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Applications of Stochastic Orders in Reliability and Mixture of Exponential family

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Abstract. In this paper, we have recalled some of the known stochastic orders and the shifted version of them, so discussed their relations. Also, we obtained some applications of proportional hazard rate ordering in reliability. Then we investigated stochastic comparisons between exponential family distributions and their mixtures with respect to the usual stochastic order, the hazard rate order, the reversed hazard rate order and the likelihood ratio order.

Keywords. Up (Down) proportional likelihood ratio order, Up (Down) proportional hazard rate order, Up proportional reversed hazard rate order, Exponential family, Log concave.

1 Introduction

Stochastic orders have been proven to be very useful in applied probability, statistics, reliability, operation research, economics and other fields. Various types of stochastic orders and associate properties have been developed rapidly over the years. A lot of research works have done on likelihood ratio, hazard rate and reversed hazard rate orders due to their properties and applications in the various sciences, for example hazard rate order is a well-known and useful tool in reliability theory and reversed hazard rate order is defined via stochastic comparison of inactivity time. Likelihood ratio order which is stronger than hazard rate and reversed hazard rate orders, introduced by Ross (1983). We can refer reader to the papers such as Shanthikumar and Yao (1986), Muller (1997), Kijima (1998), Chandra and Roy (2001), Gupta and Nanda (2001), Nanda and Shaked (2001), Kochar et al. (2002), Kayid and Ahmad (2004), Ahmad et al. (2005) and Shaked and Shanthikumar (2007).

Ramos-Romero and Sordo-Diaz (2001) introduced a new stochastic order between two absolutely continuous random variables and called it proportional likelihood ratio (plr) order, which is closely related to the usual likelihood ratio order. The proportional likelihood ratio order can be used to characterize random variables whose logarithms have log-concave (log-convex) densities. Many income random variables satisfy this property and they are said to have the increasing proportional likelihood ratio ($IPLR$) and decreasing proportional likelihood ratio ($DPLR$) properties. As an application, they showed that the $IPLR$ and $DPLR$ properties are sufficient conditions for the Lorenz ordering of truncated distributions.

Jarrahiferiz et al. (2010) studied some other properties of the proportional likelihood ratio order, then extended hazard rate and reversed hazard rate orders to proportional state similar to proportional likelihood ratio order called

them proportional (reversed) hazard rate orders, and studied their properties and relations.
Shifted stochastic orders that are useful tools for establishing interesting inequalities that have been introduced and studied in Nakai (1995) and Brown and Shanthikumar (1998). Also, they have been touched upon in Belzunce et al. (2001). Lillo et al. (2001) have been studied in detail four shifted stochastic orders, namely the up likelihood ratio order, the down likelihood ratio order, the up hazard rate order and the down hazard rate order. They have compared them and obtained some basic and closure properties of them and have shown how those can be used for stochastic comparisons of order statistics. Recently, Aboukalam and Kayid (2007) obtained some new results about shifted hazard and shifted likelihood ratio orders. In this paper we recall the proportional state of stochastic orders and the shifted version of them and so obtained some applications of proportional hazard rate order. Also we studied stochastic comparisons between exponential family distributions and their mixtures.

2 Preliminaries

Let X and Y be two absolutely continuous random variables with densities f and g, distribution functions F and G, hazard rate functions r_F and r_G and reversed hazard rate functions $\tilde{r}_F$ and $\tilde{r}_G$ respectively, each with an interval support. Denote by l_X the left endpoint of the support of X, and by u_X the right endpoint of the support of X. Similarly, define l_Y and u_Y for Y. The values l_X, u_X, l_Y and u_Y may be infinite (for any $a > 0$ we define $\frac{a}{0} = \infty$). Also, let λ be any positive constant smaller than 1.

