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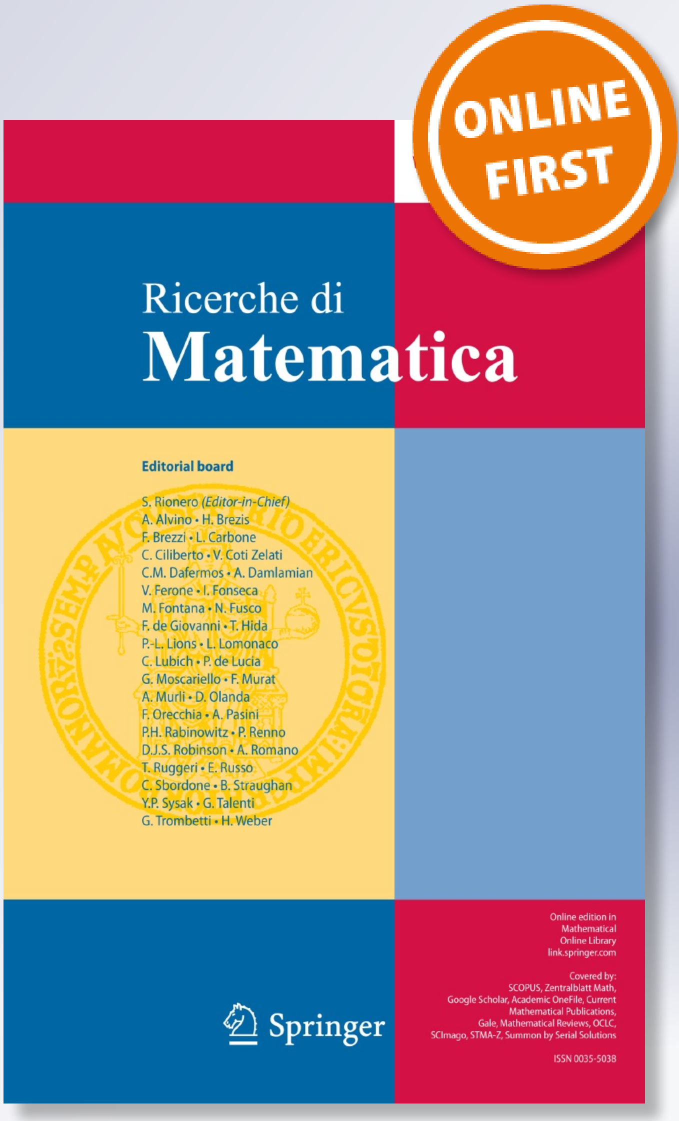
Ricerche di Matematica

ISSN 0035-5038

Ricerche mat.
DOI 10.1007/s11587-013-0172-6



Ricerche mat.
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A converse of Baer's theorem

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Received: 2 February 2013 / Revised: 16 November 2013

Abstract Schur's classical theorem states that for a group G, if $G/Z(G)$ is finite, then G' is finite. Baer extended this theorem for the factor group $G/Z_n(G)$, in which $Z_n(G)$ is the n-th term of the upper central series of G. Hekster proved a converse of Baer's theorem as follows: If G is a finitely generated group such that $\gamma_{n+1}(G)$ is finite, then $G/Z_n(G)$ is finite where $\gamma_{n+1}(G)$ denotes the $(n+1)$st term of the lower central series of G. In this paper, we generalize this result by obtaining the same conclusion under the weaker hypothesis that $G/Z_n(G)$ is finitely generated. Furthermore, we show that the index of the subgroup $Z_n(G)$ is bounded by a precisely determined function of the order of $\gamma_{n+1}(G)$. Moreover, we prove that the mentioned theorem of Hekster is also valid under a weaker condition that $Z_{2n}(G)/Z_n(G)$ is finitely generated. Although in this case the bound for the order of $\gamma_{n+1}(G)$ is not achieved.

Keywords Baer's theorem · n-Isoclinism of groups · Nilpotent groups

Mathematics Subject Classification (2000) 20B05 · 20E45

Communicated by Francesco de Giovanni.

