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Research Article

Behrooz Mashayekhy, Hanieh Mirebrahimi and Zohreh Vasagh

A topological approach to the nilpotent multipliers of a free product

Abstract: In this paper, using the topological interpretation of the Baer invariant $\mathcal{V}M(G)$ of a group G with respect to an arbitrary variety $\mathcal{V}$, we extend a result of Burns and Ellis [1] on the second nilpotent multiplier of a free product of two groups to the c-nilpotent multipliers for all $c \geq 1$. In particular, we show that $M^{(c)}(G*H) \cong M^{(c)}(G) \oplus M^{(c)}(H)$ when G and H are finite groups with some conditions or when G and H are two perfect groups.

Keywords: Baer invariant, variety of groups, nilpotent multiplier, simplicial groups, direct limit, perfect group

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Behrooz Mashayekhy, Hanieh Mirebrahimi, Zohreh Vasagh: Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran, e-mail: bmashf@um.ac.ir, h_mirebrahimi@um.ac.ir, zo_va761@stu-mail.um.ac.ir

1 Introduction

Let $G \cong F/R$ be a free presentation of a group G, and $\mathcal{V}$ be a variety of groups defined by a set of laws V. Then the Baer invariant of G with respect to $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) \cong \frac{R \cap V(F)}{[RV^*F]},$$

where $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \langle v(f_1,\dots,f_{i-1},f_ir,f_{i+1},\dots,f_n)(v(f_1,\dots,f_n))^{-1} \mid r \in R,\ f_i \in F,\ 1 \leq i \leq n,\ v \in V,\ n \in \mathbb{N}\rangle.$$

Note that the Baer invariant of G is always abelian and independent of the presentation of G (see [10]). In particular, if $\mathcal{V}$ is the variety of abelian groups, then the Baer invariant of G is the well-known notion, the Schur multiplier of G, which is isomorphic to the second homology group $H_2(G,\mathbb{Z})$ of G (see [9]). If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, then the Baer invariant of the group G is called the c-nilpotent multiplier of G which is denoted by $M^{(c)}(G)$, and will be

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,{}_cF]},$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F and $[R,{}_1F] = [R,F]$, $[R,{}_cF] = [[R,{}_{c-1}F],F]$, inductively.

Burns and Ellis in [1], using simplicial homotopy theory, introduced a topological interpretation for the c-nilpotent multiplier of G and gave an interesting formula for the second nilpotent multiplier of the free product of two groups as follows:

$$M^{(2)}(G*H) \cong M^{(2)}(G) \oplus M^{(2)}(H) \oplus \big(M(G)\otimes H^{\mathrm{ab}}\big) \oplus \big(G^{\mathrm{ab}} \otimes M(H)\big) \oplus \mathrm{Tor}(G^{\mathrm{ab}},H^{\mathrm{ab}}). \tag{1}$$

In [3], Franco extended the above topological interpretation to the Baer invariant of a group G with respect to any variety $\mathcal{V}$. In this paper, first, we give a topological proof to show that the Baer invariant functor $\mathcal{V}M(G)$ commutes with the direct limits of a directed system of groups. Second, we intend to extend the formula (1) to the c-nilpotent multiplier of the free product of two groups for all $c \geq 1$. In particular, we show that $M^{(c)}(G*H) \cong M^{(c)}(G) \oplus M^{(c)}(H)$, whenever one of the following conditions holds:

(i) G and H are finite abelian groups with coprime order.
(ii) G and H are finite groups with $(|G|, |H^{ab}|) = (|G^{ab}|, |H|) = 1$.
(iii) G and H are two finite groups with $(|G^{ab}|, |H^{ab}|) = (|M(G)|, |H^{ab}|) = (|G^{ab}|, |M(H)|) = 1$.
(iv) G and H are two perfect groups.

2 Preliminaries and the notation

In this section, we recall some basic notations and properties of simplicial groups which will be needed in the sequel. We refer the reader to [2] or [4] for further details.

