusion

In this work, we proposed a new distance between two triangular IVFNs. The applicability of the proposed method was investigated by using a numerical data set.

6 References

[1] K. Atanassov. *Intuitionistic Fuzzy Sets, Theory and Applications*. Physica-Verlag, New York, 1999.

[2] J. H. Chen and S. M. Chen. A new method to measure the similarity between interval-valued fuzzy numbers. In *Proc. of the Sixth International Conference on Machine Learning and Cybernetics, Hong Kong*, volume 3, pages 1403–1408, 2007.

[3] G. Deschrijver and E. E. Kerre. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets and Systems*, 133(2):227–235, 2003.

[4] G. Deschrijver and E. E. Kerre. On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision. *Information Sciences*, 177(8):1860–1866, 2007.

[5] G. Deschrijver. Arithmetic operators in interval-valued fuzzy set theory. *Information Sciences*, 177(14):2906–2924, 2007.

[6] B. Gorzafczany. Approximate inference with interval-valued fuzzy sets-an outline. In *Proc. of the Polish Symp. on Interval and Fuzzy Math*, pages 89–95, Poznan, Poland, 1983.

[7] M. B. Gorzalczany. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Seystems*, 21:1–17, 1987.

[8] D. H. Hong and S. Lee. Some algebraic properties and a distance measure for interval-valued fuzzy numbers. *Information Sciences*,

148(1–4):1–10, 2002.

[9] C. Li. Distances between interval-valued fuzzy sets. In Proc. of the 28th IEEE North American Fuzzy Information Processing Society Annual Conference (NAFIPS2009), Cincinnati, Ohio, USA, pages 1–3, 2009.

[10] G. Przemyslaw. Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the hausdorff metric. *Fuzzy Sets and Systems*, 148(2):319–328, 2004.

[11] I. B. Turksen. Interval valued fuzzy sets based on normal forms. *Fuzzy Sets and Systems*, 20(2):191–210, 1986.

[12] G. J. Wang and Y. Y. He. Intuitionistic fuzzy sets and l-fuzzy sets. *Fuzzy Sets and Systems*, 110(2):271–274, 2000.

[13] G. Wang and X. Li. The applications of interval-valued fuzzy numbers and interval-distribution numbers. *Fuzzy Sets and Systems*, 98(3):331–335, 1998.

[14] G. Wang and X. Li. Correlation and information energy of interval-valued fuzzy numbers. *Fuzzy Sets and Systems*, 103(1):169–175, 1999.

[15] Y. M. Wang, J. B. Yang, D. L. Xu, and K. S. Chin. On the combination and normalization of interval-valued belief structures. *Information Sciences*, 177:189–200, 2007.

[16] R. Xu and C. Li. Multidimensional least-squares fitting with a fuzzy model. *Fuzzy Sets and Systems*, 119(2):215–223, 2001.

[17] R. Xu. A linear regression model in fuzzy environment. *Adv. Modelling Simulation*, 27:31–40, 1991.

[18] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

[19] L. Zhixin and J. Hongmei. Effectiveness and relevancy measures under modal cardinality for interval-valued fuzzy sets. In *Proc. of the 3rd IEEE International Conference on Advanced Computer Theory and Engineering (ICACTE), Chengdu, China*, volume 1, pages 400–402, 2010.