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Published online: 04 December 2013

1 Introduction

For a group G there is a relation between the central factor group $G/Z(G)$ and the subgroup G', which is stated by Schur. He proved that if $G/Z(G)$ is finite, then G' is also finite. This statement was widely studied under different points of view. One of the propounded important results in this subject area is the extension of Schur's theorem stated by Baer [1] as follows:
For a group G, if $G/Z_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite.

A question that naturally arises from Baer's theorem is whether the converse of theorem is valid. An extra special p-group of infinite order is a counterexample for the case $n = 1$. In other cases, Hall [3] constructed a nilpotent p-group as a counterexample. This example will be discussed in the next section.

Some of the authors tried to find conditions under which the converse of Schur's theorem would be true. For example, Isaacs in [5] proved that if G is capable and $|G'| = n$, then $G/Z(G)$ is finite and its order is bounded by a function of $|G'|$. Also Hall in [3] confirmed that if $|G'| < \infty$, then $G/Z_2(G)$ is finite and $G/Z_2(G)$ is bounded above in terms of $|G'|$. Also, for a group with trivial Frattini subgroup, Halasi and Podoski in [2] proved the converse of Schur's theorem. On the other hand, concerning the converse of Baer's theorem, Hekster [4] showed that for a finitely generated group G, if $\gamma_{n+1}(G)$ is finite, then $G/Z_n(G)$ is finite.

The main theorem of the paper is:

1.1 Main theorem

If G is a group, $\gamma_{n+1}(G)$ is finite and $G/Z_n(G)$ is finitely generated, then

$$\left|\frac{G}{Z_n(G)}\right| \leq |\gamma_{n+1}(G)|^{d(G/Z_n(G))^n},$$

where $d(X)$ is the minimal number of generators of the group X.

The main theorem illustrates that with a weaker condition, we can deduce not only the Hekster's conclusion [4, Theorem 2.10] but also an upper bound for the order of $G/Z_n(G)$ in terms of $|\gamma_{n+1}(G)|$. Furthermore, if we ignore the upper bound for the order of $G/Z_n(G)$, we can prove the Hekster's result by considering a weaker hypothesis of being finitely generated for $Z_{2n}(G)/Z_n(G)$. The main theorem also generalizes the work of Niroomand [6]. But one should notice that the applied technique in the paper is completely different from the ones of Hekster and Niroomand.

2 Main theorem

As mentioned in the introduction, for every $n > 1$, there is a counterexample for the converse of Baer's theorem introduced by Hall [3]. We refer to his example as follows:

Example 1 Let p be a prime number. If $p = 2$, consider F to be the central product of a countable infinite copies of the quaternion group. For $p > 2$, let F be the central product of a countable infinite copies of the nonabelian group of order p^3 and exponent p. Put $G = F \wr \mathbf{Z}_p$, where "$\wr$" denotes the standard wreath product. Hall claimed that G is a nilpotent group of class $2p$. Also G has the following properties:

(i) $Z_i(G) = \gamma_{2p-i+1}(G)$, for every $0 \leq i \leq 2p$.
(ii) $Z_i(G)/Z_{i-1}(G)$ is of order p, if $1 \leq i \leq p$ and of infinite order for $p+1 \leq i \leq 2p$.

Let $n > 1$. Consider a prime p such that $p \leq n < 2p$. Then by the above properties $\gamma_{n+1}(G)$ is finite whereas $Z_{n+1}(G)/Z_n(G)$ is infinite.

To prove our main theorem, we need the following well known concept of n-isoclinism and some related results from [4].

Definition 1 Let $n \geq 0$ and let G and H be two groups. An n-isoclinism from G to H is a pair of homomorphisms (α, β) with $\alpha : G/Z_n(G) \to H/Z_n(H)$ and $\beta : \gamma_{n+1}(G) \to \gamma_{n+1}(H)$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\frac{G}{Z_n(G)} \times \ \ldots \ \times \frac{G}{Z_n(G)} & \xrightarrow{\gamma(n,G)} & \gamma_{n+1}(G) \\
\alpha^n \downarrow & & \beta \downarrow \\
\frac{H}{Z_n(H)} \times \ \ldots \ \times \frac{H}{Z_n(H)} & \xrightarrow{\gamma(n,H)} & \gamma_{n+1}(H)
\end{array}$$

Whenever the groups G and H are n-isoclinic, we write $G \sim_n H$.