Definition 2.1. A *simplicial set* $K_.$ is a sequence of sets $K_0, K_1, K_2, \dots$ together with maps $d_i : K_n \to K_{n-1}$ (faces) and $s_i : K_n \to K_{n+1}$ (degeneracies) for each $0 \le i \le n$, with the following conditions:

$$d_j d_i = d_{i-1} d_j \quad \text{for } j < i, \qquad s_j s_i = s_{i+1} s_j \quad \text{for } j \le i, \qquad d_j s_i = \begin{cases} s_{i-1} d_j & \text{for } j < i, \\ \text{identity} & \text{for } j = i, i+1, \\ s_i d_{j-1} & \text{for } j > i+1. \end{cases}$$

A *simplicial map* $f : K_. \to L_.$ means a sequence of functions $f_n : K_n \to L_n$, such that $f \circ d_i = d_i \circ f$, i.e. the following diagram commutes:

$$\begin{array}{ccccc} K_{n+1} & \xleftarrow{s_i} & K_n & \xrightarrow{d_i} & K_{n-1} \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ K_{n+1} & \xleftarrow{s_i} & K_n & \xrightarrow{d_i} & K_{n-1}. \end{array}$$

Similar to topological spaces, the homotopy of two simplicial maps between simplicial sets and the homotopy groups of simplicial sets are defined. The category of simplicial sets and topological spaces can be related by two functors as follows:

- The *geometric realization*, $|-|$, is a functor from the category of simplicial sets to the category of CW complexes.
- The *singular simplicial*, $S_*(-)$, is a functor from the category of topological spaces to the category of simplicial sets.

A simplicial set $K_.$ is called a *simplicial group* if each K_i is a group and all faces and degeneracies are homomorphisms.

There is a basic property of simplicial groups which is due to Moore [14], its homotopy groups $\pi_*(G_.)$ can be obtained as the homology of a certain chain complex $(NG_., \partial)$.

Definition 2.2. If $G_.$ is a simplicial group, then the *Moore complex* $(NG_., \partial)$ of $G_.$ is the (nonabelian) chain complex defined by $(NG)_n = \bigcap_{i=0}^{n-1} \ker d_i$ with $\partial_n : NG_n \to NG_{n-1}$ which is a restriction of d_n.

A simplicial group $G_.$ is said to be *free* if each G_n is a free group and degeneracy homomorphisms s_i send the free basis of G_n into the free basis for G_{n+1}.

Definition 2.3. For a reduced simplicial set $K_.$ (i.e. $K_0 = *$), let $\mathbb{G}K_.$ be the simplicial group defined by $(\mathbb{G}K)_n$ which is the free group generated by $K_{n+1} \backslash s_0(K_n)$, and the face and degeneracy operators are group homomorphisms such that

$$d_0^{\mathbb{G}K} k = (d_1 k)(d_0 k)^{-1}, \quad d_i^{\mathbb{G}K} k = d_{i+1} k, \quad s_i^{\mathbb{G}K} k = s_{i+1} k$$

for $i > 0$ and $k \in K_{n+1}$. We can consider the above notion as a functor from reduced simplicial sets to free simplicial groups which is called *Kan's functor*.

Definition 2.4. A free simplicial resolution of G consists of a free simplicial group $K_.$ with $\pi_0(K_.) = G$ and $\pi_n(K_.) = 0$ for all $n \ge 1$.

Theorem 2.5. *Let $K_{\cdot}$ and $L_{\cdot}$ be free simplicial resolutions of G, then for every $n \geq 0$, $\pi_n(T(K_{\cdot})) \cong \pi_n(T(L_{\cdot}))$, where T is a functor in the category of groups with $T(e) = e$ (for more details see [7]).*

Let us recall some results that will be needed in the sequel.

Theorem 2.6. (i) [2, Proposition 2.2] *For every simplicial group $G_{\cdot}$, the homotopy group $\pi_n(G_{\cdot})$ is abelian even for $n = 1$.*
(ii) [2, Lemma 3.2] *Every epimorphism between simplicial groups is a fibration.*
(iii) [2, Theorem 3.7] *Let $G_{\cdot}$ be a simplicial group, then $\pi_*(G_{\cdot}) \cong H_*(NG_{\cdot})$.*
(iv) [2, Theorem 3.16] *For every simplicial set $K_{\cdot}$, $\mathbb{G}K_{\cdot} \simeq \Omega|K_{\cdot}|$ where ΩX is the loop space of X.*
(v) [2, Proposition 5.6] *Let $G_{\cdot}$ and $H_{\cdot}$ be simplicial Abelian groups, then $H_n(N(G_{\cdot} \otimes H_{\cdot})) \cong H_n(N(G_{\cdot}) \otimes N(H_{\cdot}))$.*

