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On the Nilpotency of a Pair of Groups*

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Abstract. This paper is devoted to suggest that the extensive theory of nilpotency, upper and lower central series, residually nilpotency and Hopficity of groups could be extended in an interesting and useful way to a theory for pairs of groups. Also this yields some information on nilpotent groups.

Keywords: Pair of groups; Nilpotent group; Central series; Hopfian; Residually nilpotent.

1. Introduction

In this paper, we intend to study more one of the most famous, fundamental and oldest branches in group theory, namely the theory of nilpotency of groups, but not in the usual aspect to the notion of nilpotency which generally has been studied by authors. In other words, our main goal in this paper is to establish a new notion of nilpotency which is lain between the usual notion of nilpotency for a group and its subgroup. More exactly, for a group G and a normal subgroup N of G, we introduce the concept of nilpotency for the pair (G, N) (the pair (G, N) is known as a *pair of groups*). This new concept will be defined in such a way that the nilpotency of G implies that of (G, N) and the nilpotency of (G, N) forces N to be nilpotent.

Pairs of groups have been studied by many authors during these recent two decades and has some applications in group theory. For instance Ellis introduced

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the Schur multiplier [2] and also capability [1] for a pair of groups and attained some of their properties. In particular, he proved some group theoretic results of pairs of groups (see [2, Section 7]). Also Salemkar, Moghaddam and Chitti [6] obtained more properties of the Schur multiplier of a pair such as finding a covering pair for a pair of groups under some conditions. The notion of isoclinism for pairs was introduced by Salemkar, Saeedi and Karimi [7] as well. Recently Pourmirzaei, Hokmabadi and the last author [4] have attained a criterion for characterizing the capability of a pair and also have given a complete classification of finitely generated abelian capable pairs. Now, this paper verifies a new notion of nilpotency for pairs of groups which has some interesting results in the usual theory of nilpotency.

This article contains four sections. In the next section, we introduce the notion of a central series and then the nilpotency for a pair of groups. The concepts of lower, upper and derived series for a pair will be also introduced in this section. Actually in Section 2, we shall be interested mainly in deriving various elementary properties required for subsequent investigations. In the third section, a generalization of Robinson Theorem for a pair of groups plays an important role to find landmark theorems connecting to the idea of nilpotency in category of groups and the same concept in the larger category of pairs. Furthermore, nontriviality of the center of a nilpotent group G is obtained by replacing the nilpotency of G with the nilpotency for the pair (G, N), which shall be a weaker condition. In the sequel of Section 3, using the new notion of nilpotency, we can find both a criterion for nilpotency of G and a sufficient condition to deduce the nilpotency of G from G', where G' is the derived subgroup of G. In the last section, the Mal'cev Theorem will be extended to pairs of groups. In addition, we define a residually nilpotent pair of groups and find a necessary and sufficient condition for residually nilpotency of a pair. Using these results, we are able to show that the residually nilpotency of the pair (G, G') is equivalent to that of G. Finally a Hopfian pair of groups is defined and a special type of Hopfian pairs is introduced.

Now, in the rest of this section we try to introduce some notations which shall be used throughout the paper. The class of all pairs forms a category with the following morphisms. A *morphism* from a pair (G_1, N_1) to a pair (G_2, N_2) is a group homomorphism from G_1 to G_2 which sends N_1 into N_2. The pairs (G_1, N_1) and (G_2, N_2) are called *equivalent* and this notion is denoted by $(G_1, N_1) \cong (G_2, N_2)$, if there exists an equivalence from (G_1, N_1) to (G_2, N_2). For a family of pairs $\{(G_i, N_i)\}_{i\in I}$, the pair $(\Pi_{i\in I} G_i, \Pi_{i\in I} N_i)$ of products of the given groups, is a product in the category of pairs. This pair is called *direct product* of the family $\{(G_i, N_i)\}_{i\in I}$. If in addition G_i's are subgroups of a group G, then we refer to $(\Pi_{i\in I} G_i, \Pi_{i\in I} N_i)$ as the *internal direct product* of the family $\{(G_i, N_i)\}_{i\in I}$. If we identify the pair (G, G) by the group G, for any group G, then the category of groups will be a subcategory of the pairs. Throughout this paper, we assume that (G, N) is a pair of groups in which N is nontrivial. For a pair (G, N), the pair (H, M) is called a *subpair* of (G, N) whenever $H \leq G$ and

$M \leq N$. The *center of a pair* (G, N) is also defined to be the subgroup

$$\{n \in N | n^g = n, \forall g \in G\}$$

of N in which n^g denotes the conjugate of n by g, and it is denoted by $Z(G, N)$.