Definition 2.1 *(1) We say X is smaller than Y in the usual stochastic order ($X \leq_{st} Y$), if $P(X > x) \leq P(Y > x) \quad \forall x$.*
(2) X is said to be smaller than Y in the likelihood ratio order ($X \leq_{lr} Y$), if $\frac{g(x)}{f(x)}$ increases over the union of the supports of X and Y.
(3) X is smaller than Y in the hazard rate order ($X \leq_{hr} Y$), if $r_F(x) \geq r_G(x)$, which is equivalent to $\frac{\bar{F}(x)}{\bar{G}(x)}$ decreases in x.
(4) X is smaller than Y in the reversed hazard rate order ($X \leq_{rh} Y$), if $\tilde{r}_F(x) \geq \tilde{r}_G(x)$, which is equivalent to $\frac{F(x)}{G(x)}$ decreases in x.
Let X and Y be two absolutely continuous random variables as above:
(5) We say X is smaller than Y in the up likelihood ratio order ($X \leq_{lr\uparrow} Y$), if $X - x \leq_{lr} Y, \quad \forall x$.
(6) X is smaller than Y in the up hazard rate order ($X \leq_{hr\uparrow} Y$), if $X - x \leq_{hr} Y, \quad \forall x$.
(7) X is said to be smaller than Y in the up reversed hazard rate order ($X \leq_{rh\uparrow} Y$), if $X - x \leq_{rh} Y, \quad \forall x$.
Let X and Y be two non-negative absolutely continuous random variables as above:
(8) We say X is smaller than Y in the down likelihood ratio order ($X \leq_{lr\downarrow} Y$), if $X \leq_{lr} [Y - x | Y > x], \quad \forall x \geq 0$.
(9) X is smaller than Y in the down hazard rate order ($X \leq_{hr\downarrow} Y$), if $X \leq_{hr} [Y - x | Y > x], \quad \forall x \geq 0$.
(10) X is said to be smaller than Y in the down reversed hazard rate order ($X \leq_{rh\downarrow} Y$), if $X \leq_{rh} [Y - x | Y > x], \quad \forall x \geq 0$.

The proportional likelihood ratio order is an extension version of the likelihood ratio order that is studied by Ramos-Romero and Sordo-Diaz (2001). Also, they introduced the class of the random variables based on the proportional likelihood ratio order and derived properties and results due to them.

Definition 2.2 *If X and Y are non-negative absolutely continuous random variables, then we say that X is smaller than Y in the proportional likelihood ratio order ($X \leq_{plr} Y$), if $\frac{g(\lambda x)}{f(x)}$ is increasing in x over the union of the supports of X and Y.*

Equivalently, $X \leq_{plr} Y$ if,

$$g(\lambda x)f(y) \leq g(\lambda y)f(x), \quad \forall x \leq y. \tag{1}$$

Definition 2.3 *If X is a non-negative absolutely continuous random variable, then X is said to be increasing proportional likelihood ratio ($IPLR$), if $\frac{f(\lambda x)}{f(x)}$ is increasing in x.*

Ramos-Romero and Sordo-Diaz (2001) obtained some results of proportional likelihood ratio order such as: if $X \leq_{plr} Y$ then, $l_X \leq l_Y$, $u_X \leq u_Y$ and $\mu_X \leq \mu_Y$ or $X \leq_{plr} Y$ if and only if $X \leq_{lr} aY$ for all $a > 1$. Also, they showed that if Y has a log-concave density function, then, $X \leq_{lr} Y \Longrightarrow X \leq_{plr} Y$. Jarrahiferiz et al. (2010) developed hazard rate order and reversed hazard rate orders to proportional state similarly and studied their properties and relations.

Definition 2.4 *Let X and Y be non-negative absolutely continuous random variables. We say that X is smaller than Y in the proportional hazard rate order ($X \leq_{phr} Y$), if $r_F(t) \geq \lambda r_G(\lambda t), \quad t > 0$.*

We can see the relations between above orders as follow:

$$\begin{array}{ccccccccc}
 & & & & \leq_{lr\uparrow} & \Rightarrow & \leq_{hr\uparrow} & & \\
 & & & & \Downarrow & & \Downarrow & & \\
\leq_{phr} & \Leftarrow & \leq_{plr} & \Rightarrow & \leq_{lr} & \Rightarrow & \leq_{hr} & \Rightarrow & \leq_{st} \\
 & & & & \Uparrow & & \Uparrow & & \\
 & & & & \leq_{lr\downarrow} & \Rightarrow & \leq_{hr\downarrow} & &
\end{array}.$$

3 Shifted proportional hazard rate order and its applications

Here, we want to shift proportional hazard rate order and study its properties and relations. Note that, all of the following results are hold for the proportional reversed hazard rate ordering similarly.

Definition 3.1 *We say X is smaller than Y in the up proportional hazard rate order ($X \leq_{phr\uparrow} Y$), If $[X - x|X > x] \leq_{phr} Y, \quad \forall x \geq 0$.*
Actually, $X \leq_{phr\uparrow} Y$ if and only if, $\frac{\bar{G}(\lambda t)}{\bar{F}(t+x)}, \forall x \geq 0$, is increasing in $t \in (-\infty, \frac{u_Y}{\lambda})$, equivalently, $\bar{G}(\lambda z)\bar{F}(w+x) \leq \bar{G}(\lambda w)\bar{F}(z+x), \quad \forall z \leq w$.

Consider two coherent systems C_1 and C_2, each consisting of n *iid* components. Suppose that the lifetime of components from C_1 and C_2 have distribution functions F and G respectively. In the following result, the preservation of up proportional hazard rate is proved for a coherent system with *iid* components, that is an analogous of *Theorem* 3.1 of Aboukalam and Kayid (2007).
Remark Let X and Y be non-negative and absolutely continuous random variables. If $X \leq_{phr\uparrow} Y$ then, $\bar{G}(t+x) \leq \bar{F}(\lambda x)$.