3 Topological approach to Baer invariants

Let $X = K(G, 1)$ be the Eilenberg–MacLane space of G. Then Burns and Ellis [1, Proposition 4.1] presented an isomorphism $M^{(c)}(G) \cong \pi_1(K_{\cdot}/\gamma_{c+1}(K_{\cdot}))$, where $K_{\cdot}$ is the free simplicial group obtained from X by applying Kan's functor to the reduced singular simplicial set of X. Burns and Ellis' interpretation for $c = 1$ is $M(G) \cong \pi_1(K_{\cdot}/\gamma_2(K_{\cdot}))$. Moreover, Kan [8] proved that $\pi_*(\mathbb{G}L_{\cdot}/\gamma_2(\mathbb{G}L_{\cdot})) \cong H_{*+1}(L_{\cdot})$, where $\mathbb{G}$ is Kan's functor. Hence, we have Hopf's formula $H_2(G) \cong R \cap F/[R, F] = M(G)$, where $G = F/R$ is a free presentation of G.

Using the above notions and similar to Burns and Ellis' interpretation we can give a topological interpretation for the Baer invariant of a group G with respect to any variety $\mathcal{V}$. We recall that the following theorem was proved categorically by Franco in [3, Theorem 7].

Theorem 3.1. *Let $K_{\cdot}$ be a free simplicial resolution of G and $\mathcal{V}$ be a variety of groups defined by a set of laws V. Then the following isomorphisms hold:*

$$\pi_1(K_{\cdot}/V(K_{\cdot})) \cong \mathcal{V}M(G), \quad \pi_0(K_{\cdot}/V(K_{\cdot})) \cong G/V(G).$$

Proof. Let $G \cong F/R$ be a free presentation of G. Then for the simplicial group $K_{\cdot}$ obtained by applying Kan's functor to the reduced of $S_*(X)$, we have $|K_{\cdot}| \simeq \Omega X$ using Theorem 2.6 (iv). Therefore, $(K_{\cdot})_0 = F$ and $(K_{\cdot})_1 = R \rtimes F$ and $d_0^1(r, f) = f$ and $d_1^1(r, f) = rf$ (see [1, Proposition 4.1]). Hence, $(K_{\cdot}/V(K_{\cdot}))_0 = F/V(F)$, $(K_{\cdot}/V(K_{\cdot}))_1 = R/[RV^*F] \rtimes F/V(F)$ and $\bar{d}_0^1$ and $\bar{d}_1^1$ are induced by d_1^0 and d_1^1, respectively. We consider the Moore chain complexes $N(K_{\cdot}/V(K_{\cdot}))$ and $N(V(K_{\cdot}))$. By Theorem 2.6 (iii) we have $\pi_0(K_{\cdot}/V(K_{\cdot})) \cong G/V(G)$ and $\pi_0(V(K_{\cdot})) \cong V(F)/[RV^*F]$. By Theorem 2.6 (ii) the following exact sequence of simplicial groups is a fibration:

$$0 \to V(K_{\cdot}) \to K_{\cdot} \to \frac{K_{\cdot}}{V(K_{\cdot})} \to 0.$$

Thus it induces the long exact sequence in homotopy groups as follows:

$$\cdots \to \pi_1(K_{\cdot}) \to \pi_1\Big(\frac{K_{\cdot}}{V(K_{\cdot})}\Big) \to \pi_0\big(V(K_{\cdot})\big) \overset{\pi_0(\subseteq)}{\to} \pi_0(K_{\cdot}) \to \pi_0\Big(\frac{K_{\cdot}}{V(K_{\cdot})}\Big) \to 0.$$

Also $\pi_1(K_{\cdot}) \cong \pi_1(\Omega X) \cong \pi_2(X) = 0$ and, similarly, $\pi_0(K_{\cdot}) \cong \pi_1(X) \cong G$. Hence, $\pi_1(K_{\cdot}/V(K_{\cdot})) \cong \ker(\pi_0(\subseteq)) \cong V(F) \cap R/[RV^*F]$. By Theorem 2.6 (iv), $K_{\cdot}$ is a free simplicial resolution of G, therefore by Theorem 2.5 the result holds. □

Using the above topological interpretation of Baer invariants, we intend to study, by the topological approach, the behavior of Baer invariants with direct limits. First, we need to find the behavior of homotopy groups of simplicial groups with respect to the direct limit.

