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Non-equilibrium entanglement dynamics of a two-qubit Heisenberg XY system in the presence of an inhomogeneous magnetic field and spin-orbit interaction

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Abstract. Entanglement dynamics of an open two-qubit anisotropic XY Heisenberg system is investigated in the presence of an inhomogeneous magnetic field and spin-orbit interaction. We suppose that each qubit interacts with a separate thermal reservoir which is held in its own temperature. The asymptotical and the dynamical behavior of entanglement are analyzed. To distinguish between entanglement induced by the environment and entanglement due to the presence of inter-qubit interaction, the effects of spin-orbit parameter D and temperature difference parameter ΔT on the entanglement of the system have been investigated. We show that for a fixed set of the system parameters, entanglement can be produced just by adjusting the temperature difference between the reservoirs. The size of this entanglement, which is induced by temperature difference of reservoirs, increases as the spin-orbit parameter D increases. Also we find that, this environment induced entanglement can be improved if the qubit influenced by the weaker magnetic field is in contact with the hotter reservoir, i.e. indirect geometry of connection. In this case, the amount of asymptotic entanglement increases as D increases. Regardless of the geometry of connection, increasing D causes the appearance of entanglement in the larger regions of $T_M - \Delta T$ plane, therefore entanglement can exist in higher temperatures and temperature differences. Furthermore, increasing D enhances the amount of entanglement in these regions. We also show that the state of the system can be found in the maximally entangled state for the case of zero temperature reservoirs and large amount of the spin-orbit parameter.

1 Introduction

Entanglement is a central theme in quantum information processing which was first noted by Einstein [1] and Schrödinger [2]. It strongly affects our conceptual implication on physics, and force us to change significantly our perspective of Nature. Entanglement implies that the best knowledge of the whole of a composite system may not include complete knowledge of its parts [2]. In mathematical sense a pure state of pair of quantum systems is called entangled if it is unfactorizable. A mixed state ρ of a bipartite system is said to be separable or classically correlated if it can be expressed as a convex combination of uncorrelated states ρ_A and ρ_B of each subsystems i.e. $\rho = \sum_i \omega_i \rho_A^i \otimes \rho_B^i$ such that $\omega_i \geq 0$ and $\sum_i \omega_i = 1$, otherwise ρ is entangled [3–5]. Entanglement has no classical analog and can be considered as a uniquely quantum mechanical resource that plays a key role in many of the most interesting applications of quantum computation and quantum information processing such as: quantum teleportation, entanglement teleportation, quantum cryptography, and etc. [3,4]. Performance the above mentioned tasks needs to quantifying and optimizing the amount of the entanglement in a suitable multipartite quantum system. Many measures of entanglement have been introduced and analyzed [3,6,8], but the one most relevant to this work is the *entanglement of formation*, which is intended to quantify the resources need to create a given entangled state [6,7]. For the case of a two-qubit system Wootters [6,7] has shown that the entanglement of formation can be obtained explicitly as: $E(\rho) = \Xi[C(\rho)] = h\left(\frac{1+\sqrt{1+C^2}}{2}\right)$, where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function and $C(\rho) = \max\{0, 2\lambda_{max} - \sum_{i=1}^4 \lambda_i\}$ is the concurrence of the state, where λ_is are square roots of the eigenvalues of the non-Hermitian matrix $R = \rho\tilde{\rho}$, and $\tilde{\rho}$ is defined by $\tilde{\rho} := (\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$. The function Ξ is a monotonically increasing function and ranges from 0 to 1 as C goes from 0 to 1, so that one can take

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the concurrence as a measure of entanglement in its own right. In the case that the state of the system is pure i.e. $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, the above formula is simplified to $C(|\psi\rangle) = 2\,|ad - bc|$.

Real quantum systems are not isolated from their environment. Unavoidable interaction between the system and its environment causes the leakage of the coherence of the system to environment and hence quantum-to-classical transition occurs (see [9] and references therein). More precisely, this interaction will, in general, create some entanglement between the states of quantum system and the huge states of environment. Consequently, quantum coherence initially localized within the system will become a shared property of the composite system-environment state and can no longer be observed at the level of the system, leading to decoherence. Decoherence destroys the quantumness of the system and hence will decrease the useful entanglement between the parts of the system. Decoherence due to thermal interactions of the system and environment is important in many situations, especially in the solid-state systems. Thus a clear relation between quantum correlations and thermal magnitudes has to be elucidated. Almost all of studies have been focused on systems in thermal equilibrium, both at zero and finite temperatures. However, real systems are not in equilibrium [10]. This impose one to address the question of quantum feature survival in noisy as well as nonequilibrium conditions.

Entanglement properties of Heisenberg systems at thermal equilibrium (thermal entanglement) are extensively studied after Nielsen [11], who first studied the thermal entanglement of a two-qubit Heisenberg XXX chain (see [12] and references therein). For non-equilibrium thermal entanglement in spin systems, Eisler and Zimboras [13] calculated the von Neumann entropy of a block of spins in XX spin chain in the presence of the energy current and showed that the enhancement of the amount of entanglement due to an energy current is possible. After them, the non-equilibrium thermal entanglement for steady state of some systems has been studied in a number of works [15–17]. For example, Quiroga and Rodríguez in reference [18] considered a simple spin chain system (XXX-Heisenberg) which is in contact with two different heat reservoirs and showed that for the steady state a temperature gradient can increase or decrease entanglement depending on the internal coupling strength between spins. Dynamics of non-equilibrium thermal entanglement of the same system has been studied by Sinaysky et al. [19]. They have derived an analytical expression for the density matrix of the system at a finite time. They also have shown that the system converges to a steady state, asymptotically, and the amount of entanglement of the steady state takes its maximum value for unequal bath temperatures and also the local energy levels can maintain the entanglement at higher temperatures. However, the non-equilibrium thermal entanglement dynamics of more involved spin systems has not been considered yet.

Two coupled qubits in contact with different thermal baths is a system which is interesting both from theoretical and empirical point of view. For electrons in a solid-state environment, recent progress has been linked to fabrication technology for nanoscale devices. For example, in semiconductor quantum dots (QDs), the transfer of quantum information between nuclear spins and electronic spins has been considered [20–22]. The nuclear spins and the electronic spins are coupled to their environment (a great variety of degrees' of freedom within the solid) in completely different manner and hence they experience different effects from environment. Today, with the help of NMR and quantum optical techniques, it is possible to cool the nuclear spins in a controlled manner without affecting the temperature of the electron spins [23,24], thus creating two thermal baths with effective different temperatures. As an another example we address to a two spin electron confined in two coupled quantum dots (CQDs) initially proposed by Loss et al. [25–28], where qubit is represented by the spin of a single electron in a QD, which can be initialized, manipulated, and read out by extremely sensitive devices. Such systems are more scalable and have a longer coherence time than other systems such as quantum optical and NMR systems. Lithography techniques allow us to couple the dots to a different source and drain electrodes, and hence enable us to drive the qubits in a nonequilibrium manner [29,30].

Motivated by this, in the following we investigate dynamics of non-equilibrium thermal entanglement of an open two qubit system which is realized by the spin of two electrons confined in two CQDs interacting with two separate reservoirs. Because of weak vertical or lateral confinement, electrons can tunnel from one dot to the other and spin-spin and spin-orbit interactions between the two qubits exist. Indeed, the spin-orbit interaction produces an isotropic part of exchange interaction between localized conduction-band electrons in semiconductor structures that lack inversion symmetry, including practically all low dimensional structures and also bulk semiconductors with zinc-blende and wurtzite type of crystal lattice. The main part of the anisotropic interaction has the form of the Dzyaloshinski-Moriya (DM) interaction and may be as strong as several percent of the isotropic exchange [31–35]. In this sense, the Hamiltonian of the inter-qubit interaction can be modelled by the Heisenberg Hamiltonian including the DM interaction. Thus we model the inter-qubit interaction by anisotropic XY Heisenberg model in the presence of the inhomogeneous magnetic field and the DM interaction. On the other hand, in what follows we model the environment by a thermal reservoir and assume that the inter dot separation is large enough such that each dot couples to a separate thermal reservoir (bosonic bath) (see Fig. 1). Here the bathes are assumed to be in thermodynamical equilibrium at different temperature $\beta_i = \frac{1}{k_B T_i}$. In general, there are two different ways for connecting the quantum dots to the reservoirs: (i) "*direct geometry*"; where a high temperature bath couples to the QD which is in the stronger magnetic field i.e. $b\Delta T > 0$ and (ii) "*indirect geometry*"; where a high