2. Basic Definitions and Elementary Results

In this section, invoking the definition of a central series for a pair, the concept of nilpotency for a pair of groups is introduced. The usual notions of lower and upper central series are also generalized to a pair of groups. Using these new concepts, some elementary results shall be given.

Definition 2.1. *Let (G, N) be a pair of groups. A normal series $1 = N_0 \leq N_1 \leq \ldots \leq N_t = N$ is called a central series of the pair (G, N) if each N_i is a normal subgroup of G and N_{i+1}/N_i is contained in the center of $(G/N_i, N/N_i)$, for all i. A pair of groups (G, N) is called nilpotent if it has a central series. The length of a shortest central series of the pair (G, N) is called nilpotent class of (G, N).*

The first important and fundamental note on the definition of nilpotency of a pair (G, N) is the observation that the introduced notion of nilpotency lies between the usual concept of nilpotency of G and N. In other words, if G is nilpotent, then (G, N) is nilpotent and if (G, N) is nilpotent, then so is N. This note helps us to improve some known results in nilpotent groups, with considering weaker conditions (see Section 3). Finally one may easily see that in a nilpotent pair of groups (G, N) of class 0, N has to be trivial, while for a nilpotent pair of groups (G, N) of class at most 1, N is central in G.

The following statement provides some general information on the class of nilpotent pairs.

Theorem 2.2. *The set of nilpotent pairs of groups forms a class which is closed under the formation of subpairs, and finite direct products.*

Definition 2.3. *Let $(G, N)^{(1)} = [N, G]$, and assume that for $i \geq 1$, $(G, N)^{(i)}$ is defined inductively. Then $(G, N)^{(i+1)}$ is defined to be the commutator subgroup $[(G, N)^{(i)}, (G, N)^{(i)}]$, for all $i \geq 1$. Assuming $(G, N)^{(0)} = N$, we have the normal series*

$$N \geq (G, N)^{(1)} \geq (G, N)^{(2)} \geq \ldots$$

which we call it the derived series of (G, N).

The lower central series for a pair of groups is also defined inductively as follows.

Definition 2.4. *Put $\gamma_1(G,N) = (G,N)^{(0)} = N$, and let $\gamma_i(G,N)$ be defined inductively for $i \geq 1$. Then $\gamma_{i+1}(G,N)$ is defined as the subgroup $[\gamma_i(G,N),G]$. The obtained series*

$$N = \gamma_1(G,N) \geq \gamma_2(G,N) \geq \dots$$

is called the lower central series of (G,N).

Note that $\gamma_n(G,N)/\gamma_{n+1}(G,N)$ lies in the center of $(G/\gamma_{n+1}(G,N), N/\gamma_{n+1}(G,N))$ and that each $\gamma_n(G,N)$ is a normal subgroup of G.

Also for a pair (G,N), there is an ascending series which is dual to the lower central series in the sense that the center is dual to the first term of the derived series. This is the upper central series

$$1 = Z_0(G,N) \leq Z_1(G,N) \leq Z_2(G,N) \leq \dots$$

which is defined by

$$\frac{Z_{n+1}(G,N)}{Z_n(G,N)} = Z(\frac{G}{Z_n(G,N)}, \frac{N}{Z_n(G,N)}).$$

Note that this series need not reach N. One should also note that neither $\gamma_n(G,N)$ is necessarily a fully-invariant nor $Z_n(G,N)$ is necessarily a characteristic subgroup of N. A counterexample to both of these claims is (G,N), where G is the dihedral group of order 8, and N is a Klein 4-group. Then $\gamma_2(G,N) = Z_1(G,N) = Z(G)$ which is not characteristic in N.