Lemma 1 *Let G be a group and $H \leq G$. If $G = HZ_n(G)$, then $G \sim_n H$.*

In the following we prove our main theorem which is a converse of Baer's theorem by considering a condition. The bound presented in the theorem for the index of $Z_n(G)$ in G is sharp for the case $n = 1$ (consider a finite extra-special p-group of order p^5). But unfortunately we do not know whether the obtained bound is the best for $n > 1$.

2.1 Proof of the main theorem

For convenience set $t = d(G/Z_n(G))$. The proof is done by induction on n.
Let $n = 1$. If $G/Z(G) =< x_1Z(G), x_2Z(G), \ldots, x_tZ(G) >$, then the group $H =< x_1, x_2, \ldots, x_t >$ is a subgroup of G. By the previous lemma $G \sim H$. Therefore $G/Z(G) \cong H/Z(H)$ and $\gamma_2(G) \cong \gamma_2(H)$. Hence we can replace G with H. Note that H is finitely generated and $t = d(H)$. Therefore,

$$[H : Z(H)] = \left[H : \cap_{i=1}^{t} C_H(x_i)\right] \leq \prod_{i=1}^{t} [H : C_H(x_i)] = \prod_{i=1}^{t} |x_i^H|,$$

where by $C_H(x_i)$ we mean the centralizer of x_i in H. Also x_i^H denotes the conjugacy class of x_i in H. Since $x_i^h = x_i[x_i, h]$ for every $h \in H$, then $|x_i^H| \leq |x_iH'| = |H'|$. Thus

$$[H : Z(H)] \leq \prod_{i=1}^{t} |H'| = |H'|^t.$$

Now, assume that $|G/Z_{n-1}(G)| \leq |\gamma_n(G)|^{t^{n-1}}$, for every group G satisfying the assumptions of theorem. Let $G/Z_n(G) =< x_1 Z_n(G), x_2 Z_n(G), \ldots, x_t Z_n(G) >$ and $\gamma_{n+1}(G)$ be finite. Suppose that H denotes the group generated by $x_1, x_2, \ldots, x_t$, which is n-isoclinic to G. Obviously $H/Z(H)$ is finitely generated and from [4, 2.10] one can easily see that the finiteness of $\gamma_{n+1}(H)$ implies that $\gamma_n(H/Z(H))$ is finite. Therefore, $H/Z(H)$ satisfies the induction hypothesis. Hence

$$\left|\frac{H}{Z_n(H)}\right| = \left|\frac{H/Z(H)}{Z_{n-1}(H/Z(H))}\right| \leq \left|\gamma_n\left(\frac{H}{Z(H)}\right)\right|^{t^{n-1}}.$$

But,

$$\gamma_n\left(\frac{H}{Z(H)}\right) \cong \frac{\gamma_n(H)}{\gamma_n(H)\cap Z(H)} \cong \frac{\gamma_n(H)}{\cap_{i=1}^t C_{\gamma_n(H)}(x_i)},$$

which implies

$$\begin{aligned}\left|\gamma_n\left(\frac{H}{Z(H)}\right)\right| &= \left[\gamma_n(H) : \cap_{i=1}^t C_{\gamma_n(H)}(x_i)\right] \leq \prod_{i=1}^t \left[\gamma_n(H) : C_{\gamma_n(H)}(x_i)\right] \\ &= \prod_{i=1}^t |x_i \gamma_{n+1}(H)| = |\gamma_{n+1}(H)|^t.\end{aligned}$$

Hence, we have

$$\left|\frac{G}{Z_n(G)}\right| = \left|\frac{H}{Z_n(H)}\right| \leq \left(|\gamma_{n+1}(H)|^t\right)^{t^{n-1}} = |\gamma_{n+1}(H)|^{t^n} = |\gamma_{n+1}(G)|^{t^n}.$$

It can be seen that for the case $n = 1$, using the concept of isoclinism, we have a simple proof. Now compare the proof of both Niroomasnd's [6] and Sury's [7].