Theorem 3.2. *Let $\{{}^jG_{\cdot}, \varphi_i^j \mid i, j \in J\}$ be a direct system of simplicial groups $\{{}^jG_{\cdot}\}$ indexed by a directed set J. Then*

$$\pi_n(\varinjlim_{j\in J} {}^jG_{\cdot}) \cong \varinjlim_{j\in J} \pi_n({}^jG_{\cdot}).$$

Proof. Let ${}^j d_i^k : {}^j G_k \to {}^j G_{k-1}$ and ${}^j s_i^k : {}^j G_k \to {}^j G_{k+1}$ be faces and degeneracies for $0 \leq i \leq k$. Recall that the direct limit of simplicial groups can be considered as a simplicial group as

$$(\varinjlim_{j\in J} {}^j G_{\bullet})_n = \varinjlim_{j\in J} ({}^j G_{\bullet})_n, \quad d_i^n = \varinjlim_{j\in J} ({}^j d_i^n), \quad s_i^n = \varinjlim_{j\in J} ({}^j s_i^n).$$

We have the commutative diagram

$$\begin{array}{ccccc}
\varinjlim_{j\in J} ({}^j G_{\bullet})_{n+1} & \xleftarrow{\varinjlim_{j\in J} ({}^j s_i^n)} & \varinjlim_{j\in J} ({}^j G_{\bullet})_n & \xrightarrow{\varinjlim_{j\in J} ({}^j d_i^n)} & \varinjlim_{j\in J} ({}^j G_{\bullet})_{n-1} \\
\uparrow {\scriptstyle ({}^k\theta)_{n+1}} & & \uparrow {\scriptstyle ({}^k\theta)_n} & & \uparrow {\scriptstyle ({}^k\theta)_{n-1}} \\
{}^k G_{n+1} & \xleftarrow{{}^k s_i^n} & {}^k G_n & \xrightarrow{{}^k d_i^n} & {}^k G_{n-1} \\
\uparrow {\scriptstyle (\varphi_l^k)_{n+1}} & & \uparrow {\scriptstyle (\varphi_l^k)_n} & & \uparrow {\scriptstyle (\varphi_l^k)_{n-1}} \\
{}^l G_{n+1} & \xleftarrow{{}^l s_i^n} & {}^l G_n & \xrightarrow{{}^l d_i^n} & {}^l G_{n-1}.
\end{array}$$

Consider the Moore chain complex $N(\varinjlim_{j\in J} {}^j G_{\bullet})$ as follows:

$$\cdots \xrightarrow{\varinjlim {}^j d_3^3} \ker \varinjlim_{j\in J} {}^j d_0^2 \cap \ker \varinjlim_{j\in J} {}^j d_1^2 \xrightarrow{\varinjlim {}^j d_2^2} \ker \varinjlim_{j\in J} {}^j d_0^1 \xrightarrow{\varinjlim {}^j d_1^1} \varinjlim_{j\in J} ({}^j G_{\bullet})_0.$$

Since the direct limit of a directed system preserves exact sequences and

$$\varinjlim_{j\in J} (\ker {}^j d_k^i) \cap \varinjlim_{j\in J} (\ker {}^j d_{k'}^i) = \varinjlim_{j\in J} (\ker {}^j d_k^i \cap \ker {}^j d_{k'}^i),$$

we obtain the chain complex

$$\cdots \xrightarrow{\varinjlim {}^j d_3^3} \varinjlim_{j\in J} (\ker {}^j d_0^2 \cap \ker {}^j d_1^2) \xrightarrow{\varinjlim {}^j d_2^2} \varinjlim_{j\in J} \ker {}^j d_0^1 \xrightarrow{\varinjlim {}^j d_1^1} \varinjlim_{j\in J} ({}^j G_{\bullet})_0.$$

Hence,

$$N(\varinjlim_{j\in J} {}^j G_{\bullet}) \cong \varinjlim_{j\in J} N({}^j G_{\bullet})$$

when J is a directed set. Also, the homology functor preserves the direct limits of directed systems of simplicial groups. Therefore, using Theorem 2.6 (ii), we have

$$\pi_n(\varinjlim_{j\in J} {}^j G_{\bullet}) \cong H_n(N(\varinjlim_{j\in J} {}^j G_{\bullet})) \cong \varinjlim_{j\in J} H_n(N({}^j G_{\bullet})) \cong \varinjlim_{j\in J} \pi_n({}^j G_{\bullet}).$$ □

Remark 3.3. Note that homotopy groups do not commute with the direct limits of topological spaces in general and hence Theorem 3.2 does not hold in the category of topological spaces. To prove this, Goodwillie [5] gives the following interesting example.

Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle. Let $A_n = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1,\ x \leq 1 - 1/n\}$ be a sequence of closed arcs in S^1 such that A_n is in the interior of A_{n+1} and such that the union U of all the A_n is the complement of a point in S^1. Let X_n be S^1/A_n. The direct limit of the diagram of circles $X_1 \to X_2 \to \cdots$ is S^1/U, a two-point space in which only one of the points is closed. Or if one prefers to form the colimit in the category of Hausdorff or T_1 spaces, then the colimit is a point. Either way, π_1 does not commute with the direct limit.

Note that homotopy groups preserve the direct limit of filtered based spaces (for more details see [12, p. 75]).

Now we are in the position to give a topological proof for the following theorem which was proved algebraically in [13].

Theorem 3.4. *Let $\{G_i, \varphi_i^j \mid i \in I\}$ be a directed system of groups. Then*

$$\varinjlim_{i\in I} \mathcal{V}M(G_i) \cong \mathcal{V}M(\varinjlim_{i\in I} (G_i)).$$

Proof. Let $K_{i_\bullet}$ be a free simplicial group corresponding to G_i. By [13, Lemma 3.2], $\varinjlim_{i\in I} K_{i_\bullet}$ is a free simplicial group and Theorem 3.2 implies that $\varinjlim_{i\in I} K_{i_\bullet}$ is a free simplicial resolution corresponding to $\varinjlim_{i\in I} G_i$. Hence, we have

$$\mathcal{V}M(\varinjlim_{i\in I}(G_i)) \cong \pi_1\Big(\frac{\varinjlim K_{i_\bullet}}{V(\varinjlim K_{i_\bullet})}\Big) \cong \pi_1 \varinjlim\Big(\frac{K_{i_\bullet}}{V(K_{i_\bullet})}\Big) \cong \varinjlim \pi_1\Big(\frac{K_{i_\bullet}}{V(K_{i_\bullet})}\Big) \cong \varinjlim_{i\in I} \mathcal{V}M(G_i).$$ □

4 Main results

In this section, by considering the variety of nilpotent groups, we intend to compute the nilpotent multipliers of the free product of two groups.

Proposition 4.1. *Let $F = K * L$ be the free product of two free groups K and L and let $\varphi : F \to K \times L$ be the natural epimorphism. Then for all $c \geq 1$, there exists the short exact sequence*

$$0 \to \ker\bar{\varphi}_c \to \frac{F}{\gamma_{c+1}(F)} \xrightarrow{\bar{\varphi}_c} \frac{K}{\gamma_{c+1}(K)} \times \frac{L}{\gamma_{c+1}(L)} \to 0,$$

where $\ker\bar{\varphi}_c \cong \frac{[K,L]^F}{[K,L,_{c-1}F]^F}$ *which satisfies the exact sequence*

$$0 \to \frac{[K,L,_{c-2}F]^F}{[K,L,_{c-1}F]^F} \to \ker\bar{\varphi}_c \to \frac{[K,L]^F}{[K,L,_{c-2}F]^F} \to 0.$$

Moreover, we have the isomorphism

$$\frac{[K,L,_{c-2}F]^F}{[K,L,_{c-1}F]^F} \cong \oplus \sum_{\text{for some } i+j=c} \underbrace{K^{\mathrm{ab}} \otimes \cdots \otimes K^{\mathrm{ab}}}_{i \text{ times}} \otimes \underbrace{L^{\mathrm{ab}} \otimes \cdots \otimes L^{\mathrm{ab}}}_{j \text{ times}}.$$

Proof. Clearly, the natural epimorphism $\varphi : F \to K \times L$ induces an epimorphism

$$\bar{\varphi}_c : F/\gamma_{c+1}(F) \to K/\gamma_{c+1}(K) \times L/\gamma_{c+1}(L)$$

given by

$$\bar{\varphi}_c(\omega\gamma_{c+1}(F)) = (\omega_1\gamma_{c+1}(K), \omega_2\gamma_{c+1}(L))$$

for all $c \geq 1$, where $\varphi(\omega) = (\omega_1, \omega_2)$. Therefore, we have

$$\ker\bar{\varphi}_c \cong \frac{[K,L]^F\gamma_{c+1}(F)}{\gamma_{c+1}(F)} \cong \frac{[K,L]^F}{[K,L]^F \cap \gamma_{c+1}(F)} \cong \frac{[K,L]^F}{[K,L,_{c-1}F]^F}.$$