Some properties of these central series are displayed in the next result.

Theorem 2.5. *Let $1 = N_0 \leq N_1 \leq \dots \leq N_t = N$ be a central series of a nilpotent pair of groups (G,N). Then*

(1) *$\gamma_i(G,N) \leq N_{t-i+1}$, so that $\gamma_{t+1}(G,N) = 1$,*
(2) *$N_i \leq Z_i(G,N)$, so that $Z_t(G,N) = N$,*
(3) *The nilpotent class of (G,N) = the length of the upper central series = the length of the lower central series.*

Proof. (1) The proof is based on induction. If $i = 1$, this is clear. Since the above series is central, we have $[N_{t-i+1}, G] \leq N_{t-i}$. Thus $\gamma_{i+1}(G,N) = [\gamma_i(G,N),G] \leq [N_{t-i+1}, G] \leq N_{t-i}$ as required.

(2) The proof is easy by induction.

(3) By (i) and (ii) the upper and lower central series are shortest central series of G. ∎

One may see that a pair of groups (G,N) is nilpotent if and only if the lower central series reaches the identity after a finite number of steps, or equivalently the upper central series after a finite number of steps reaches the normal subgroup N.

The following lemma is needed for the next result and its proof is straightforward.

Lemma 2.6. *Let (G, N) be any pair of groups and let i and j be positive integers. Then*

(1) $[\gamma_i(G, N), \gamma_j(G, N)] \leq \gamma_{i+j}(G, N)$,
(2) $\gamma_i(\gamma_j(G, N)) \leq \gamma_{ij}(G, N)$,
(3) $[\gamma_i(G, N), Z_j(G, N)] \leq Z_{j-i}(G, N)$ *if* $j \geq i$,
(4) $Z_i(G/Z_j(G, N), N/Z_j(G, N)) = Z_{i+j}(G, N)/Z_j(G, N)$.

Theorem 2.7. *If (G, N) is any pair of groups, then $(G, N)^{(i)} \leq \gamma_{2^i}(G, N)$. If (G, N) is nilpotent of class $c \geq 1$, its derived length is at most* $[\log_2 c] + 1$

Proof. The first part follows on applying induction and Lemma 2.6 (i). Now let (G, N) be nilpotent of class $c \geq 1$. If d is the derived length of (G, N) then $(G, N)^{(i)} \leq \gamma_{2^i}(G, N) \leq \gamma_{c+1}(G, N) = 1$ which implies $2^i \geq c + 1$. The smallest such i is $[\log_2 c] + 1$. Therefore $d \geq [\log_2 c] + 1$. ∎

Remark 2.8. Note that if (G, N) is a nilpotent pair of groups of class greater than 1 and $n \in N$, then the nilpotency class of $\langle n, [N, G]\rangle$ is smaller than that of (G, N). We can now deduce that (G, N) can be expressed as an internal product of some nilpotent subpairs of smaller classes.

3. Further Properties of Nilpotent Pairs of Groups

We shall now embark on investigation some more properties of nilpotent pairs of groups. Let us begin with a well known property of a nilpotent group. A nontrivial normal subgroup of a nilpotent group G intersects nontrivially the center of G. Now, this property can be proved with the weaker condition than the nilpotency for an arbitrary pair adopted of G.

Theorem 3.1. *If (G, N) is a nilpotent pair of groups and M is a nontrivial normal subgroup of G such that $M \cap N$ is nontrivial, then $M \cap Z(G, N) \neq 1$.*

Proof. Since (G, N) is nilpotent there exists a positive integer c such that $N = Z_c(G, N)$. Let i be a least integer such that $M \cap Z_i(G, N) \neq 1$. Now, $[M \cap Z_i(G, N), G] \leq M \cap Z_{i-1}(G, N) = 1$ and $M \cap Z_i(G, N) \leq M \cap Z_1(G, N)$. Hence $M \cap Z_1(G, N) = M \cap Z_i(G, N) \neq 1$. ∎

The interesting point in Theorem 3.1 and in the two following statements is that, with a weaker condition we reach to a sharper conclusion. This happens since the center of a pair lies in the center of the group itself.