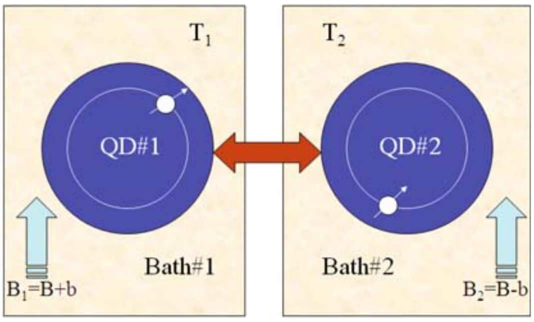

Fig. 1. (Color online) Two interacting qubits which are realized by two electron spin confined in two quantum dots, respectively. Each quantum dot is embedded in a separate thermal bath which is held in its own temperature.

temperature bath couples to the QD which is in the weaker magnetic field i.e. $b\Delta T < 0$. The results show that the non-equilibrium thermal entanglement dynamics depends on the geometry of the system. The influence of the parameters of the system (i.e. magnetic field (B), inhomogeneity of magnetic field (b), partial anisotropy (χ), mean coupling (J) and the spin-orbit interaction parameter (D)) and environmental parameters (i.e. temperatures T_1 and T_2 or equally T_M and ΔT, and the couplings γ_1 and γ_2) on the entanglement of the system is investigated. We have shown that, there is a steady state entanglement in large time limit. The size of this steady state (asymptotic) entanglement and the dynamical behavior of entanglement depend on the parameters of the model and also on the geometry of the connection. We show that, the amount of asymptotic entanglement decreases as the temperature difference ΔT and the mean temperature T_M increase. We have also shown that, the size of $T_M^{cr.}$ (temperature over which the entanglement vanishes) and the amount of entanglement can be improved by adjusting the value of the spin-orbit interaction parameter D. The maximum entanglement ($C = 1$) can be achieved in the case of large values of D and zero temperature reservoirs ($T_1 = T_2 = 0$). Furthermore, we find that the indirect geometry of connection is more suitable for creating and maintaining the entanglement. In the absence of spin-orbit interaction and for both symmetric and nonsymmetric cases, the entanglement exists only in a small interval of temperature difference. Introducing spin-orbit interaction causes this temperature difference interval to broad such that for large values of D, the entanglement exists for any allowable value of ΔT in the mean temperature interval $T_M \in [0, T_M^{cr.}]$. Also, the amount of entanglement in this interval increases with D.

The paper is organized as follows. In Section 2, we introduce the Hamiltonian of the whole system-reservoir under the rotating wave approximation and then write the Markovian master equation governed on the system by tracing out the reservoirs' degrees of freedom. Ultimately, given some initial states, the density matrix of the system at a later time is derived exactly. The effects of initial conditions and system parameters on the dynamics of entanglement and entanglement of asymptotic state of the system are presented in Section 3. Finally in Section 4 a discussion concludes the paper.

2 The model and Hamiltonian

The total Hamiltonian of the a qubit-system which is interacting with two heat reservoirs is described by

$$\hat{H} = \hat{H}_S + \hat{H}_{B1} + \hat{H}_{B2} + \hat{H}_{SB1} + \hat{H}_{SB2}, \qquad (1)$$

where $\hat{H}_S$ is the Hamiltonian of the system, $\hat{H}_{Bj}$ is the Hamiltonian of the jth bath ($j = 1, 2$) and $\hat{H}_{SBj}$ denotes the system-bath interaction Hamiltonian. The system consists of two spin electrons confined in a two coupled quantum dots, described by a two-qubit anisotropic Heisenberg XY-model in the presence of an inhomogeneous magnetic field and DM interaction with the following Hamiltonian [12,36,37]

$$\hat{H}_S = \frac{1}{2}(J_x\,\sigma_1^x\sigma_2^x + J_y\,\sigma_1^y\sigma_2^y + \mathbf{B}_1\cdot\boldsymbol{\sigma}_1 + \mathbf{B}_2\cdot\boldsymbol{\sigma}_2 + \mathbf{D}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2) + \delta\,\boldsymbol{\sigma}_1\cdot\overline{\boldsymbol{\Gamma}}\cdot\boldsymbol{\sigma}_2), \qquad (2)$$

where $\boldsymbol{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$ is the vector of Pauli matrices, $\mathbf{B}_j\,(j = 1, 2)$ is the magnetic field on site j, $J_\mu\,(\mu = x, y)$ are the real coupling coefficients (the chain is antiferromagnetic (AFM) for $J_\mu > 0$ and ferromagnetic (FM) for $J_\mu < 0$) and $\mathbf{D}$ is Dzyaloshinski-Moriya vector, which is of first order in spin-orbit coupling and is proportional to the coupling coefficients (J_μ) and $\overline{\boldsymbol{\Gamma}}$ is a symmetric tensor which is of second order in spin-orbit coupling [31–35]. For simplicity we assume $\mathbf{B}_j = B_j\,\hat{\boldsymbol{z}}$ such that $B_1 = B + b$ and $B_2 = B - b$, where b indicates the amount of inhomogeneity of the magnetic field. If we take $\mathbf{D} = JD\,\hat{\boldsymbol{z}}$ and ignore the second order spin-orbit coupling, then the above Hamiltonian can be written as[1]:

$$\hat{H}_S = J\chi(\sigma_1^+\sigma_2^+ + \sigma_1^-\sigma_2^-) + J(1 + iD)\sigma_1^+\sigma_2^- + J(1 - iD)\sigma_1^-\sigma_2^+ + \left(\frac{B + b}{2}\right)\sigma_1^z + \left(\frac{B - b}{2}\right)\sigma_2^z, \qquad (3)$$

where $J := \frac{J_x + J_y}{2}$, is the mean coupling coefficient in the XY-plane, $\chi := \frac{J_x - J_y}{J_x + J_y}$, specifies the amount of anisotropy in the XY-plane (partial anisotropy, $-1 \le \chi \le 1$) and $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ are lowering and raising operators. The spectrum of H_S is easily obtained as

$$|\varepsilon_{1,2}\rangle = |\Psi^\pm\rangle = N^\pm\left(\left(\frac{b \pm \xi}{J(1 - iD)}\right)|01\rangle + |10\rangle\right), \quad \varepsilon_{1,2} = \pm\xi,$$
$$|\varepsilon_{3,4}\rangle = |\Sigma^\pm\rangle = M^\pm\left(\left(\frac{B \pm \eta}{J\chi}\right)|00\rangle + |11\rangle\right), \quad \varepsilon_{3,4} = \pm\eta. \qquad (4)$$

[1] The parameters D and δ are dimensionless. In the system like coupled GaAs quantum dots D is of order of a few percent, while the order of δ is 10^{-4} and is negligible [31]. The spin-orbit interaction and the coupling magnetic fields are expected to be orders of magnitude stronger in InAs compared to GaAs [38].

Here the eigenstates are expressed in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $|0\rangle$ indicates the spin state parallel to the magnetic field (z direction) and $|1\rangle$ illustrates the spin state in the reverse direction. In the above equations $N^{\pm} = (1 + \frac{(b\pm\xi)^2}{J^2+(JD)^2})^{-1/2}$ and $M^{\pm} = (1+(\frac{B\pm\eta}{J\chi})^2)^{-1/2}$ are the normalization constants. Here we define, $\xi := (b^2+J^2+(JD)^2)^{1/2}$ and $\eta := (B^2+(J\chi)^2)^{1/2}$, for later convenience. The Hamiltonian of the reservoirs for each spin $j = 1, 2$ are given by

$$\hat{H}_{Bj} = \sum_n \omega_n \hat{b}^{\dagger}_{nj}\hat{b}_{nj}\,, \tag{5}$$

where $\hat{b}^{\dagger}_{nj}$ ($\hat{b}_{nj}$) is the creation (annihilation) operator of the jth bath mode. The interaction between the system and the jth bath has the following general form

$$\hat{H}_{SBj} = (\sigma_j^+ + \sigma_j^-)\left(\sum_n g_n^{(j)}\hat{b}_{n,j} + g_n^{(j)*}\hat{b}^{\dagger}_{n,j}\right) + \sigma_j^z \sum_n h_n^{(j)}(\hat{b}_{n,j} + \hat{b}^{\dagger}_{n,j}). \tag{6}$$