Hence, there exists the exact sequence

$$0 \to \frac{[K,L,_{c-2}F]^F}{[K,L,_{c-1}F]^F} \to \ker\bar{\varphi}_c \to \frac{[K,L]^F}{[K,L,_{c-2}F]^F} \to 0.$$

Moreover, let K and L be free groups on $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_{m+n}\}$, respectively. Then by a theorem of Hall [6, Theorem 11.2.4] which says that $\gamma_c(F)/\gamma_{c+1}(F)$ is a free abelian group with the basis of all basic commutators of the weight c on $\{x_1, \ldots, x_{m+n}\}$, it is easy to show that $[K,L,_{c-2}F]^F/[K,L,_{c-1}F]^F$ is a free abelian group with the basis $\bar{B} = \{b[K,L,_{c-1}F]^F \mid b \in B\}$, where $B = B_1 - B_2 - B_3$ in which B_1, B_2, B_3 are the set of all basic commutators of the weight c on $\{x_1, \ldots, x_m, \ldots, x_{m+n}\}$, $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_{m+n}\}$, respectively. Now by the universal property of free abelian groups and tensor products, we have the isomorphism

$$\frac{[K,L,_{c-2}F]^F}{[K,L,_{c-1}F]^F} \cong \oplus \sum_{\text{for some } i+j=c} \underbrace{K^{\mathrm{ab}} \otimes \cdots \otimes K^{\mathrm{ab}}}_{i \text{ times}} \otimes \underbrace{L^{\mathrm{ab}} \otimes \cdots \otimes L^{\mathrm{ab}}}_{j \text{ times}}.$$

Note that the number of copies in the above direct sum is the number of all basic commutators of the weight c on K and L. □

Theorem 4.2. *Let G and H be two groups with*

$$G^{ab} \otimes H^{ab} = M^{(1)}(G) \otimes H^{ab} = M^{(1)}(H) \otimes G^{ab} = \mathrm{Tor}(G^{ab}, H^{ab}) = 0.$$

Then the following isomorphism holds for all $c \geq 1$:

$$M^{(c)}(G * H) \cong M^{(c)}(G) \oplus M^{(c)}(H).$$

Proof. For $c = 1$, by a well-known result on the Schur multiplier of the free product (see [9, Theorem 2.6.8]), we have the isomorphism

$$M^{(1)}(G * H) \cong M^{(1)}(G) \oplus M^{(1)}(H).$$

Now we discuss in more detail Burns and Ellis' method in [1, Proposition 2.13], and extend the method to any $c \geq 2$. Let $K_{.}$ and $L_{.}$ be the free simplicial groups corresponding to $X = K(G, 1)$ and $Y = K(H, 1)$, respectively. By the van Kampen theorem we have $X \vee Y \cong K(G * H, 1)$ so that the free simplicial group $F_{.}$ obtained by applying Kan's functor to the reduced singular simplicial set of $X \vee Y$ is equal to $K_{.} * L_{.}$. Therefore, we have $M^{(c)}(G * H) \cong \pi_1(F_{.}/\gamma_{c+1}(F_{.}))$. By Proposition 4.1, consider the following short exact sequence of simplicial groups:

$$0 \to (\ker \bar{\varphi}_c)_{.} \to \frac{F_{.}}{\gamma_{c+1}(F_{.})} \xrightarrow{\bar{\varphi}_c} \frac{K_{.}}{\gamma_{c+1}(K_{.})} \times \frac{L_{.}}{\gamma_{c+1}(L_{.})} \to 0,$$

where $(\ker \bar{\varphi}_c)_{.}$ is a simplicial group defined by $((\ker \bar{\varphi}_c)_{.})_n = \ker(\bar{\varphi}_c)_n$. Theorem 2.6 (ii) implies the long exact sequence