Corollary 3.2. *If (G, N) is a nilpotent pair of groups with $N \neq 1$, then $Z(G, N) \neq 1$.*

Theorem 3.3. *Let (G, N) be a nilpotent pair of groups and N has an element of prime order p, then so does $Z(G, N)$.*

Proof. Since N is nilpotent, the subgroup

$$H = \{x \in N \mid x^{p^\alpha} = 1, \quad \text{for some} \quad \alpha\}$$

is fully-invariant in N (see [5]). Thus $H \trianglelefteq G$ and $H \cap Z(G, N) \neq 1$. So the result holds. ■

Now we state several theorems in which the nilpotency of a pair (G, N) provides some properties for the group G and its subgroups. First recall that a subnormal subgroup of G is a subgroup which has a subnormal series begin by it and reaches G.

Theorem 3.4. *If (G, N) is a nilpotent pair of groups, then every maximal subgroup of G which does not contain N is normal.*

Proof. Since $Z_c(G, N) = N$, for some c, thus for every maximal subgroup M of G, the following series is a subnormal series for M.

$$M \leq MZ_1(G, N) \leq MZ_2(G, N) \leq \ldots \leq MZ_c(G, N) = MN = G$$

Since $Z_c(G, N) = N$, for some c, thus for every maximal subgroup M of G, the following series is a subnormal series for M.

$$M \trianglelefteq MZ_1(G, N) \trianglelefteq MZ_2(G, N) \trianglelefteq \ldots \trianglelefteq MZ_c(G, N) = MN = G$$

If $c = 1$, obviously $M \trianglelefteq G$. Otherwise by maximality of M there exists a least positive integer j where $MZ_j(G, N) = G$. Then $MZ_{j-1}(G, N) = M$ which yields that $M \trianglelefteq G$. ■

Robinson showed how the first lower central factor $G_{ab} = G/G'$ exerts a very strong influence on subsequent lower central factors of a group G (for details see [5, Theorem 5.2.5]). Now, we state another main theorem of this section which is a wide generalization of Robinson Theorem for a pair of groups (G, N).

Theorem 3.5. [Generalized Robinson Theorem] *Let (G, N) be a pair of groups and let $F_i = \gamma_i(G, N)/\gamma_{i+1}(G, N)$. Then the map*

$$F_i \otimes \frac{G}{[N, G]} \to F_{i+1}$$
$$n\gamma_{i+1}(G, N) \otimes g[N, G] \mapsto [n, g]\gamma_{i+2}(G, N)$$

is a well-defined epimorphism.

Note that in the above theorem we have faced to nonabelian tensor product with trivial actions which is isomorphic to their abelian tensor product.

In the next two theorems and corollary, using Theorem 3.5, we intend to give some sufficient conditions under which the second term of a pair of groups can be finite or finite exponent.

Theorem 3.6. *If (G, N) is a nilpotent pair of groups such that $G/[N, G]$ is finite, then N is finite.*

Proof. Let $F_i = \gamma_i(G, N)/\gamma_{i+1}(G, N)$ be finite. Then by Theorem 3.5, F_{i+1} is also finite. Whence every lower central factor is finite. Since (G, N) is nilpotent, then $\gamma_{c+1}(G, N) = 1$, for some c. As finiteness is closed under forming extensions, thus N is finite. ∎

Theorem 3.7. *Suppose that (G, N) is an arbitrary pair of groups such that $G/[N, G]$ has exponent m. Then for all positive integers n, $\gamma_n(G, N)/\gamma_{n+1}(G, N)$ has exponent dividing m.*

Proof. The result follows from Theorem 3.5 and from the fact that the exponent of a tensor product divides the exponents of each of the factors. ∎