Theorem 3.2 *Let $h(p)$ be the reliability function of coherent system of n independent and identical components having first and second derivatives $h'(p)$ and $h''(p)$ respectively. If $\frac{ph''(p)}{h'(p)}$ is decreasing and $\bar{G} \leq_{phr\uparrow} \bar{F}$, then, $h(\bar{G}) \leq_{phr\uparrow} h(\bar{F})$.*

Proof We must show that, $\frac{\lambda f(\lambda t)h'(\bar{F}(\lambda t))}{g(t+x)h'(\bar{G}(t+x))}$ is increasing in $t > 0$, equivalently,

$$\lambda[\lambda f'(\lambda t)g(t+x) - g'(t+x)f(\lambda t)] + \lambda[f(\lambda t)g(t+x)][\frac{g(t+x)h''(\bar{G}(t+x))}{h'(\bar{G}(t+x))} - \frac{\lambda f(\lambda t)h''(\bar{F}(\lambda t))}{h'(\bar{F}(\lambda t))}] \geq 0,$$

thus,

$$\begin{aligned}&\lambda[\lambda f'(\lambda t)g(t+x) - g'(t+x)f(\lambda t)] \\ &+\lambda[f(\lambda t)g(t+x)][\frac{g(t+x)}{\bar{G}(t+x)}\frac{\bar{G}(t+x)h''(\bar{G}(t+x))}{h'(\bar{G}(t+x))} - \frac{\lambda f(\lambda t)}{\bar{F}(\lambda t)}\frac{\bar{F}(\lambda t)h''(\bar{F}(\lambda t))}{h'(\bar{F}(\lambda t))}] \geq 0,\end{aligned} \tag{2}$$

which is non-negative because the both terms are non-negative by assumption. □

Theorem 3.3 *Let X and Y be non-negative absolutely continuous random variables. If $X \leq_{phr\uparrow} Y$, then, there exists a random variable Z that is $UIPLRS$, such that $X \leq_{phr} Z \leq_{phr} Y$.*

Proof If $u_X \leq l_Y$ then take Z to be any random variables that is $UIPLRS$ on $[u_X, l_Y]$. Therefore, suppose that $l_Y \leq u_X$.

$$\begin{aligned} X \leq_{phr\uparrow} Y &\iff r_X(t+x) \geq \lambda r_Y(\lambda t), \quad \forall x \geq 0, t \in (l_Y, u_X - x) \\ &\iff r_X(t') \geq \lambda r_Y(\lambda t), \quad \forall l_Y \leq t < t' \leq u_X \end{aligned} \tag{3}$$

Define $r^*(t) = max_{\nu \leq t} r_Y(\nu), t \in (l_Y, u_Y)$. By Lillo et al. (2001), r^* defines a hazard rate function on (l_Y, u_Y) and it is sufficient that consider Z has the hazard rate function r^*. By assuming r^* is increasing in t, so, $r^*(t') \geq \lambda r^*(\lambda t)$, and therefore, $Z \leq_{phr} Y$. Finally, from (4), it follows that $r_X(t) \geq \lambda r^*(\lambda t)$, for all $t \in (l_Y, u_X)$, so, $X \leq_{phr} Z$. □

Consider a system of n independent and not necessarily identical components in which the ith component has survival function $\bar{F}_i(t) = 1 - F_i(t)$, $i = 1, 2, ..., n$. Let $h(\mathbf{P}) = h(p_1, p_2, ..., p_n)$ be the system reliability function. In the following theorem we compare the random lifetimes of two systems according to the up proportional hazard rate order.

Theorem 3.4 *If* $\sum\limits_{i=1}^{n} \frac{p_i \partial h/\partial p_i}{h(\mathbf{p})}$ *is decreasing in* p_i *and* $X_i \leq_{phr\uparrow} Y$, *for all* $i = 1, 2, ..., n$ *then,* $h(\mathbf{X}) \leq_{phr\uparrow} h(\mathbf{Y})$.