$$\begin{aligned} \cdots \to \pi_2((\ker \bar{\varphi}_c)_{.}) \to \pi_2\Big(\frac{F_{.}}{\gamma_{c+1}(F_{.})}\Big) &\xrightarrow{\pi_2(\bar{\varphi}_c)} \pi_2\Big(\frac{K_{.}}{\gamma_{c+1}(K_{.})}\Big) \oplus \pi_2\Big(\frac{L_{.}}{\gamma_{c+1}(L_{.})}\Big) \\ \to \pi_1((\ker \bar{\varphi}_c)_{.}) \to \pi_1\Big(\frac{F_{.}}{\gamma_{c+1}(F_{.})}\Big) &\xrightarrow{\pi_1(\bar{\varphi}_c)} \pi_1\Big(\frac{K_{.}}{\gamma_{c+1}(K_{.})}\Big) \oplus \pi_1\Big(\frac{L_{.}}{\gamma_{c+1}(L_{.})}\Big) \to \cdots . \end{aligned}$$

Let $\alpha_n^K : \pi_n(F_{.}/\gamma_{c+1}(F_{.})) \to \pi_n(K_{.}/\gamma_{c+1}(K_{.}))$ and $\alpha_n^L : \pi_n(F_{.}/\gamma_{c+1}(F_{.})) \to \pi_n(L_{.}/\gamma_{c+1}(L_{.}))$ be homomorphisms induced by continuous maps from $X \vee Y$ to X and Y, respectively. Since $\pi_n(K_{.}/\gamma_{c+1}(K_{.})) \oplus \pi_n(L_{.}/\gamma_{c+1}(L_{.}))$ is a coproduct in the category of abelian groups, there exists a unique homomorphism

$$\psi_n : \pi_n(K_{.}/\gamma_{c+1}(K_{.})) \oplus \pi_n(L_{.}/\gamma_{c+1}(L_{.})) \to \pi_n(F_{.}/\gamma_{c+1}(F_{.}))$$

such that $\alpha_n^{K_{.}} \circ \psi_n = p_n^{K_{.}}$ and $\alpha_n^{L_{.}} \circ \psi_n = p_n^{L_{.}}$, where $p_n^{L_{.}}$ and $p_n^{K_{.}}$ are projection maps. Therefore, $\psi_n \circ \pi_n(\bar{\varphi}_c) = \mathrm{id}$ and, consequently,

$$\pi_1(\ker \bar{\varphi}_c) \oplus \pi_1(K_{.}/\gamma_{c+1}(K_{.})) \oplus \pi_1(L_{.}/\gamma_{c+1}(L_{.})) \cong \pi_1(F_{.}/\gamma_{c+1}(F_{.})).$$

By Proposition 4.1, we have the following exact sequence of simplicial groups:

$$0 \to \frac{[K_{.}, L_{.}, {}_{c-2}F_{.}]^{F_{.}}}{[K_{.}, L_{.}, {}_{c-1}F_{.}]^{F_{.}}} \to (\ker \bar{\varphi}_c)_{.} \to \frac{[K_{.}, L_{.}]^{F_{.}}}{[K_{.}, L_{.}, {}_{c-2}F_{.}]^{F_{.}}} \to 0.$$

Theorem 2.6 (ii) implies the following long exact sequence of homotopy groups which in low dimension takes the form

$$\cdots \to \pi_1\Big(\frac{[K_{.}, L_{.}, {}_{c-2}F_{.}]^{F_{.}}}{[K_{.}, L_{.}, {}_{c-1}F_{.}]^{F_{.}}}\Big) \to \pi_1(\ker \bar{\varphi}_c)_{.} \to \pi_1(\ker \bar{\varphi}_{c-1})_{.} \to \cdots .$$

By induction on c, we prove that $\pi_1(\ker \bar{\varphi}_c)_{.} = 0$. For $c = 2$, Burns and Ellis [1, Lemma 4.2] proved that $(\ker \bar{\varphi}_2)_{.} \cong K_{.}^{ab} \otimes L_{.}^{ab}$. Hence,

$$\begin{aligned} \pi_1(\ker \bar{\varphi}_2)_{.} &\cong H_1(N(K_{.}^{ab} \otimes L_{.}^{ab})) \\ &\cong H_1(N(K_{.}^{ab}) \otimes N(L_{.}^{ab})) \\ &\cong H_1(N(K_{.}^{ab})) \otimes H_0(N(L_{.}^{ab})) \oplus H_0(N(K_{.}^{ab})) \otimes H_1(N(L_{.}^{ab})) \oplus \mathrm{Tor}(H_0(N(K_{.}^{ab})), H_0(N(L_{.}^{ab}))) \\ &\cong M^{(1)}(G) \otimes H^{ab} \oplus M^{(1)}(H) \otimes G^{ab} \oplus \mathrm{Tor}(G^{ab}, H^{ab}). \end{aligned}$$