Corollary 3.8. *Suppose that (G, N) is a nilpotent pair of groups of class c such that $G/[N, G]$ has exponent m. Then N has finite exponent dividing m^c.*

Proof. This follows from the preceding theorem. ∎

It is known that an extension of a nilpotent group by another nilpotent group may not be nilpotent in general. P.Hall [3] obtained a criterion under which such an extension can be nilpotent. In what follows we state a similar theorem for pairs of groups. Note that the notion of a trivial *G-module*, that is, each element of G acts like the identity automorphism, will be used in the proof. Also by a *polytrivial G-module* we mean a G-module having a series with G-trivial factors polytrivial. Note also that if A and B are polytrivial G-modules, then $A \otimes B$ is also a polytrivial G-module (see [5, Theorem 5.2.11]). Furthermore, one can easily see that a homomorphic image of a polytrivial G-module is again polytrivial.

Theorem 3.9. *Let M be a normal subgroup of G contained in N. If (N, M) and $(G/[M, N], N/[M, N])$ are nilpotent pairs of groups, then the pair of groups (G, N) is nilpotent.*

Proof. Since $(G/[M, N], N/[M, N])$ is nilpotent, then it has a central series as follows:

$$1 = \frac{H_0}{[M,N]} \leq \frac{H_1}{[M,N]} \leq \ldots \leq \frac{H_t}{[M,N]} = \frac{N}{[M,N]}. \quad (1)$$

Now we construct the following series for $(N/[M,N])_{ab} = N/[M,N]N'$:

$$1 = \frac{H_0N'}{[M,N]N'} \leq \frac{H_1N'}{[M,N]N'} \leq \ldots \leq \frac{H_tN'}{[M,N]N'} = \frac{N}{[M,N]N'}.$$

It is routine to check that by the above central series $N/[M,N]N'$ is a polytrivial G-module. Put $F_i = \gamma_i(N,M)/\gamma_{i+1}(N,M)$. Now by induction we prove that every lower central factor of (N,M) is a polytrivial G-module. Since M is nilpotent, M/M' is polytrivial and since a homomorphic image of a polytrivial G-module is again polytrivial, thus $F_1 = M/[M,N]$ is polytrivial. Suppose that F_i is a polytrivial G-module, then $F_i \otimes N/[M,N]N'$ is polytrivial (see [5, 5.2.11]). As we know $F_i \otimes N/[M,N]N' \cong F_i \otimes N/[M,N]$ and the property of being polytrivial G-module is closed under the image of tensor product, thus it will follow by Theorem 3.5 that F_{i+1} is polytrivial. By nilpotency of (N,M), there exists a positive integer c such that $\gamma_{c+1}(N,M) = 1$. Now, combining the lower central series of (N,M) and (1) we obtain

$$1 = \gamma_{c+1}(N,M) \leq \ldots \leq \gamma_2(N,M) = [M,N] = H_0 \leq \ldots \leq H_t = N.$$

By the fact that F_i is a polytrivial G-module, there is a series

$$1 = K_{i_1} \leq \ldots \leq K_{i_r} = F_i$$

such that $K_{i_{j+1}}/K_{i_j}$ is a trivial G-module. By considering the preimage of each term of the latest series under the canonical homomorphism $\gamma_i(N,M) \to F_i$, we obtain a central series of (G,N) which provides the nilpotency of (G,N), as required. ■

As it is shown, if G is nilpotent, then so is (G,N) and the nilpotency of (G,N) implies nilpotency of N. But the interesting questions that usually arise in such situations are knowing the conditions under which the inverse of the mentioned statements might be true. Theorem 3.9 provides some simple but useful conclusions related to the reverse statements.

Corollary 3.10. *If (G,N) and $G/[N,G]$ are nilpotent, then so is G. In particular, G is nilpotent if and only if (G,G') is nilpotent.*

Corollary 3.11. *Let N and $(G/N', N/N')$ be nilpotent. Then so is (G,N).*

Corollary 3.12. *Let G' be nilpotent. Then G is nilpotent if $(G/G'', G'/G'')$ is nilpotent.*

In the next theorem, invoking the notion of nilpotency for a pair, a sufficient condition will be provided for a group to be nilpotent. Although the theorem may be concluded from Corollary 3.10, we can also prove it by constructing a central series for G using the nilpotency properties for (G, N) and G/N.