The first term in the above equation describes the dissipation process induced by the environment and the term associated with σ^z indicates the phase relaxation process stimulated by the environment (dephasing). In order to able to solve the problem analytically, we just consider the influence of energy exchange between the system and environment and exclude the dephasing. The effects of pure dephasing process on the nonequilibrium thermal entanglement of a simple spin chain system have been considered in references [39,40], recently. By ignoring the dephasing process and using the rotating wave approximation (RWA), the system-bath coupling Hamiltonian can be written as [14,18,41–44]

$$\hat{H}_{SBj} = \sigma_j^+ \sum_n g_n^{(j)}\hat{b}_{n,j} + \sigma_j^- \sum_n g_n^{(j)*}\hat{b}^{\dagger}_{n,j} \equiv \sum_\mu (\hat{V}^+_{j,\mu}\hat{f}_{j,\mu} + \hat{V}^-_{j,\mu}\hat{f}^{\dagger}_{j,\mu}). \tag{7}$$

The system operators $\hat{V}^{\pm}_{j,\mu}$ are chosen to satisfy $[\hat{H}_S, \hat{V}^{\pm}_{j,\mu}] = \pm\omega_{j,\mu}\hat{V}^{\pm}_{j,\mu}$, and the $\hat{f}_{j,\mu}$'s are the random operators of reservoirs and act on the bath degrees of freedom. Physically, the index μ corresponds to transitions between the internal levels of the system induced by the bath. The dynamics of the whole system+reservoirs is described by a density operator ($\hat{\sigma}$) satisfying the Liouville equation $\dot{\hat{\sigma}} = -i[\hat{H}, \hat{\sigma}]$. If the coupling strengths of the system and the environment are weak, the evolution of the system does not influence the states of the reservoirs and one can write $\hat{\sigma}(t) = \hat{\rho}(t)\hat{\rho}_{B1}(0)\hat{\rho}_{B2}(0)$ (irreversibility hypothesis), where $\hat{\rho}(t)$ is the reduced density matrix describing the system and each bath is described by a canonical density matrix of the form $\hat{\rho}_{Bj} = e^{-\beta_j \hat{H}_{Bj}}/Z$, where $Z = \mathrm{Tr}(e^{-\beta_j \hat{H}_{Bj}})$ is the partition function of the jth bath.

In the Born-Markov approximation the master equation describing the dynamics of the reduced density matrix of the system is [14,45]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_S, \hat{\rho}] + \mathcal{L}_1(\hat{\rho}) + \mathcal{L}_2(\hat{\rho}), \tag{8}$$

where $\mathcal{L}_j(\hat{\rho})$ are *dissipators* given by [45]

$$\mathcal{L}_j(\hat{\rho}) \equiv \sum_{\mu,\nu} J^{(j)}_{\mu,\nu}(\omega_{j,\nu})\{[\hat{V}^+_{j,\mu}, [\hat{V}^-_{j,\nu}, \hat{\rho}]] - (1 - e^{\beta_j\omega_{j,\nu}})[\hat{V}^+_{j,\mu}, \hat{V}^-_{j,\nu}\hat{\rho}]\}. \tag{9}$$

Here $J^{(j)}_{\mu,\nu}(\omega_{j,\nu})$ is the spectral density of the jth reservoir,

$$J^{(j)}_{\mu,\nu}(\omega_{j,\nu}) = \int_0^\infty d\tau e^{i\omega_{j,\nu}\tau} G_{\alpha\beta}(\tau), \tag{10}$$

where $G_{\alpha\beta}(\tau)$ is the environment self-correlation function,

$$G_{\alpha\beta}(\tau) = \mathrm{Tr}_{Bj}[\rho_{Bj}\bar{f}_{j,\nu}(\tau)\hat{f}_{j,\mu}] \tag{11}$$

and $\bar{f}_{j,\nu}(\tau) = e^{-iH_{Bj}\tau}\hat{f}^{\dagger}_{j,\mu}e^{iH_{Bj}\tau}$. Spectral densities encapsulate the physical properties of the environment and play an immensely important role in the theoretical and experimental studies of the decoherence. In this paper, we will consider the bosonic thermal bath as an infinite set of harmonic oscillators and apply a Weisskpof-Wignner-like expression for spectral density such as $J^{(j)}(\omega_\mu) = \gamma_j(\omega_\mu)n_j(\omega_\mu)$, where $n_j(\omega_\mu) = (e^{\beta_j\omega_\mu} - 1)^{-1}$ denotes the thermal mean value of the number of excitations in the jth reservoir at frequency ω_μ and temperature $T_j = \frac{1}{\beta_j}$ and $\gamma_j(\omega_\mu)$ is the coupling strength of system and the jth reservoir and $J^{(j)}(-\omega_\mu) = e^{\beta_j\omega_\mu}J^{(j)}(\omega_\mu)$. For simplicity we take $\gamma_j(\omega_\mu) = \gamma_j$. Thus, the dissipators $\mathcal{L}_j(\hat{\rho})$ become

$$\mathcal{L}_j(\hat{\rho}) = \sum_{\mu=1}^{4} J^{(j)}(-\omega_\mu)(2\hat{V}^+_{j,\mu}\hat{\rho}\hat{V}^-_{j,\mu} - \{\hat{\rho}, \hat{V}^-_{j,\mu}\hat{V}^+_{j,\mu}\}_+)) + \sum_{\mu=1}^{4} J^{(j)}(\omega_\mu)(2\hat{V}^-_{j,\mu}\hat{\rho}\hat{V}^+_{j,\mu} - \{\hat{\rho}, \hat{V}^+_{j,\mu}\hat{V}^-_{j,\mu}\}_+)), \tag{12}$$

with the transition frequencies

$$\omega_1 = \xi - \eta, \qquad \omega_4 = -\omega_1, \qquad \omega_2 = \xi + \eta, \qquad \omega_3 = -\omega_2, \tag{13}$$

and the transition operators

$$\begin{aligned} \hat{V}^+_{j,1} &= a_{j,1}|\Psi^+\rangle\langle\Sigma^+|, & \hat{V}^-_{j,1} &= a^*_{j,1}|\Sigma^+\rangle\langle\Psi^+|,\\ \hat{V}^+_{j,2} &= a_{j,2}|\Psi^+\rangle\langle\Sigma^-|, & \hat{V}^-_{j,2} &= a^*_{j,2}|\Sigma^-\rangle\langle\Psi^+|,\\ \hat{V}^+_{j,3} &= a_{j,3}|\Psi^-\rangle\langle\Sigma^+|, & \hat{V}^-_{j,3} &= a^*_{j,3}|\Sigma^+\rangle\langle\Psi^-|,\\ \hat{V}^+_{j,4} &= a_{j,4}|\Psi^-\rangle\langle\Sigma^-|, & \hat{V}^-_{j,4} &= a^*_{j,4}|\Sigma^-\rangle\langle\Psi^-|, \end{aligned} \tag{14}$$

where

$$|a_{j,1}|^2 = |a_{j,4}|^2 = \frac{1}{2\xi\eta}(\xi\eta + J^2\chi + (-1)^j Bb),$$
$$|a_{j,2}|^2 = |a_{j,3}|^2 = \frac{1}{2\xi\eta}(\xi\eta - J^2\chi - (-1)^j Bb), \quad (15)$$

can be obtained from the spectrum of the system Hamiltonian. Note that, the transition operators $\hat{V}^{\pm}_{j,\mu}$ defined in equation (14) just describe the energy exchange between the system and environment (dissipative coupling), including both excitation and de-excitation of the qubits and the absence of the transitions $\Sigma^+ \leftrightarrow \Sigma^-$ and $\Psi^+ \leftrightarrow \Psi^-$ (which are a consequence of the dephasing process), stems in the omittance of the σ^z coupling in the system-bath interaction Hamiltonian.

It should be noted that the validity of the independence of the reservoirs during the evolution of the system may be violated, although the interaction of the spins and their associated baths is local and remains local during the evolution, the nonlocal inter-qubit interaction leads to entanglement distribution between the spins and the baths. And consequently, the initially independent reservoirs may become correlated via the inter-qubit interaction at later times but the temperature of the reservoirs are still held constant during the evolution, i.e. the system does not reach thermal equilibrium and remain in the nonequilibrium regime.