Similarly, we can prove that

$$\pi_0(K_{\bullet}^{ab} \otimes L_{\bullet}^{ab}) \cong G^{ab} \otimes H^{ab}.$$

Now let $\pi_1(\ker \bar{\varphi}_{c-1})_{\bullet} = 0$. We are going to show that $\pi_1(\ker \bar{\varphi}_c)_{\bullet} = 0$. Since

$$\frac{[K_{\bullet}, L_{\bullet}, {}_{c-2}F_{\bullet}]^{F_{\bullet}}}{[K_{\bullet}, L_{\bullet}, {}_{c-1}F_{\bullet}]^{F_{\bullet}}} \cong \oplus \sum_{\text{for some } i+j=c} \underbrace{K_{\bullet}^{ab} \otimes \cdots \otimes K_{\bullet}^{ab}}_{i \text{ times}} \otimes \underbrace{L_{\bullet}^{ab} \otimes \cdots L_{\bullet}^{ab}}_{j \text{ times}},$$

it is enough to compute

$$\pi_1(\underbrace{K_{\bullet}^{ab} \otimes \cdots \otimes K_{\bullet}^{ab}}_{i \text{ times}} \otimes \underbrace{L_{\bullet}^{ab} \otimes \cdots \otimes L_{\bullet}^{ab}}_{j \text{ times}}).$$

Since $i, j \neq 0$, we have

$$\begin{aligned}\pi_1(\underbrace{K_{\bullet}^{ab} \otimes \cdots \otimes K_{\bullet}^{ab}}_{i \text{ times}} \otimes \underbrace{L_{\bullet}^{ab} \otimes \cdots \otimes L_{\bullet}^{ab}}_{j \text{ times}}) &\cong \pi_1(K_{\bullet}^{ab} \otimes L_{\bullet}^{ab}) \otimes \pi_0(\underbrace{K_{\bullet}^{ab} \otimes \cdots \otimes K_{\bullet}^{ab}}_{(i-1) \text{ times}} \otimes \underbrace{L_{\bullet}^{ab} \otimes \cdots \otimes L_{\bullet}^{ab}}_{(j-1) \text{ times}}) \\ &\oplus \pi_0(K_{\bullet}^{ab} \otimes L_{\bullet}^{ab}) \otimes \pi_1(\underbrace{K_{\bullet}^{ab} \otimes \cdots \otimes K_{\bullet}^{ab}}_{(i-1) \text{ times}} \otimes \underbrace{L_{\bullet}^{ab} \otimes \cdots \otimes L_{\bullet}^{ab}}_{(j-1) \text{ times}}) \\ &\oplus \operatorname{Tor}(\pi_0(K_{\bullet}^{ab} \otimes L_{\bullet}^{ab}), \pi_0(\underbrace{K_{\bullet}^{ab} \otimes \cdots \otimes K_{\bullet}^{ab}}_{(i-1) \text{ times}} \otimes \underbrace{L_{\bullet}^{ab} \otimes \cdots \otimes L_{\bullet}^{ab}}_{(j-1) \text{ times}})).\end{aligned}$$

By hypothesis, we have $\pi_1(K_{\bullet}^{ab} \otimes L_{\bullet}^{ab}) = \pi_0(K_{\bullet}^{ab} \otimes L_{\bullet}^{ab}) = 0$. Hence, $\pi_1(\ker \bar{\varphi}_c)_{\bullet} = 0$. □

Corollary 4.3. *Let G and H be two groups. Then, for all $c \geq 1$, we have the isomorphism*

$$M^{(c)}(G * H) \cong M^{(c)}(G) \oplus M^{(c)}(H)$$

if one of the following conditions holds:

(i) *G and H are two abelian groups with coprime orders.*

(ii) *G and H are two finite groups with* $(|G|, |H^{ab}|) = (|G^{ab}|, |H|) = 1$.

(iii) *G and H are two finite groups with*

$$(|G^{ab}|, |H^{ab}|) = (|M(G)|, |H^{ab}|) = (|G^{ab}|, |M(H)|) = 1.$$

(iv) *G and H are two perfect groups.*

Note that parts (i)–(iii) of the above corollary are vast generalizations of a result of the first author (see [11, Theorem 2.5]).

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