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# THE JACOBSON GRAPH OF COMMUTATIVE RINGS

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Abstract. Some graph theoretical properties of the Jacobson graph of a finite commutative ring with non-zero identity including its connectivity, planarity and perfectness is obtained. Also, we compute some numerical invariants of Jacobson graphs, namely diameter, dominating number, independence number and vertex chromatic numbers and give an estimate for their edge chromatic number.

## 1. Introduction

Associating a graph to a ring has been the interest of many authors recently. Studying graph theoretical aspects of rings was first considered by Beck in 1988. Beck [3] defines the zero-divisor graphs of rings and characterizes all rings which are finitely colorable. The work of Beck is further continued by Anderson and Naseer in [1] and, for other graph theoretical aspects, by Anderson and Livingston in [2]. See [4, 5] for further results on zero-divisor and other graphs.

Let $R$ be a commutative ring with non-zero identity. The Jacobson radical of $R$ is defined by

$$J(R) = \bigcap\{\mathfrak{m} : \mathfrak{m} \text{ is a maximal ideal of } R\}.$$

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* Speaker.

It is known that an element $r \in R$ belongs to $J(R)$ if and only if $1-rx$ is invertible for all $x \in R$. We remind that $R$ is semi-simple if $J(R)=0$ and hence the quotient ring $R/J(R)$ is always a semi-simple ring.

The *Jacobson graph* of $R$, denoted by $\mathfrak{J}_R$, is defined as a graph with vertex set $V(\mathfrak{J}_R)=R\setminus J(R)$ in such a way that two distinct vertices $x,y \in V(\mathfrak{J}_R)$ are adjacent if and only if $1-xy \notin U(R)$, where $U(R)$ denotes the group of units of $R$.

## 2. Main results

**Theorem 2.1.** *Let $(R,\mathfrak{m})$ be a finite local ring with associated field $F$. Then the connected components of $\mathfrak{J}_R$ are either complete graphs of size $|\mathfrak{m}|$ or complete bipartite graphs $K_{|\mathfrak{m}|,|\mathfrak{m}|}$. Moreover,*

(1) *if $|F|$ is odd, then $\mathfrak{J}_R$ has two complete components and $(|F|-3)/2$ complete bipartite components, and*
(2) *if $|F|$ is even, then $\mathfrak{J}_R$ has one complete component and $(|F|-2)/2$ complete bipartite components.*

**Theorem 2.2.** *Let $R$ be a finite ring. The graph $\mathfrak{J}_R$ is a complete graph if and only if $R$ is a local ring with associated field of order 2.*

**Theorem 2.3.** *Let $R$ be a finite non-local ring. Then $\mathfrak{J}_R$ is a connected graph and $\operatorname{diam}(\mathfrak{J}_R)\leqslant 3$.*

**Theorem 2.4.** *Let $R$ be a finite ring . Then $\mathfrak{J}_R$ is planar if and only if either $R$ is a field, or $R$ is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_4$, $\mathbb{Z}_2\oplus\mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2)$ *of order* 4,
(ii) $\mathbb{Z}_6$ *of order* 6,
(iii) $\mathbb{Z}_8$, $\mathbb{Z}_2\oplus\mathbb{Z}_4$, $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x,x^2)$, $\mathbb{Z}_2\oplus\mathbb{Z}_2[x]/(x^2+x+1)$, $\mathbb{Z}_4[x]/(2x,x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$ *of order* 8, *and*
(iv) $\mathbb{Z}_9$, $\mathbb{Z}_3\oplus\mathbb{Z}_3$, $\mathbb{Z}_3[x]/(x^2)$ *of order* 9.

A $k$-vertex coloring of $\Gamma$ is an assignment of $k$ colors to the vertices of $\Gamma$ such that no two adjacent vertices have the same color and the *chromatic number* of $\Gamma$, denoted by $\chi(\Gamma)$, is the smallest number $k$ for which $\Gamma$ has a $k$-coloring. Using these notions, a graph is *perfect* if the clique number and the chromatic number of its induced subgraphs are equal. The complement of a graph $\Gamma$ is denoted by $\overline{\Gamma}$.

**Theorem 2.5.** *Let $R$ be a finite ring. Then $\mathfrak{J}_R$ is perfect if and only if*

(1) *$R$ is a local ring,*
(2) *$R/J(R)\cong\mathbb{Z}_3\oplus\mathbb{Z}_3$ or $\mathbb{Z}_2\oplus F$,*
(3) *$R/J(R)\cong\mathbb{Z}_3\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ or $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$,*

(4) $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
*where $F$ is a finite field.*

A *dominating set* in a graph $\Gamma$ is a set of vertices $X$ such that every vertex of $\Gamma$ either belongs to $X$ or is adjacent to some vertex in $X$. The size of a smallest dominating subset of $\Gamma$ is called the *dominating number* of $\Gamma$ and is denoted by $\gamma(\Gamma)$.