The master equation (8) has an important property, when the spectrum of $\hat{H}_s$ (see Eq. (4)) is non-degenerate, i.e. in the energy basis $\{|\varepsilon_i\rangle\}_{i=1}^4$, the equations for diagonal elements decouple from nondiagonal ones [14]. Furthermore, nondiagonal elements are not coupled and the time dependence of these elements has the simple form

$$\rho_{i,j}(t) = \rho_{i,j}(0)e^{\alpha_{ij}t}, \quad (16)$$

where $\alpha_{i,j} \in \mathbb{C}$ are determined by the system parameters. The equations for diagonal elements have the following form

$$\dot{R}(t) = BR(t), \quad (17)$$

where dot denotes the time derivative, $R(t) = (\rho_{11}(t), \rho_{22}(t), \rho_{33}(t), \rho_{44}(t))^T$ and B is the time independent 4×4 matrix

$$B = \begin{pmatrix} -(X_1^- + Y_2^-) & 0 & X_1^+ & Y_2^+ \\ 0 & -(X_1^+ + Y_2^+) & Y_2^- & X_1^- \\ X_1^- & Y_2^+ & -(X_1^+ + Y_2^-) & 0 \\ Y_2^- & X_1^+ & 0 & -(X_1^- + Y_2^+) \end{pmatrix}, \quad (18)$$

where

$$X_\mu^{\pm} = 2\sum_{j=1,2} J^{(j)}(\mp\omega_\mu) \mid a_{j,1} \mid^2,$$
$$Y_\mu^{\pm} = 2\sum_{j=1,2} J^{(j)}(\mp\omega_\mu) \mid a_{j,2} \mid^2 . \quad (19)$$

The analytical solution of equation (17) in the energy basis is given by

$$R(t) = M(t)R(0), \quad (20)$$

where $M(t) = [m_{ij}]_{4\times4}$, and the elements m_{ij} are defined by

$$m_{11} = \frac{1}{X_1Y_2}(X_1^+ + X_1^- e^{-tX_1})(Y_2^+ + Y_2^- e^{-tY_2}),$$
$$m_{12} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})(1 - e^{-tY_2})X_1^+Y_2^+,$$
$$m_{13} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})X_1^+(Y_2^+ + Y_2^- e^{-tY_2}),$$
$$m_{14} = \frac{1}{X_1Y_2}(X_1^+ + X_1^- e^{-tX_1})(1 - e^{-tY_2})Y_2^-,$$
$$m_{21} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})(1 - e^{-tY_2})X_1^-Y_2^-,$$
$$m_{22} = \frac{1}{X_1Y_2}(X_1^- + X_1^+ e^{-tX_1})(Y_2^- + Y_2^+ e^{-tY_2}),$$
$$m_{23} = \frac{1}{X_1Y_2}(X_1^- + X_1^+ e^{-tX_1})(1 - e^{-tY_2})Y_2^-,$$
$$m_{24} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})X_1^-(Y_2^- + Y_2^+ e^{-tY_2}),$$
$$m_{31} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})X_1^-(Y_2^+ + Y_2^- e^{-tY_2}),$$
$$m_{32} = \frac{1}{X_1Y_2}(X_1^- + X_1^+ e^{-tX_1})(1 - e^{-tY_2})Y_2^+,$$
$$m_{33} = \frac{1}{X_1Y_2}(X_1^- + X_1^+ e^{-tX_1})(Y_2^+ + Y_2^- e^{-tY_2}),$$
$$m_{34} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})(1 - e^{-tY_2})X_1^-Y_2^+,$$
$$m_{14} = \frac{1}{X_1Y_2}(X_1^+ + X_1^- e^{-tX_1})(1 - e^{-tY_2})Y_2^-,$$
$$m_{42} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})X_1^+(Y_2^- + Y_2^+ e^{-tY_2}),$$
$$m_{43} = \frac{1}{X_1Y_2}(1 - e^{-tX_1})(1 - e^{-tY_2})X_1^+Y_2^-,$$
$$m_{44} = \frac{1}{X_1Y_2}(X_1^+ + X_1^- e^{-tX_1})(Y_2^- + Y_2^+ e^{-tY_2}). \quad (21)$$

Here we have defined $X_\mu = X_\mu^+ + X_\mu^-$ and $Y_\mu = Y_\mu^+ + Y_\mu^-$.

There is a singular point $\xi = \eta$, for which the spectrum (4) becomes degenerate and the above solution is not valid. The state of the system is not well defined at this critical point. This critical point assigns a critical value for the parameters of the system such as critical magnetic field (B_c), critical parameter of inhomogeneity of magnetic field (b_c), critical spin-orbit interaction parameter (D_c) and etc. The behavior of the entanglement of the system changes abruptly when the parameters cross their critical values (see Sect. 3).

In the following we will examine a class of bipartite density matrices having the following standard form as the initial state of the system

$$\hat{\rho}_s(0) = \begin{pmatrix} \mu_+ & 0 & 0 & \nu \\ 0 & w_1 & z & 0 \\ 0 & z^* & w_2 & 0 \\ \nu & 0 & 0 & \mu_- \end{pmatrix}. \quad (22)$$

Here the subscript s denotes the standard computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. This kind of density matrices are called X states and arise naturally in a wide variety of physical situations. The X states contain some important subsets like the pure Bell states, Werner states, mixture of Bell states, and so on. If the initial state $\hat{\rho}_s(0)$ belongs to the set of X states (22), then equations (16) and (20) guarantee that $\hat{\rho}_s(t)$ also belongs to the same set. Therefore, the only nonvanishing off-diagonal components of the density matrix in the energy basis are

$$\rho_{12}(t) = \rho_{12}(0)e^{-2i\xi t - t(X_1+Y_2)/2}, \qquad \rho_{21}(t) = \rho_{12}(t)^*,$$
$$\rho_{34}(t) = \rho_{34}(0)e^{-2i\eta t - t(X_1+Y_2)/2}, \qquad \rho_{43}(t) = \rho_{34}(t)^*. \quad (23)$$

Knowing the density matrix we can calculate the concurrence $C(\rho(t)) = \max\{0, 2\lambda_{max}(t) - \sum_{i=1}^{4} \lambda_i(t)\}$ where,

$$\lambda_{1,2}(t) = |\sqrt{\rho_{s11}(t)\rho_{s44}(t)} \pm |\rho_{s14}(t)||,$$

$$\lambda_{3,4}(t) = |\sqrt{\rho_{s22}(t)\rho_{s33}(t)} \pm |\rho_{s23}(t)||. \quad (24)$$

Evidently, the $\lambda_i(t)$s depend on the parameters involved. This prevents one from writing an analytical expression for the concurrence. But it is possible to evaluate the concurrence numerically, for a given set of the parameters. The results are shown in Figures 2–8. Figures 2–4 depict the dynamical behavior of concurrence versus parameters of the system and reservoirs and Figures 5–8 illustrate the asymptotical behavior of concurrence versus parameters of the system and reservoirs. Without loss of generality we can assume that $J > 0$ since the above formula are invariant under substitution $J \longrightarrow -J$. This means that the dynamical behavior of FM chain is the same as AFM chain.

