The Extended Abstracts of
The 44$^{\text{th}}$ Annual Iranian Mathematics Conference
27-30 August 2013, Ferdowsi University of Mashhad, Iran.

A REVIEW AND A COMPARISION OF RADON TRANSFORM WITH FOURIER TRANSFORM

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Abstract. The aim of this talk is to review some important properties of Radon transform. More precisely, definition and properties of Radon transform are illustrated and these properties are compared with those of Fourier transform. In addition, this transform are discussed on locally compact group G and semi direct product of locally compact group.

1. Introduction

When J. Radon wrote his paper [2] in 1917, nobody thought that this paper will be introduced as the base for solving applied problems in various areas such as physics, medical sciences, electricity engineering, image and signal processing. Of course it was undercover for almost half of a century, still Soviet and American authors [Shtein (1972); Vainshtein and Orlov (1972); Vest and Cormack (1973)] have pointed out Radon's work as the foundation to the problem of reconstruction from projections.

2000 *Mathematics Subject Classification.* Primary: 44A12, Secondary: 58C35.
Key words and phrases. Radon transform, inversion formula, Fourier transform.
* Speaker.

2. Line integral and Radon transform

Definition 2.1. Let f be a function on some domain $D \subseteq \mathbb{R}^n$, and let L be a line in the plane, the line integral

$$\check{f} = Rf = \int_L f(x,y)ds$$

is said the Radon transform of f.

Consider Figure. 2.1, to get the line equation. We have $\sin(\frac{\pi}{2} - \phi) = \frac{x}{x_0}$ and $\cos(\frac{\pi}{2} - \phi) = \frac{y}{y_0}$, so $xsin\phi + ycos\phi = p$. Suppose

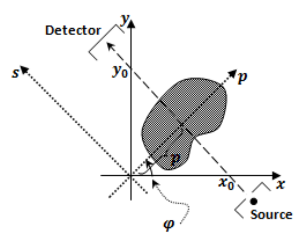

***Figure* 2.1.** *Emission – Photon Model to Describe Line Integral*

now a new coordinate system is introduced with axes rotated by the angle ϕ, then we have

$$\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

thus

$$\check{f}(p,\phi) = \int_{-\infty}^{+\infty} f(pcos\phi - ssin\phi, psin\phi + scos\phi)ds.$$

The first theorem in Radons' 1917 paper asserts that the above integral is well-defined.

Theorem 2.2. [2]. *The integral of f along the stright line L with the equation $xcos\phi + ysin\phi = p$, given by*

$$\check{f}(p,\phi) = \int_{-\infty}^{+\infty} f(pcos\phi - ssin\phi, psin\phi + scos\phi)ds.$$

is in general well-defined.

In the following example the Radon transform over the unit circle is illustrated.

Example 2.3. Let $f(x,y)$ be defined by $f(x,y) = (1-x^2-y^2)^{\lambda-1}$ for $x^2+y^2 \leqslant 1$ and f is zero outside the unit circle, then

$$\check{f}(p,\phi) = \frac{\sqrt{\pi}\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}(1-p^2)^{\lambda-\frac{1}{2}}; -1 \leqslant p \leqslant 1.$$

Furthermore, if we introduce unit vector $\xi := (cos\phi, sin\phi)$ and $X := (x,y)$ then the line equation is $p = \xi \cdot X$ and

$$\check{f}(p,\phi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(X)\delta(p-\xi\cdot X)dxdy.$$

where δ is Dirac delta function.

3. Extension of Radon transform to higher dimension and its properties

Definition 3.1. Let f be a function on some domain $D \subseteq \mathbb{R}^n$ and $X = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $dX = dx_1dx_2\cdots dx_n$, and more let ξ be the normal vector for hyperplan $p = \xi \cdot X$ in $\mathbb{R}^n$. The Radon transform of f is defined as

$$\check{f} = Rf = \int_{\mathbb{R}^n} f(X)\delta(p-\xi\cdot X)dX$$

is said the Radon transform of f.

This definition used by Gel'fand, Greav and Vilenkin in 1966. Note that $\check{f}$ is defined on $\mathbb{R} \times S^{n-1}$, where S^{n-1} is the unit sphere. Because (p,ϕ) and $(-p,-\phi)$ represent the same hyperplane in $\mathbb{R}^n(p = \xi \cdot X$ if and only if $-p = -\xi \cdot X)$ thus the Radon transform is not a one to one transform. In the following theorem we collect some properties of Radon transform .

Theorem 3.2. *Let $R = \check{f}(p,\xi)$ be the Radon transform then we have the following properties*
1) $\check{f}(sp, s\phi) = \det s\check{f}(p,\xi)$,
2) $R(c_1f + c_2g) = c_1Rf + c_2Rg$,
3) $Rf(AX) = \det(A^{-1})\check{f}(p,(A^{-1})^t\xi)$,
4) $Rf(\lambda X) = \frac{1}{\lambda^n}\check{f}(p,\frac{1}{\lambda}\xi)$,
5) $Rf(X-\vec{a}) = \check{f}(p-\xi\cdot\vec{a},\xi)$,
6) $R(\Delta_x f(x)) = \frac{\partial^2\check{f}(p,\xi)}{\partial p^2}$,
7) $\frac{\partial}{\partial\xi_k}f(X) = -\frac{\partial}{\partial p}R(x_kf(X))$,
8) $R(f * g) = \check{f} * \check{g}$.

Theorem 3.3. [1]. *Let* F_1 *denote the one dimensional and* F_n *denote the* $n-$*dimensional Fourier transforms on* $\mathbb{R}^n$, *we have*

$$Rf = {\mathrm{F}_1}^{-1}\mathrm{F}_\mathrm{n} f.$$

Theorem 3.4. [1]. *For any odd number* $n \in \mathrm{N}$, *the inversion formula for Radon transform is given by*

$$f(X) = \frac{1}{2^n}\Big(\frac{\sqrt{\Delta_x}}{\pi i}\Big)^{n-1} \int_{|\xi|=1} \check{f}(p,\xi)d\xi$$

and for any even number $n \in \mathrm{N}$, *we have*

$$f(X) = \Big(\frac{\sqrt{\Delta_x}}{2\pi i}\Big)^n \int_{|\xi|=1} d\xi \int_{-\infty}^{+\infty} dp \frac{(\frac{\partial}{\partial p})^{n-1}\check{f}(p,\xi)}{p-\xi\cdot X}.$$

Proof. Use the Courant and Hilbert (1962) identity. □

In the rest of this paper we are going to study the Radon transform on semi direct product of locally compact groups.

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