When G is nilpotent, we get a stronger result in the following.

Corollary 1 *Let G be a nilpotent group such that $G/Z_n(G)$ is finitely generated. If $\gamma_{n+1}(G)$ is finite, then*

$$\left|\frac{G}{Z_n(G)}\right| \mid |\gamma_{n+1}(G)|^{d(G/Z_n(G))^n}.$$

Proof By the Main Theorem, $G/Z_n(G)$ is finite. Let π be the set of primes in the decomposition of $|G/Z_n(G)|$. Since $G/Z_n(G)$ is a finite nilpotent π-group, then (by [4, 7.9]) there exists a finite nilpotent π-group H n-isoclinic to G. Therefore, $H = \prod_{i=1}^m P_i$, where P_i is the Sylow p_i-subgroup of H. One can easily see that $H/Z_n(H) = \prod_{i=1}^m (P_i/Z_n(P_i))$ and since $P_i/Z_n(P_i)$ is a finite p_i-group, then $|P_i/Z_n(P_i)| \mid |\gamma_{n+1}(P_i)|^{t^n}$ which yields that

$$\left|\frac{H}{Z_n(H)}\right| \mid |\gamma_{n+1}(H)|^{t^n}.$$

□

The following example shows that the nilpotency of G in Corollary 1 is necessary.

Example 2 Consider the Symmetric group S_3. We know that $Z_n(S_3) = 1$ and $\gamma_{n+1}(S_3) = A_3$, for every $n \in \mathbf{N}$. Therefore $|S_3/Z_n(S_3)|$ does not divide $|\gamma_{n+1}(S_3)|^{2^n}$.

This is a well-known fact that a subgroup of a finitely generated group may not be finitely generated. The following corollary provides a sufficient condition for the n-th term of the upper central series of a finitely generated group to be finitely generated.

Corollary 2 *Let G be a finitely generated group. If for some positive integer n, $\gamma_{n+1}(G)$ is finite, then the subgroup $Z_n(G)$ is finitely generated.*

Finally we show that a converse of Baer's theorem can be deduced with a weaker condition. Theorem 1 actually generalizes the Yadav's result [8].

Theorem 1 *Suppose that for a group G, $Z_{2n}(G)/Z_n(G)$ is finitely generated. If $\gamma_{n+1}(G)$ is finite, then $G/Z_n(G)$ is finite.*

Proof By a theorem of Hall [3], if $\gamma_{n+1}(G)$ is finite, then $G/Z_{2n}(G)$ is finite. Now the isomorphism

$$\frac{G}{Z_{2n}(G)} \cong \frac{G/Z_n(G)}{Z_{2n}(G)/Z_n(G)}$$

implies the assertion. □

References

1. Baer, R.: Endlichkeitskriterien für kommutatorgruppen. Math. Ann. **124**, 161–177 (1952)
2. Halasi, Z., Podoski, K.: Bounds in groups with trivial Frattini subgroup. J. Alg. **319**(3), 893–896 (2008)
3. Hall, P.: Finite-by-nilpotent groups. Proc. Camb. Phil. Soc. **52**, 611–616 (1956)
4. Hekster, N.S.: On the structure of n-isoclinism classes of groups. J. Pure Appl. Algebra **40**(1), 63–85 (1986)
5. Isaacs, I.M.: Derived subgroups and centers of capable groups. Proc. Amer. Math. Soc. **129**(10), 2853–2859 (2001)
6. Niroomand, P.: The converse of Schur's theorem. Arch. Math. (Basel) **94**(5), 401–403 (2010)
7. Sury, B.: A generalization of a converse of Schur's theorem. Arch. Math. **95**, 317–318 (2010)
8. Yadav, M.K.: A note on the converse of Schur's theorem. arxive:1011.2083v2 [math. GR] (19 January 2011)