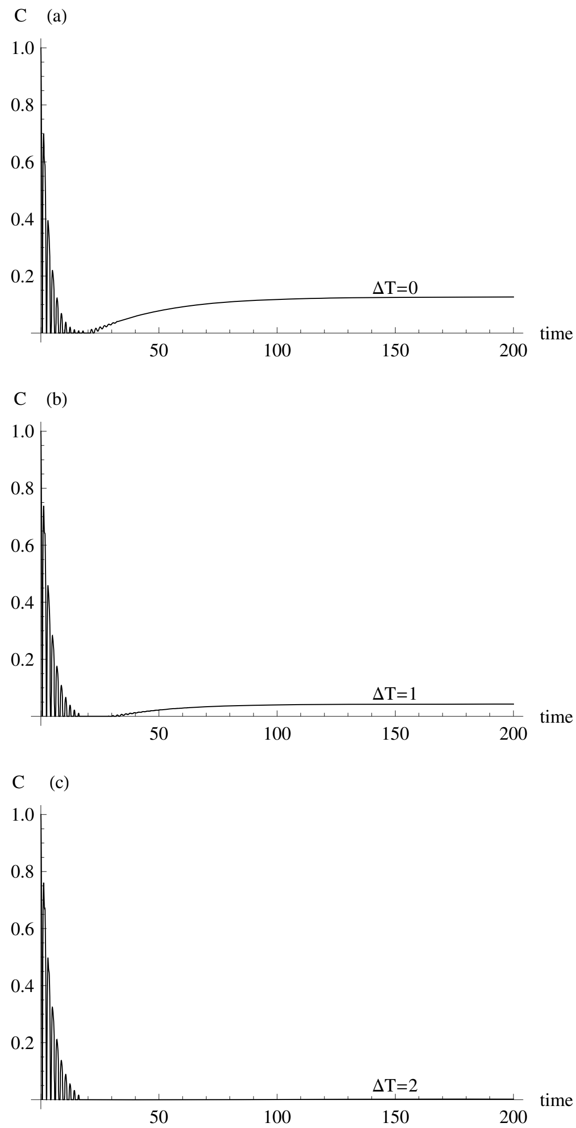

Fig. 2. Dynamics of non-equilibrium concurrence for the initial reduced density matrix $\rho_s(0) = \frac{1}{\sqrt{2}}(|01\rangle\langle 01| + |10\rangle\langle 10|)$. The parameters of the model are chosen to be $\gamma_1 = \gamma_2 = 0.02$, $J = 1$, $\chi = 0.9$, $B = 2$, $b = 1$, $T_M = 1.5$ and for different values of temperature difference ΔT: (a) $\Delta T = 0$ (b) $\Delta T = 1$ (c) $\Delta T = 2$. All parameters are dimensionless.

2.1 Asymptotic case

For a class of states $\hat{\rho}_{st}$, the dissipative and decoherence mechanisms, i.e the second and third terms in the master equation (8), compensate the unitary dynamics which is governed by system Hamiltonian, i.e. $i[\hat{H}_S, \hat{\rho}_{st}] = \pounds_1(\hat{\rho}_{st}) + \pounds_2(\hat{\rho}_{st})$ or $\frac{d}{dt}\hat{\rho}_{st} = 0$. These states are called stationary states because they are constant in time. If there exist such a stationary state solution for the master equation, the system tends to it asymptotically in large time limit i.e. $\lim_{t\to\infty} \hat{\rho}(t) \longrightarrow \hat{\rho}_{asym.} = \hat{\rho}_{st}$. For the present system, the nondiagonal elements (16) vanish asymptotically at large time limit and hence $\hat{\rho}(t)$ converges to a diagonal density matrix (in the energy basis) with elements not depending on the initial conditions

$$\hat{\rho}_{asym.} = \frac{1}{X_1 Y_2}\text{diagonal}(X_1^+ Y_2^+, X_1^- Y_2^-, X_1^- Y_2^+, X_1^+ Y_2^-). \quad (25)$$

The asymptotic concurrence is given by $C(\rho_{asym.}) = C^\infty = \max\{0, 2\lambda_{max} - \sum_{i=1}^{4} \lambda_i\}$ with

$$\lambda_{1,2} = \left|\sqrt{\rho_{s11}^{asym.}\rho_{s44}^{asym.}} \pm |\rho_{s14}^{asym.}|\right|,$$

$$\lambda_{3,4} = \left|\sqrt{\rho_{s22}^{asym.}\rho_{s33}^{asym.}} \pm |\rho_{s23}^{asym.}|\right|, \quad (26)$$

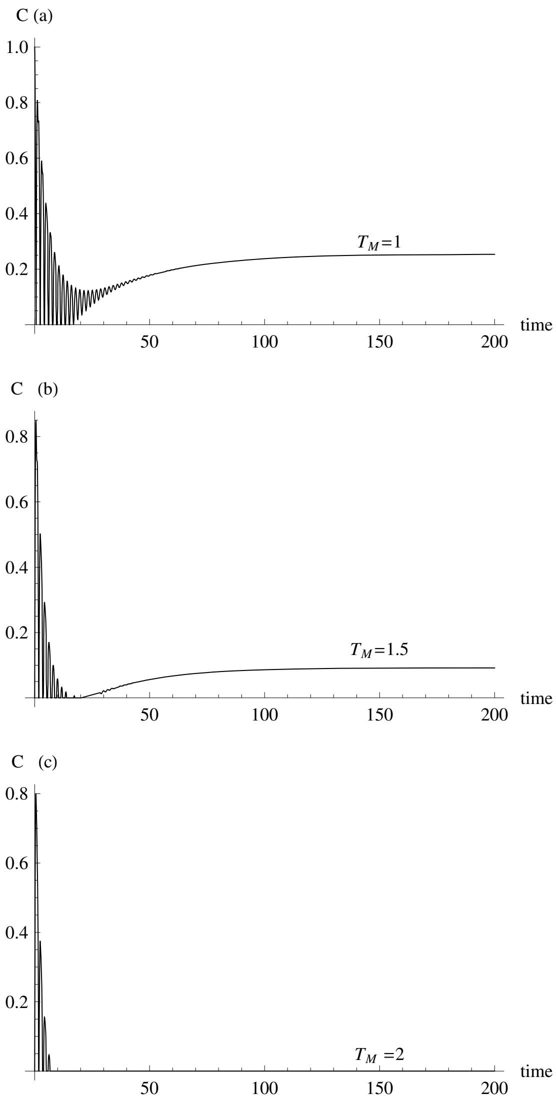

Fig. 3. Dynamics of non-equilibrium concurrence for the initial reduced density matrix $\rho_s(0) = \frac{1}{\sqrt{2}}(|01\rangle\langle 01| + |10\rangle\langle 10|)$. The parameters of the model are chosen to be $\gamma_1 = \gamma_2 = 0.02$, $J = 1$, $\chi = 0.9$, $B = 2$, $b = 1$, $\Delta T = 0.5$ and for different values of mean temperature T_M: (a) $T_M = 1$ (b) $T_M = 1.5$ (c) $T_M = 2$. All parameters are dimensionless.

where

$$\begin{aligned}
\rho_{s11}^{asymp.} &= \frac{1}{2\eta X_1 Y_2}((\eta + B)X_1^- Y_2^+ + (\eta - B)X_1^+ Y_2^-),\\
\rho_{s14}^{asymp.} &= \frac{J\chi}{2\eta X_1 Y_2}(X_1^- Y_2^+ - X_1^+ Y_2^-) = \rho_{s41}^{asymp.},\\
\rho_{s22}^{asymp.} &= \frac{1}{2\xi X_1 Y_2}((\xi + b)X_1^+ Y_2^+ + (\xi - b)X_1^- Y_2^-),\\
\rho_{s23}^{asymp.} &= \frac{J(1+iD)}{2\xi X_1 Y_2}(X_1^+ Y_2^+ - X_1^- Y_2^-) = (\rho_{s32}^{asymp.})^*,\\
\rho_{s33}^{asymp.} &= \frac{1}{2\xi X_1 Y_2}((\xi - b)X_1^+ Y_2^+ + (\xi + b)X_1^- Y_2^-),\\
\rho_{s44}^{asymp.} &= \frac{1}{2\eta X_1 Y_2}((\eta - B)X_1^- Y_2^+ + (\eta + B)X_1^+ Y_2^-).
\end{aligned} \tag{27}$$

There is an interesting limiting case for which the coupled QDs are in contact with two independent reservoirs at identical temperatures ($\beta_1 = \beta_2 = \beta$). In this case, it is easy to show that

$$\frac{X_1^+}{X_1} = \frac{e^{\beta\omega_1}}{e^{\beta\omega_1}+1}, \qquad \frac{X_1^-}{X_1} = \frac{1}{e^{\beta\omega_1}+1},$$
$$\frac{Y_2^+}{Y_2} = \frac{e^{\beta\omega_2}}{e^{\beta\omega_2}+1}, \qquad \frac{Y_2^-}{Y_2} = \frac{1}{e^{\beta\omega_2}+1}.$$

By substituting these relations into equation (25), the reduced density matrix $\hat{\rho}_{asym.}$ takes the thermodynamical canonical form for a system described by the Hamiltonian $\hat{H}_S$ at temperature $T = \beta^{-1}$, as expected. This means that

$$\hat{\rho}_{asym.}(\Delta T = 0) \equiv \hat{\rho}_T = \frac{e^{-\beta H_S}}{Z}, \tag{28}$$

where $Z = \mathrm{Tr}(e^{-\beta H_S})$ is the partition function. Thermal entanglement properties of such systems have been studied substantially, in our previous work [12]. Thus, for the special case $\Delta T = 0$, the results reduce to the results of reference [12].