Theorem 3.13. *Let (G, N) be a pair of groups. If both (G, N) and G/N are nilpotent, then so is G.*

4. Nilpotent Pair of Finitely Generated Groups

It is a well known fact that finitely generated abelian groups satisfy the maximal condition. This result was generalized to nilpotent groups by Baer (for example see [7, Theorem 5.2.17]). In this section we prove a generalization of Baer Theorem to a pair of groups. A generalization of Mal'cev Theorem for a pair of groups shall also be stated. In the sequel, introducing a residually nilpotent and a Hopfian pair of groups, we show that a residually nilpotent pair of finitely generated groups is Hopfian.

Theorem 4.1. [Generalized Baer Theorem] *If (G, N) is a nilpotent pair of groups such that $(G/[N, G], N/[N, G])$ is a pair of finitely generated groups, then N satisfies the maximal condition.*

Proof. Assume $F_i = \gamma_i(G, N)/\gamma_{i+1}(G, N)$. Since the tensor product of two finitely generated groups is finitely generated, then by the Generalized Robinson Theorem each lower central factor is finitely generated. It follows that such factors satisfy the maximal condition. Since maximal condition is closed with respect to extension, the theorem holds. ■

Theorem 4.2. *A nilpotent pair of finitely generated groups (G, N) has a central series whose factors are infinite cyclic or cyclic of a prime order.*

Proof. Use Theorem 4.1 and refine the lower central series. ■

Theorem 4.3. [Generalized Mal'cev Theorem] *Let (G, N) be a pair of groups such that $Z(G, N)$ be torsion free. Then each upper central factor is torsion free.*

Proof. Let $Z(G, N) = Z_1(G, N)$ be torsion free. By Lemma 2.6 (iv) it is enough to show that $Z_2(G, N)/Z_1(G, N)$ is torsion free. Suppose that $x \in Z_2(G, N)$ and $x^m \in Z_1(G, N)$ where $m > 0$. So we have $[x^m, g] = 1$. Since $Z_1(G, N)$ is torsion free, then $[x, g] = 1$, for all $g \in G$ and $x \in Z_1(G, N)$. ■

The following consequence is deduced from Theorems 4.2 and 4.3.

Corollary 4.4. *A nilpotent pair of finitely generated groups (G, N) with torsion free center has a central series whose factors are infinite cyclic.*

Theorem 4.5. *Let (G, N) be a nilpotent pair of groups.*

(1) *If $exp(Z(G, N)) = e$, then $exp(N)|e^c$, where c is the nilpotency class of (G, N).*

(2) *If (G, N) is a pair of infinite finitely generated groups, then its center has an element of infinite order.*

Proof. (1) Assume that $N \neq Z(G, N)$. Let $x \in Z_2(G, N)$ and $g \in G$. Then $[x, g] \in Z(G, N)$ and $1 = [x, g]^e = [x^e, g]$, whence $x^e \in Z_1(G, N)$. Thus $Z_2(G, N)/Z_1(G, N)$ has finite exponent dividing e. By induction $N/Z_1(G, N)$ has finite exponent dividing e^{c-1} and N has exponent dividing e^c.
(2) If $Z(G, N)$ is a torsion group, then it is finite, since N satisfies maximal condition. So $Z(G, N)$ has finite exponent e. By (1), $exp(N)|e^c$. Theorem 4.2 implies that N is finite which is a contradiction. ■

Definition 4.6. *A pair of groups (G, N) is* residually nilpotent *if for every $1 \neq x \in N$, there exists a normal subgroup M_x of G contained in N such that $x \notin M_x$ and $(G/M_x, N/M_x)$ is nilpotent.*

Note that the residually nilpotency property of G carries over to (G, N) and that of (G, N) forces N to be residually nilpotent.