**Theorem 2.6.** *Let $R = R_1 \oplus \cdots \oplus R_n$ $(n \geq 2)$ be a decomposition of $R$ into local rings $R_i$ with associated fields $F_i$, respectively. If $F_a$ and $F_b$ are the two largest fields among $F_i$, then*

$$\gamma(\mathfrak{J}_R) = \max(|F_1| - 1, \ldots, |F_n| - 1),$$

*if $J(R) \neq 0$, or $J(R) = 0$ and $|F_a| = |F_b|$, and*

$$\gamma(\mathfrak{J}_R) = \left\lceil \frac{|F_a| + |F_b|}{2} \right\rceil - 1,$$

*otherwise.*

A subset $X$ of vertices in a graph $\Gamma$ is called an *independent set* if the induced subgraph on $X$ is empty. The size of a largest independent subset of $\Gamma$ is called the *independence number* of $\Gamma$ and is denoted by $\alpha(\Gamma)$.

In what follows, we use the notations $E(R)$ and $O(R)$ for the set of all indices $i$ such that $|F_i|$ is even and odd, respectively. Also $e(R)$ and $o(R)$ stand for the cardinals of $E(R)$ and $O(R)$, respectively.

**Theorem 2.7.** *Let $R = R_1 \oplus \cdots \oplus R_n$ be a decomposition of $R$ into local rings $(R_i, \mathfrak{m}_i)$ with associated fields $F_i$, respectively. Then*

$$\alpha(\mathfrak{J}_R) = \frac{|J(R)|}{2^n} \prod_{i \in E(R)} |F_i| \prod_{i \in O(R)} (|F_i| - 1|) + 2n - e(R) - |J(R)|.$$

**Theorem 2.8.** *Let $R = R_1 \oplus \cdots \oplus R_n$ be a decomposition of $R$ into local rings $(R_i, \mathfrak{m}_i)$ with associated fields $F_i$, respectively. Then*

$$\chi(\mathfrak{J}_R) = |J(R)| \sum_{\substack{X \subseteq \{1,\ldots,n\} \\ |X| \leq \frac{n}{2}}} \max\{\theta(X), \theta(X^c)\}$$

*if $n$ is odd, and*

$$\chi(\mathfrak{J}_R) = |J(R)| \left( \sum_{\substack{X \subseteq \{1,\ldots,n\} \\ |X| < \frac{n}{2}}} \max\{\theta(X), \theta(X^c)\} + \frac{1}{2} \sum_{\substack{X \subseteq \{1,\ldots,n\} \\ |X| = \frac{n}{2}}} \max\{\theta(X), \theta(X^c)\} \right)$$

*if $n$ is even, where $\theta(\emptyset)=0$ and*

$$\theta(Y)=\frac{2^{o(Y)}}{1+\left[\frac{1}{e(Y)+1}\right]}\prod_{i\in E(Y^c)}(|F_i|-1)\prod_{i\in O(Y^c)}(|F_i|-2)$$

*for each non-empty subset $Y$ of $\{1,\ldots,n\}$.*

**Theorem 2.9.** *Let $R$ be a finite ring with maximal ideals $\mathfrak{m}_1,\mathfrak{m}_2,\ldots,\mathfrak{m}_k$ and let $x$ be a unit of $R$. Then*

$$\deg(x)=\begin{cases}\left|\bigcup_{i=1}^k\mathfrak{m}_i\right|, & 1-x^2\in U(R),\\ \\ \left|\bigcup_{i=1}^k\mathfrak{m}_i\right|-1, & 1-x^2\notin U(R).\end{cases}$$

The edge coloring of a graph $\Gamma$ is defined in a similar manner to vertex coloring in such a way that incident edges accept different colors. The *edge chromatic number* of $\Gamma$ is denoted by $\chi'(\Gamma)$.

**Corollary 2.10.** *If $R$ is a finite ring with maximal ideals $\mathfrak{m}_1,\mathfrak{m}_2,\ldots,\mathfrak{m}_k$, then*

$$\left|\bigcup_{i=1}^k\mathfrak{m}_i\right|-1\leq\chi'(\mathfrak{J}_R)\leq\left|\bigcup_{i=1}^k\mathfrak{m}_i\right|+1.$$

*In particular,*

$$n-\varphi(n)-1\leq\chi'(\mathfrak{J}_{\mathbb{Z}_n})\leq n-\varphi(n)+1$$

A *clique* in a graph $\Gamma$ is a maximal complete subgraph and the *clique number* of $\Gamma$, denoted by $\omega(\Gamma)$, is the size of a largest clique in $\Gamma$.

*Conjecture.* Let $R=R_1\oplus\cdots\oplus R_n$ be a decomposition of $R$ into local rings $R_i$ with associated fields $F_i$, respectively. Then

$$\omega(\mathfrak{J}_R)=\frac{|R|}{\min\{|F_1|,\ldots,|F_n|\}}.$$

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