3 Results

The non-equilibrium thermal concurrence as a function of time for three values of temperature difference ($\Delta T = T_1 - T_2$) and for a fixed value of mean temperature ($T_M = \frac{T_1+T_2}{2}$) are plotted in Figure 2, in the case of "direct geometry" of connection ($b\Delta T > 0$). The presence of temperature difference has not effective influence on the dynamics of entanglement at early times of evolution, but changes effectively the behavior of the asymptotic entanglement (which is more evident form Figs. 5–8). Figure 3 depicts the variation of entanglement dynamics for some values of T_M and for a fixed ΔT in the case of "direct geometry" of connection ($b\Delta T > 0$). By increasing T_M, thermal fluctuations suppress quantum fluctuations and hence decrease coherent oscillations at early times of evolution and also decreases asymptotic entanglement.

Perhaps, surprisingly, the decoherence due to environmental interaction does not prevent the creation of a steady state level of entanglement, regardless of the initial state of the system. This is demonstrated in Figure 4 which shows the time evolution of the non-equilibrium thermal concurrence for a given set of parameters and for four different initial states: (i) a maximally entangled state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$; (ii) a separable state, $|\psi(0)\rangle = |01\rangle$; (iii) a mixed state defined as an equal mixture of a Bell state and a product state, e.g. $\rho_s(0) = \frac{1}{4}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) + \frac{1}{2}|01\rangle\langle 01|$; and finally (iv) an unpolarized state, $\rho_s(0) = \frac{1}{4}I$. Despite the presence of decoherence (due to interaction with environment) the results of Figure 3 show that for a given set of parameters, the concurrence reaches the same steady state value, C^∞ (after some oscillatory behavior), regardless of initial conditions. Clearly, the Heisenberg interaction in the

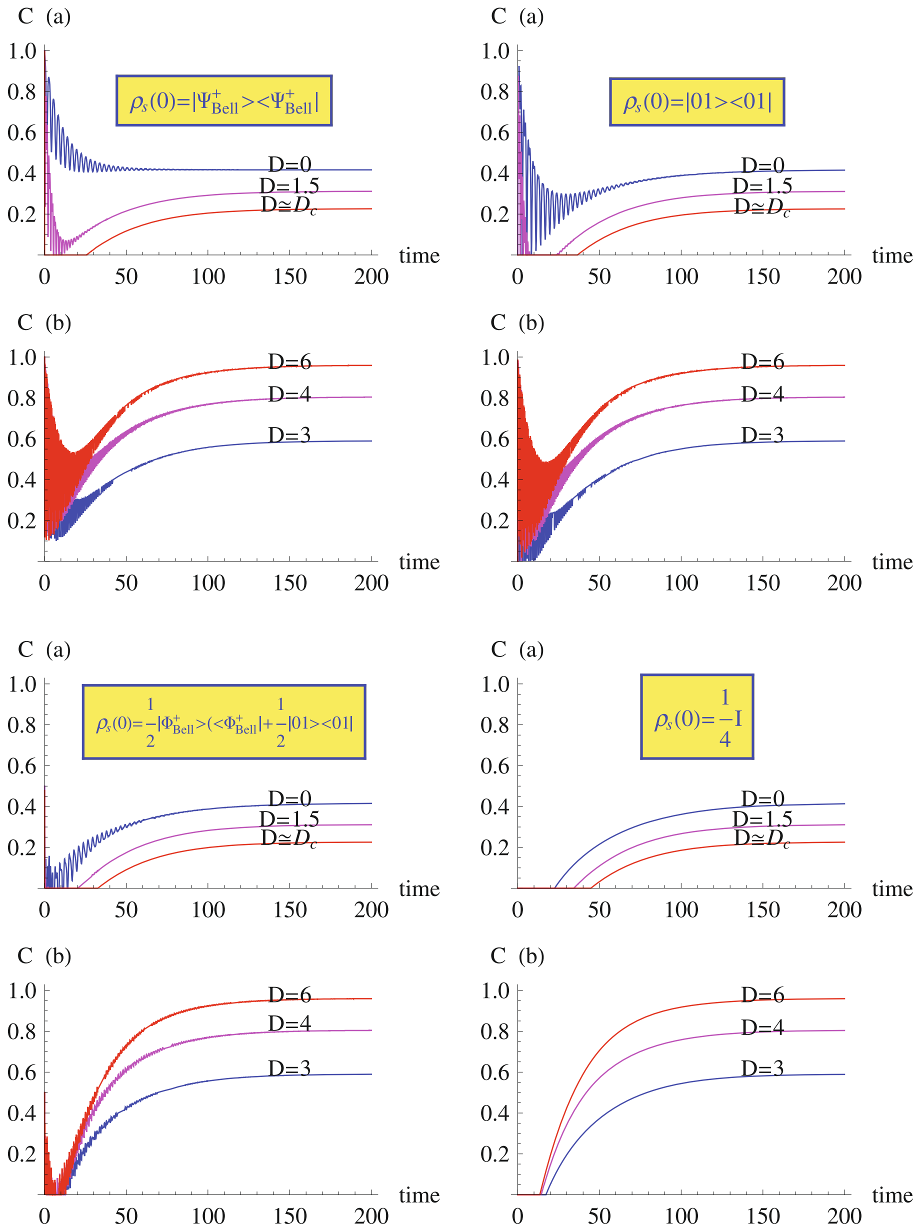

Fig. 4. (Color online) Dynamics of non-equilibrium concurrence for different values of D and different initial state. The parameters of the model are chosen to be $\gamma_1 = \gamma_2 = 0.02$, $J = 1$, $\chi = 0.9$, $B = 2$, $b = 0.5$, $T_M = 1$ and $\Delta T = 0.5$. Each plot contains two graphs for (a) $D < D_c$ (b) $D > D_c$ ($D_c \simeq 1.8868$). All parameters are dimensionless.

equation (3) serves to maintain an entangled asymptotic state despite the presence of decoherence. At early times of evolution the amount of D determines the frequency of oscillations. Each plot in Figure 4 contains two cases (i) $D < D_c$, in this case asymptotic value of entanglement decreases with D and (ii) $D > D_c$, in this case asymptotic value of entanglement increases as D increases.

In Figure 5, the asymptotic non-equilibrium thermal concurrence of the system in the case of indirect geometry of connection is plotted versus T_M and D for different values of ΔT. In the case of identical temperatures ($\Delta T = 0$) the results are the same as reference [12], as mentioned in the previous section. This figure shows that, there is a critical mean temperature ($T_M^{cr.}$) over which entanglement vanishes. The size of $T_M^{cr.}$ and the amount of entanglement can be improved by increasing D. Increasing ΔT enhances the size of $T_M^{cr.}$, i.e. entanglement can exists in higher mean temperatures due to existence of temperature difference of reservoirs (environment induced entanglement). Furthermore, the size of this environment entanglement increases with D.