Proof We know that the hazard rate function of a coherent system is:

$$r_{h(\mathbf{X})}(z+t) = \sum_{i=1}^{n} r_{X_i}(z+t)\bar{F}_{X_i}(z+t)\frac{\partial h/\partial p_i}{h(\mathbf{p})}|_{p_i=\bar{F}_{X_i}(z+t)}$$

Hence, Due to relation (4) and hypothesis $X_i \leq_{phr\uparrow} Y$, gives,

$$r_{h(\mathbf{X})}(z+t) \geq \lambda r_Y(\lambda z)\sum_{i=1}^{n} \bar{F}_{X_i}(z+t)\frac{\partial h/\partial p_i}{h(\mathbf{p})}|_{p_i=\bar{F}_{X_i}(z+t)},$$

using assuming implies that,

$$r_{h(\mathbf{X})}(z+t) \geq \lambda r_Y(\lambda z)\sum_{i=1}^{n} \bar{F}_{Y}(\lambda z)\frac{\partial h/\partial p_i}{h(p)}|_{p_i=\bar{F}_{X_i}(z+t)} \equiv \lambda r_{h(\mathbf{Y})}(\lambda z),$$

and the proof is complete. □

4 Stochastic Ordering of Exponential Family and their mixtures

In this section, we investigate stochastic comparisons between exponential family distributions and their mixtures with respect to the usual stochastic order, the hazard rate order, the reversed hazard rate order and the likelihood ratio order. For represent the results we need recall the following definition.

Definition 4.1 *We say* X *is log-concave relative to* Y, *denoted* $X \leq_{lc} Y$, *if*
(1) *Support of* X *and* Y *are intervals on* $\mathbf{R}$.
(2) *Support of* X *be sub-set if Support of* Y.
(3) $log(f(x)/g(x))$ *is concave on Support of* X.

Yu (2010) proved the following result:

Theorem 4.2 *For random variables* X *and* Y *let* $l(x) = log(f(x)/g(x))$ *is continuous and moreover concave, i.e.,* $X \leq_{lc} Y$. *Then*
(a) $X \leq_{st} Y$ *and* $X \leq_{lr} Y$ *are equivalent, and each holds if and only if* $\lim_{x\downarrow 0} l(x) \geq 0$.
(b) *Assuming* $l(x)$ *is continuously differentiable, then* $X \leq_{lr} Y$ *and* $X \leq_{rh} Y$ *are equivalent, and each holds if and only if* $\lim_{x\downarrow 0} l'(x) \leq 0$.

Consider the density of an exponential family

$$f(x;\theta) = f_0(x)\exp[b(\theta)x]h(\theta), \tag{4}$$

where θ is a parameter, and for simplicity, assume the support of $f(x;\theta)$ is interval $(0,\infty)$. Let $g(x) = \int f(x;t)d\mu(t)$ be the mixture of $f(x;\theta)$ with respect to a probability distribution μ on θ. Whitt (1985) showed that

$$\log(g(x)/f(x;\theta)) = \log\left(\int e^{[b(t)-b(\theta)]x}h(t)/h(\theta)d\mu(t)\right)$$

is a convex function of x, *i.e.*, $l(x) = \log(f(x;\theta)/g(x))$ is concave. YU (2010) computed

$$\lim_{x\downarrow 0} l(x) = -\log\left(\int h(t)/h(\theta)d\mu(t)\right),$$

and

$$\lim_{x\downarrow 0} l'(x) = \frac{\int [b(\theta) - b(t)]h(t)d\mu(t)}{\int h(t)d\mu(t)},$$

that provided the interchange of limit and integration is valid. Thus, if random variables X and Y have densities $f(x;\theta)$ and $g(x)$ respectively, then by Theorem**??**,
1. $X \leq_{st} Y$ ($X \leq_{hr} Y$) if and only if

$$\int h(t)d\mu(t) \leq h(\theta) \tag{5}$$

2. $X \leq_{lr} Y$ ($X \leq_{rh} Y$) if and only if

$$b(\theta) \leq \frac{\int b(t)h(t)d\mu(t)}{\int h(t)d\mu(t)}. \tag{6}$$

Example 4.3 *Let $X \sim Gamma(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$, which is parameterized so that the density function is*

$$f(x;\beta) = \Gamma(\alpha)^{-1}\beta^{-\alpha}x^{\alpha-1}\exp(-x/\beta), \quad x > 0,$$

*or, in the form of (**??**),*

$$f(x;\beta) = \Gamma(\alpha)^{-1}x^{\alpha-1}\exp(xb(\beta))h(\beta), \quad x > 0,$$

with $b(\beta) = -\beta^{-1}$ and $h(\beta) = \beta^{-\alpha}$. Suppose Y is a mixture of $Gamma(\alpha, t)$ with respect to a distribution $\mu(t)$ on $t \in (0, \infty)$. Then (**??**) *and* (**??**) *give*
1. $X \leq_{st} Y$ ($X \leq_{hr} Y$) if and only if

$$\int t^{-\alpha}d\mu(t) \leq \beta^{-\alpha},$$

2. $X \leq_{lr} Y$ ($X \leq_{rh} Y$) if and only if

$$\beta\int t^{-\alpha-1}d\mu(t) \leq \int t^{-\alpha}d\mu(t) < \infty.$$

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