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SHEARLET TRANSFORM: A NEW DIRECTIONAL REPRESENTATION

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Abstract. Drawbacks of classic wavelet theory in dealing with data, among them problems concerning image processing, stimulated during the past 20 years many mathematicians to seek for efficient substitutes such as shearlet theory.

In this paper we introduce shearlet representation and compare it with other representations such as wavelets and curvelets.

1. Introduction

Among the main tasks of applied harmonic analysis are, firstly optimally sparse representation of functions belonging to a specific family in terms of some building blocks such as frames or bases; secondly, identifying discontinuities of functions such as edge detection in image processing. For a long time wavelet theory was the best possible tool for dealing with these problems, but wavelets was isotropic objects, that is, wavelets act in every direction equally and this is not a desirable phenomenon since for example, in an image texture and edge is not isotropic. So it was natural to seek for new representations enjoying some directional sensitivity. The first satisfactory directional representation was curvelet introduced by Cands and Donoho [2] in 2004. The

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two main drawbacks of the curvelet approach are that, firstly, this system is not singly generated, i.e., it is not derived from the action of countably many operators applied to a single (or finite set) of generating functions; secondly, its construction involves rotations and these operators do not preserve the digital lattice, which prevents a direct transition from the continuum to the digital setting. In the year 2005, shearlets were introduced by Guo, Kutyniok, Labate, Lim, and Weiss in [3, 4]. One of the distinctive features of shearlets is the use of shearing to control directional selectivity, in contrast to rotation used by curvelets. It allows shearlet systems to be derived from a single generator, and it also ensures a unified treatment of the continuum and digital world due to the fact that the shear matrix preserves the integer lattice. In addition, similar to wavelets, shearlet theory has a group theoretical background derived from a locally compact group *shearlet group*.

Finally we note that shearlet systems provide optimally sparse approximations of anisotropic features in multivariate data.

Preliminaries and notations. *Let*$A_a = \begin{bmatrix} a & 0 \\ 0 & \sqrt{a} \end{bmatrix}$ be anisotropic (parabolic) scaling matrix and $S_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ be shear matrix acting on plane. Let $\psi \in L^2(R^2)$, and for each $a \in R^+$, $s \in R$ and $t \in R^2$ define $\psi_{a,s,t} \in L^2(R^2)$ by

$$\psi_{a,s,t}(x) = a^{-3/4}\psi\big(A_a^{-1}S_s^{-1}(x-t)\big)$$

Then the shearlet system generated by ψ is defined by

$$\{\psi_{a,s,t} : a \in R^+, s \in R, t \in R^2\}$$

The associated continuous shearlet transform of some $f \in L^2(R^2)$ is given by

$$\mathcal{SH}_\psi f : R^+ \times R \times R^2 \to \mathcal{C}$$
$$\mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{a,s,t}\rangle$$

In general, given a unitary representation U of a locally compact group G on a Hilbert space $\mathcal{H}$, a function $\psi \in \mathcal{H}$ is called admissible, if

$$\mathcal{C}_\psi := \int_G |\langle \psi, U(g)\psi\rangle|^2 d\mu_l(g) < \infty.$$

The admissibility condition is important, since it usually yields to a resolution of identity that allows the reconstruction of signals $f \in \mathcal{H}$ from the representation coefficients $(\langle \psi, U(g)\psi\rangle)_{g\in G}$.

Let the shearlet group $\mathbb{S}$ be defined to be the set $R^+ \times R \times R^2$ along with the multiplication law given by

$$(a, s, t)(a', s', t') = (aa', s + \sqrt{a}s', t + S_s A_a t')$$

Shearlet group is a locally compact group and as mentioned in [1], the left and right Haar measures, respectively, are

$$d\mu_l(a, s, t) = \frac{da}{a^3} dsdt, d\mu_r(a, s, t) = \frac{da}{a} dsdt.$$

Moreover $\sigma : \mathbb{S} \to \mathcal{U}(L^2(R^2))$ defined by

$$\sigma(a, s, t)\psi = \psi_{a,s,t}$$

is a unitary representation of $\mathbb{S}$ on $L^2(R^2)$.

Given an admissible $\psi \in L^2(R^2)$, define

$$\mathcal{C}_\psi^+ := \int_0^\infty \int_R \frac{|\hat{\psi}(\xi)|^2}{\xi_1^2} d\xi_2 d\xi_1 \quad , \quad \mathcal{C}_\psi^- := \int_{-\infty}^0 \int_R \frac{|\hat{\psi}(\xi)|^2}{\xi_1^2} d\xi_2 d\xi_1$$

2. Main results

Shearlet transform tends to be isometry:

Theorem 2.1. *If $f, \psi \in L^2(R^2)$, then*

$$\begin{aligned}\|\mathcal{SH}_\psi f\|_{L^2(\mathbb{S})}^2 &= \int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \frac{da}{a^3} dsdt \\ &= \mathcal{C}_\psi^+ \int_R \int_0^\infty |\hat{f}(\omega)|^2 d\omega_1 d\omega_2 + \mathcal{C}_\psi^- \int_R \int_{-\infty}^0 |\hat{f}(\omega)|^2 d\omega_1 d\omega_2.\end{aligned}$$

Proof. Theorem 2.5 of [1]. □

Corollary 2.2. *Let $\psi \in L^2(R^2)$ be such that*

$$\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\xi)|^2}{\xi_1^2} d\xi < \infty,$$

then ψ is admissible.

Corollary 2.3. *Given an admissible $\psi \in L^2(R^2)$, If $\mathcal{C}_\psi^- = \mathcal{C}_\psi^+ = \mathcal{C}_\psi$, then the shearlet transform is a $\mathcal{C}_\psi$-multiple of an isometry.*

A function $\psi \in L^2(R^2)$ is called a continuous shearlet, if it satisfies the admissibility condition.

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