The variation of the asymptotic non-equilibrium thermal concurrence as a function of ΔT and D for fixed values of T_M and b is illustrated in Figure 6. The behavior of concurrence depends on the geometry of connection. In the case of "direct geometry" of connection ($b\Delta T > 0$) and for $D < D_c$, no entanglement is observed but for the same geometry of connection and for $D > D_c$ the amount of

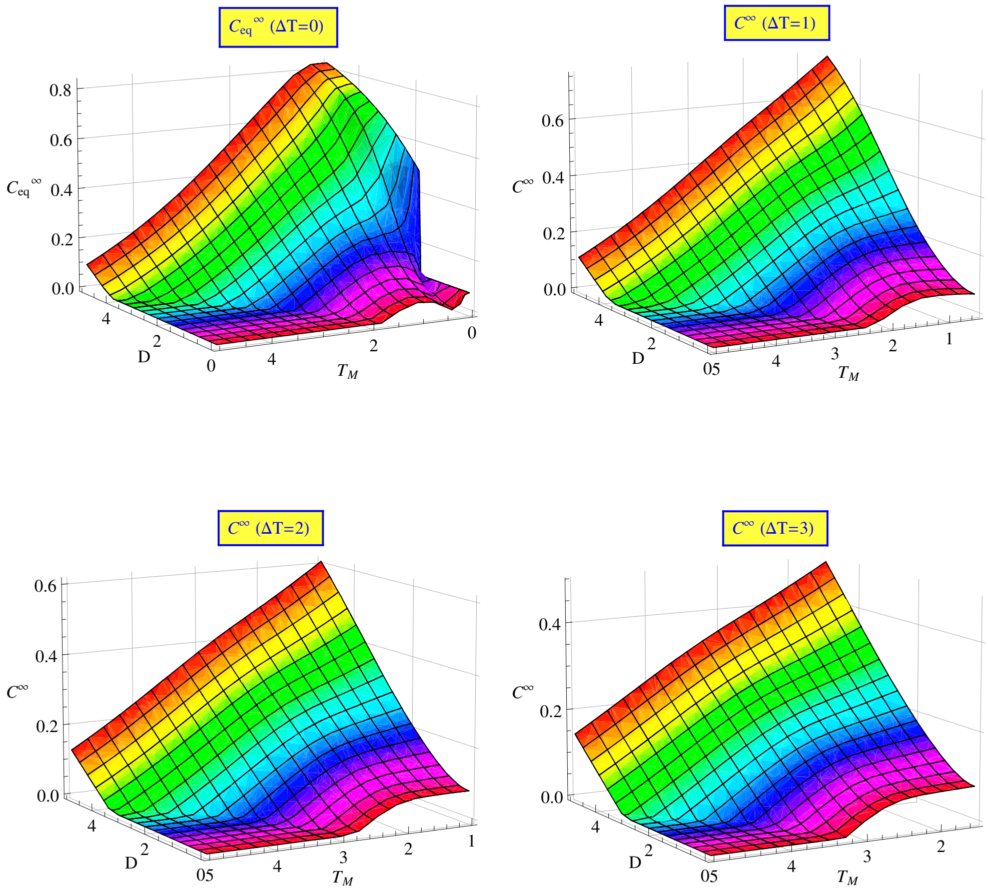

Fig. 5. (Color online) Asymptotic entanglement vs. T_M and D, The parameters of the model are chosen to be $\gamma_1 = \gamma_2 = 0.02$, $B = 4$, $J = 1$, $\chi = 0.3$ and $b = -3.5$ for (a) $\Delta T = 0$ (b) $\Delta T = 1$ (c) $\Delta T = 2$ (d) $\Delta T = 3$ ($D_c \simeq 1.68523$). All parameters are dimensionless.

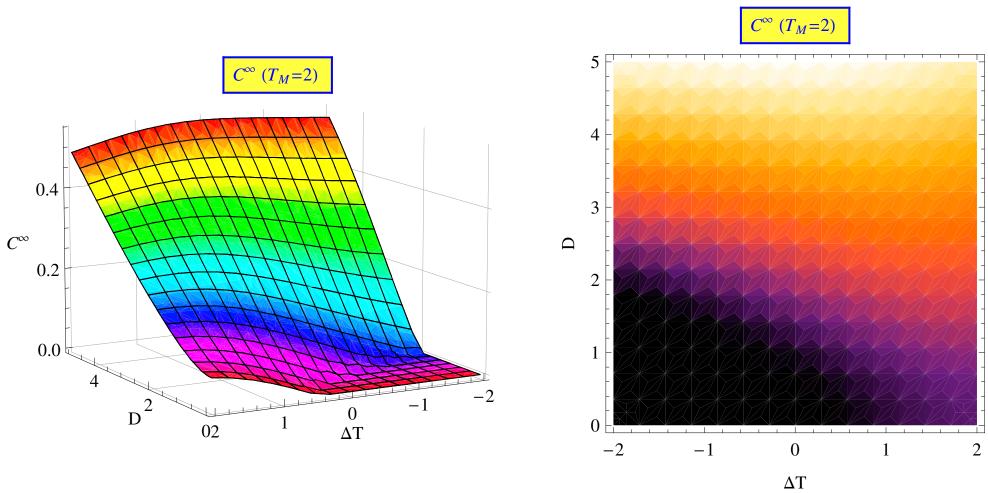

Fig. 6. (Color online) Asymptotic entanglement vs. ΔT and D. The parameters of the model are chosen to be $\gamma_1 = \gamma_2 = 0.02$, $B = 4$, $J = 1$, $\chi = 0.3$ and $b = -3.5$ and $T_M = 2$ ($D_c \simeq 1.68523$). All parameters are dimensionless.

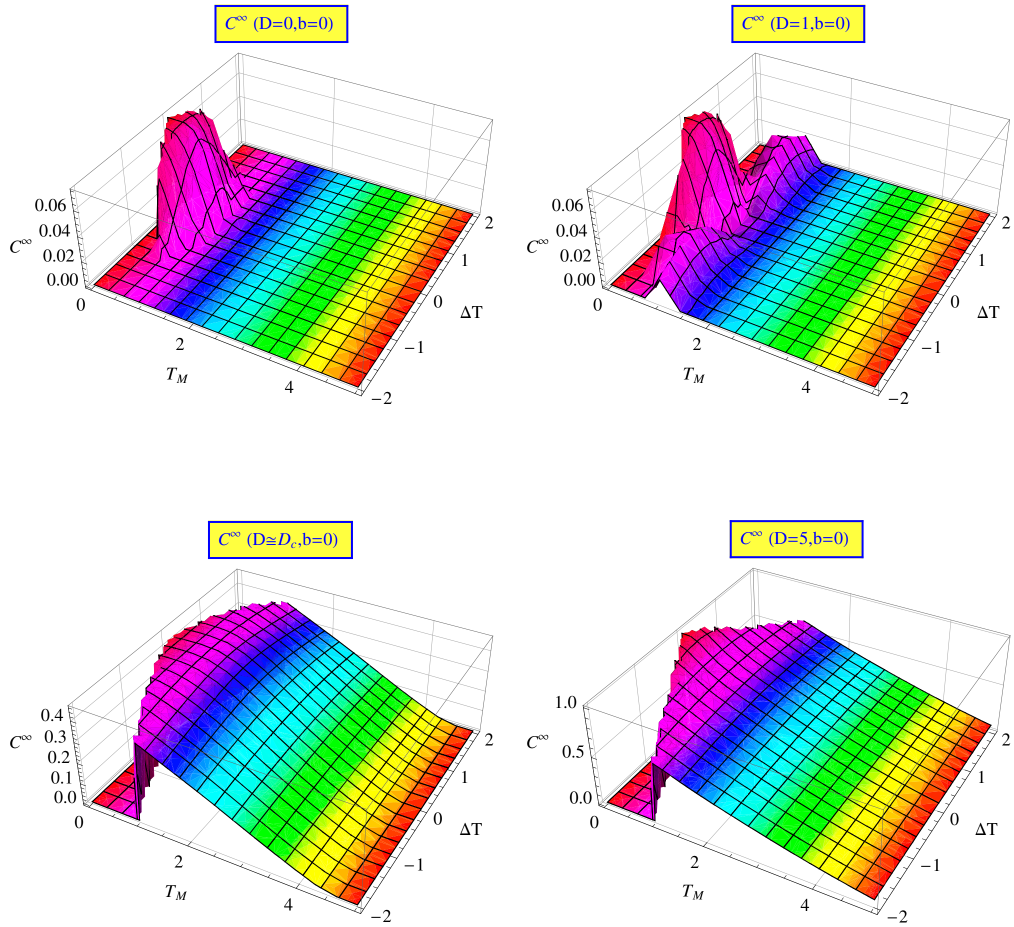

Fig. 7. (Color online) Asymptotic entanglement vs. T_M and ΔT. The parameters of the model are chosen to be $\gamma_1 = \gamma_2 = 0.02$, $B = 4$, $J = 1$, $\chi = 0.3$ and $b = 0$ for (a) $D = 0$ (b) $D = 1$ (c) $D \approx D_c$ (d) $D = 5$ ($D_c \simeq 3.88458$). All parameters are dimensionless.

entanglement is nonzero and it increases with both D or ΔT. In the case of "indirect geometry" ($b\Delta T < 0$), there is a nonzero entanglement for all values of D. The amount of entanglement is an increasing function of D. For the values of $\Delta T \le \frac{T_M}{2}$, it increases with ΔT and decreases with ΔT for the values of $\frac{T_M}{2} < \Delta T \le T_M$.