Theorem 4.7. *A pair of groups (G, N) is residually nilpotent if and only if $\cap_{n=1}^{\infty}\gamma_n(G, N) = 1$.*

Proof. Let $\cap_{n=1}^{\infty}\gamma_n(G, N) \neq 1$. Then there exists $1 \neq x \in \cap_{n=1}^{\infty}\gamma_n(G, N)$ and $M_x \trianglelefteq G$ contained in N such that $x \notin M_x$ and $(G/M_x, N/M_x)$ is nilpotent. So $\gamma_i(N/M_x, G/M_x) = 1$ for some i, that is a contradiction. The "only if" part is obvious. ■

We have already known that a free group is residually nilpotent. This fact together with Theorem 4.7 imply that a pair of free groups (F, E) is residually nilpotent. The following corollary gives us a criterion for a group to be residually nilpotent.

Corollary 4.8. *A pair of groups (G, G') is residually nilpotent if and only if G is.*

We close this section by the definition and determining a Hopfian pair.

Definition 4.9. *A pair of groups (G, N) is Hopfian if (G, N) is not isomorphic to $(G/K, N/K)$, for every $K \neq 1$.*

It is obvious that for $N = G$, Definition 4.9 reduces to the usual definition of Hopfian for groups.

Theorem 4.10. *A residually nilpotent pair of finitely generated groups (G, N) is Hopfian.*

Proof. Assume that $(G, N) \cong (G/K, N/K)$, for some normal subgroup K of G. Since $(G, N) \cong (G/K, N/K)$ and the terms of the lower central series are verbal subgroups, then

$$\frac{\gamma_n(G,N)}{\gamma_{n+1}(G,N)} \cong \frac{\gamma_n(G/K,N/K)}{\gamma_{n+1}(G/K,N/K)}$$

and $\gamma_n(G/K, N/K) = \gamma_n(G,N)K/K$, $\gamma_{n+1}(G/K,N/K) = \gamma_{n+1}(G,N)K/K$.

If $K \neq 1$, then there is a smallest integer n such that $K \subset \gamma_n(G,N)$, but K is not a subgroup of $\gamma_{n+1}(G,N)$. Therefore

$$\begin{aligned}\frac{\gamma_n(G,N)}{\gamma_{n+1}(G,N)} &\cong \frac{\gamma_n(G/K,N/K)}{\gamma_{n+1}(G/K,N/K)} = \frac{K\gamma_n(G,N)/K}{K\gamma_{n+1}(G,N)/K}\\ &\cong \frac{K\gamma_n(G,N)}{K\gamma_{n+1}(G,N)} = \frac{\gamma_n(G,N)}{K\gamma_{n+1}(G,N)}\\ &\cong \frac{\gamma_n(G,N)/\gamma_{n+1}(G,N)}{K\gamma_{n+1}(G,N)/\gamma_{n+1}(G,N)}.\end{aligned}$$

Since $\gamma_n(G,N)/\gamma_{n+1}(G,N)$ is a finitely generated abelian group, then it can not be isomorphic to a proper quotient group of itself. Therefore $K = 1$ and (G,N) is Hopfian. ■

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References

[1] G. Ellis, Capability, homology, and central series of a pair of groups, *J. Algebra* **179** (1996) 31–46.

[2] G. Ellis, The schur multiplier of a pair of groups, *Applied Categorical Structures* **6** (1998) 355–371.

[3] P. Hall, Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.* **2** (1958) 787–801.

[4] A. Pourmirzaei, A. Hokmabadi, S. Kayvanfar, Capability of a pair of groups, *Bull. Malays. Math. Sci. Soc.* **35** (1) (2011) 205–213.

[5] D.J.S. Robinson, *A Course in the Theory of Groups*, Springer Verlag, New York, 1982.

[6] A.R. Salemkar, M.R.R. Moghaddam, Some properties on the schur multiplier of a pair of groups, *J. Algebra* **312** (1) (2007) 1–8.

[7] A.R. Salemkar, F. Saeedi, T. Karimi, The structure of isoclinism classes of pair of groups, *Southeast Asian Bull. Math.* **31** (2007) 1173–1181.