Figures 7 and 8 show the behavior of the asymptotic non-equilibrium thermal concurrence versus T_M and ΔT for different values of D and for the symmetric ($b = 0$) and nonsymmetric ($b \neq 0$) cases, respectively. Both figures reveal that, increasing D causes the existence of entanglement in the larger regions of $T_M - \Delta T$ plane. The departure between symmetric (Fig. 7) and nonsymmetric (Fig. 8) cases is more obvious for $D < D_c$. The results show that, in the absence of spin-orbit interaction and for both symmetric and nonsymmetric cases, the entanglement exists only in a small interval of temperature difference. Introducing spin-orbit interaction causes this temperature difference interval to broad such that for large values of D, the entanglement exists for any allowable value of ΔT in the mean temperature interval $T_M \in [0, T_M^{cr.}]$. Also, increasing D enhances the amount of entanglement in this interval. These figures also show that, the maximum entanglement ($C = 1$) can be achieved in the case of identical temperatures ($\Delta T = 0$), zero mean temperature ($T_M = 0$) (i.e. when both of reservoirs are in the ground state ($T_1 = T_2 = 0$)) and large values of D, regardless of the geometry of connection which is in agreement with the results of [12].

4 Discussion

The dynamics of non-equilibrium thermal entanglement of an open two-qubit system is investigated. The inter-qubit interaction is considered as the Heisenberg interaction in the presence of inhomogeneous magnetic field and spin-orbit interaction, arised from the Dzyaloshinski-Moriya (DM) anisotropic antisymmetric interaction. Each qubit interacts with a separate thermal reservoir which is held in its own temperature. For physical realization of the model we address to the spin states of two electrons which are confined in two coupled quantum dots, respectively. The dots are assumed to biased via different sources and drains and hence experience different environments. The effects of the parameters of the model, including the parameters of the system (especially, the parameter of the spin-orbit

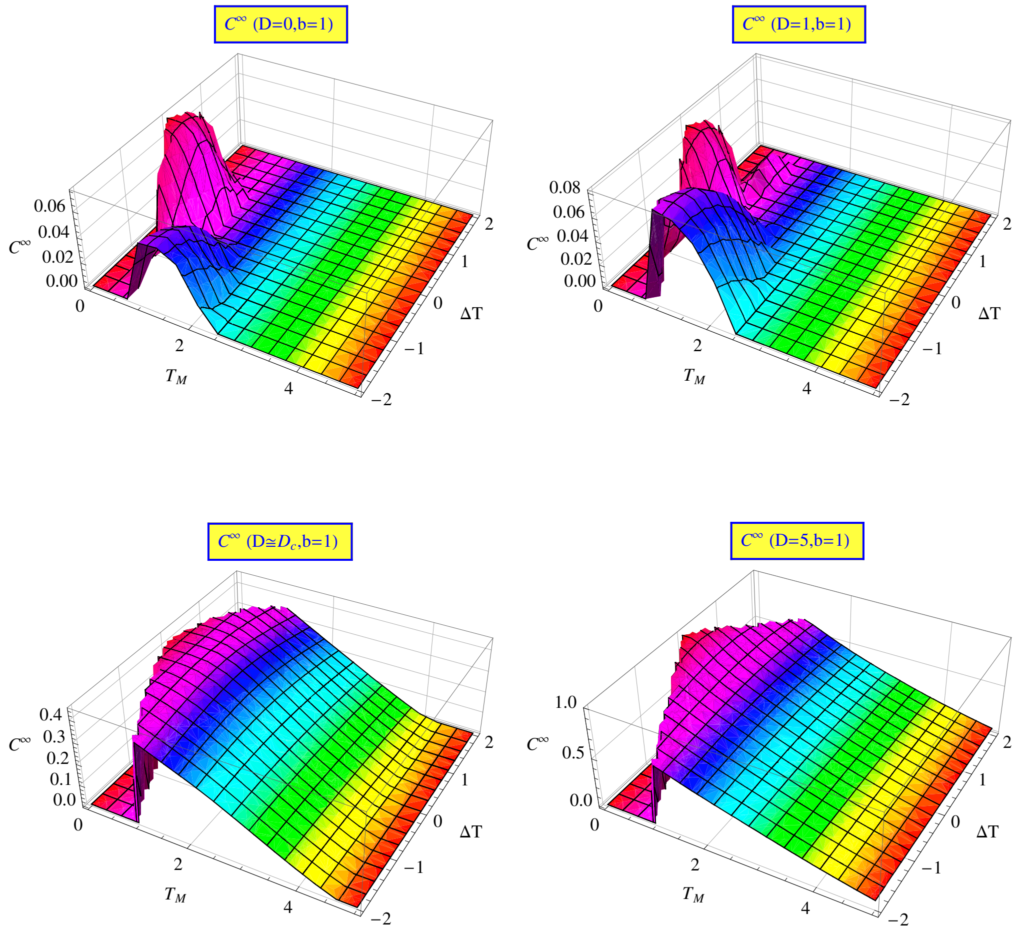

Fig. 8. (Color online) Asymptotic entanglement vs. T_M and ΔT. The parameters of the model are chosen to be $\gamma_1 = \gamma_2 = 0.02$, $B = 4$, $J = 1$, $\chi = 0.3$ and $b = 1$ for (a) $D = 0$ (b) $D = 1$ (c) $D \approx D_c$ (d) $D = 5$ ($D_c \simeq 3.75366$). All parameters are dimensionless.

interaction D) and environmental parameters (particularly, mean temperature T_M and temperature difference ΔT), on the non-equilibrium thermal entanglement dynamics of the system is investigated, by solving the quantum Markov-Born master equation of the system. An analytical solution of the master equation is derived and then entanglement dynamics and asymptotic entanglement of the system versus the parameters of the model are studied. The entanglement dynamics allowed us to distinguish between the entanglement induced by the interaction and by the environment. It is shown that in some special cases the entanglement can be produced only by increasing ΔT. Indeed, the entanglement production is proportional to the rate of change of entropy, which is known as entropy production [10] and it is well known that the existence of the temperature difference enhances the rate of entropy production. Thus, the entanglement can be enhanced by increasing ΔT. The amount of this environment induced entanglement increases as D increases. The results show that, decoherence induced by thermal bathes are competing with inter-qubit interaction terms to create a steady state level of entanglement, as measured by the concurrence. The size of this steady state entanglement and the dynamical behavior of the entanglement depend on the parameters of the model and also on the geometry of connection. Increasing mean temperature T_M, kills the asymptotic entanglement. Indeed, thermal fluctuations suppress the quantum fluctuations, i.e. all quantum effects such as the entanglement and local coherence, and hence the entanglement of the system dies at a critical temperature $T_M^{cr.}$. We have shown that, the size of $T_M^{cr.}$ and the amount of entanglement can be enhanced by choosing a suitable value of the spin-orbit interaction parameter D and the temperature difference ΔT. The maximum entanglement can be achieved in the case of large values of D and zero temperature reservoirs. Furthermore, we find that the indirect geometry of connection is more proper for creating and maintaining the entanglement. In this case, the heat current between two QDs is substantially decreased but the concurrence is enhanced by temperature difference ΔT. The results of [18,19] are also obtained in the limit of $D = \chi = 0$ and by considering $b = \epsilon_1 - \epsilon_2$.

In summary, the novel results of this paper are as follows. (i) As demonstrated in Figures 5–8, increasing D amplifies the effects of ΔT on the asymptotic entanglement, i.e the amount of the asymptotic entanglement induced by ΔT is increased by increasing D. (ii) For the both symmetric and nonsymmetric cases the results reveal that increasing D causes the appearance of entanglement in the larger regions of $T_M - \Delta T$ plane and hence entanglement exist in higher temperatures and temperature differences. In addition, increasing D enhances the amount of entanglement in these regions. (iii) The indirect geometry of connection is more suitable for creating and maintaining the entanglement. (iv) For $D < D_c$ and for indirect geometry of connection the size of the asymptotic entanglement increases with D. (v) For $D > D_c$, the asymptotic entanglement is an increasing function of D. Additionally, in this case and for both symmetric and nonsymmetric cases, a relatively large entanglement exists for any allowable value of ΔT in the mean temperature interval $T_M \in [0, T_M^{cr.}]$. The results can provide useful recipes for realistic quantum information processing in noisy and non-equilibrium environments.

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