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Capability of finite nilpotent groups of class 2 with cyclic Frattini subgroups

Azam Kaheni*, Rasoul Hatamian, Saeed Kayvanfar

Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.

Abstract. A group is called capable if it is a central factor group. Let **N** denote the set of all finite groups of nilpotency class 2 whose derived subgroups be cyclic and coincide with their Frattini subgroups. This paper is organized to provide the explicit structures of capable groups in **N**.

Keywords: Capable groups; Extra special p-groups; Nilpotent groups; Schur multiplier.

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1 Introduction

A group G is said to be capable if it is isomorphic to the group of inner automorphisms of some group K. Baer [2] characterized the capable groups that are direct sums of cyclic groups. The capability of very specific classes, such as metacyclic groups, extra special p-groups, nilpotent products of cyclic p-groups of class less than or equal to p is also investigated. Moreover, some numerical necessary and sufficient conditions for capability of p-groups of class 2 and prime exponent are known. For example see [5, Theorem 1] and [10, Theorem 5.26]. Magidin obtained some interesting results in this area, when p is an odd prime. He [11] showed that the direct sum of two non cyclic p-groups of class at most 2 and exponent p is capable if and only if each of them are capable. Furthermore, for each finite non cyclic p-group G of class at most 2 and exponent p, he proved that $G \times \mathbb{Z}_p$ is capable if and only if G is capable.

Now, let **N** denote the set of all finite groups of nilpotency class 2 whose derived subgroups be cyclic and coincide with their Frattini subgroups. In this paper, we will

*Correspondence to: Azam Kaheni, Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran. Email: `azam.kaheni@stu-mail.um.ac.ir, azamkaheni@yahoo.com.`
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determine the exact structures of capable groups in **N**. Since, a nilpotent group is capable if and only if each of its Sylow subgroups is capable, we reduce the problem to a restricted subclass. In other words, we shall first focus on p-groups with at least two generators, then conclude the main result for finite nilpotent groups in **N**. One should notice that a classification of two generated finite capable p-groups of nilpotency class 2 was given by Bacon and Kappe [1] for an odd prime p. In 2006, Magidin [9] used the classification of two generated finite 2-groups of nilpotency class 2 that was given by Kappe *et al.* [7] to achieve the set of all two generated finite capable 2-groups of nilpotency class 2. Notice that the commutator subgroup of a finite two generated p-group of class 2 is always cyclic. But a finite p-group of class 2 with cyclic commutator subgroup may not be two generated such as extra special p-groups. For this reason, Yadav [15] interested in studying finite capable p-groups of class 2 with cyclic commutator subgroups. He proved this problem for a group G for which $Z(G) \subseteq \Phi(G)$. Actually, Yadav [15] showed that each capable group of nilpotency class 2 that its central elements are nongenerator should be two generators. Since finite capable two generated p-groups of class 2 had been listed by Bacon, Kappe and Magidin, the result can be concluded immediately. Accordingly, our result for p-groups gives an answer to Yadav's problem [15] and also generalizes the results of Magidin [10, 11], somehow. The main result also is the generalization of the work in [12].

2 Preliminaries

For a group G, recall from [3] that the epicenter of G is denoted by $Z^*(G)$ and defined to be the intersection of all $\varphi(Z(E))$, where (E, φ) is a central extension of G. A relation between $Z^*(G)$ and the notion of capability is provided by Beyl *et al.* [3] as follows.

Theorem 2.1. $Z^*(G)$ is the smallest central subgroup of G whose factor group is capable. In particular, G is capable if and only if the epicenter of G is trivial.

Obviously, the class of all capable groups is neither subgroup closed nor under homomorphic image. But this class is closed under direct product (see [3, Proposition 6.1]), and therefore $Z^*(\prod_{i\in I} G_i) \subseteq \prod_{i\in I} Z^*(G_i)$, for each family $\{G_i\}$ of groups. One should also note that the inclusion is proper in general. Beyl *et al.* [3] gave a sufficient condition forcing equality as follows.

Proposition 2.1. Let $G = \prod_{i\in I} G_i$. Assume that for $i \neq j$ the maps $\upsilon_i \otimes 1 : Z^*(G_i) \otimes G_j/G'_j \to G_i/G'_i \otimes G_j/G'_j$ are zero, where υ_i is the natural map $Z^*(G_i) \to G_i \to G_i/G'_i$. Then $Z^*(G) = \prod_{i\in I} Z^*(G_i)$.

This certainly indicates that a finite nilpotent group is capable if and only if its Sylow subgroups are capable.

Isaacs [6] proved that if G is a finite capable group such that G' is cyclic and all elements of order 4 in G' are central in G, then $[G : Z(G)] \leq |G'|^2$. In 2005, Podoski and Szegedy [13] showed that the assumption about elements of order 4 can be omitted. In fact they proved the following theorem.

Theorem 2.2. If G is a finite capable group and G' is cyclic, then $[G : Z(G)] \leq |G'|^2$.

As an immediate consequence of Theorem 2.2, we can say that if an extra special p-group is capable, then its order is p^3. Of course, the exact structures of capable extra special p-groups are given by Beyl *et al.* [3] as follows.

Theorem 2.3. An extra special p-group is capable if and only if it is either D_8 or E_1, where D_8 is the dihedral group of order 8 and E_1 is the extra special p-group of order p^3 and exponent p $(p > 2)$.

Let G be a group with a free presentation $1 \to R \to F \to G \to 1$, where F is a free group. Recall that the Schur multiplier of G, which is denoted by $H_2(G)$ is defined to be $(R \cap F')/[R, F]$. One can see that the Schur multiplier of the group G is always abelian and independent of the choice of the free presentation of G.

Proposition 2.2. For every finite groups H and K, we have

$$H_2(H \times K) \cong H_2(H) \times H_2(K) \times (\tfrac{H}{H'} \otimes \tfrac{K}{K'}).$$

It is known that every cyclic group has trivial Schur multiplier. Indeed, if $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$ is a finite abelian group such that $n_{i+1} \mid n_i$, for all $1 \leq i < k$, then using the above fact one can obtain the Schur multiplier of G as follows.

$$H_2(G) \cong \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}^{(2)} \times \ldots \times \mathbb{Z}_{n_k}^{(k-1)},$$

where $\mathbb{Z}_n^{(m)}$ denotes the direct product of m copies of the cyclic group $\mathbb{Z}_n$.

Proposition 2.3. Let G be an extra special p-group of order p^{2m+1}, Q_8 be the quaternion group of order 8 and E_2 be the extra special p-group of order p^3 and exponent p^2 $(p > 2)$. Then

(i) if $m \geq 2$, then $|H_2(G)| = p^{2m^2-m-1}$,

(ii) if $m = 1$, then the order of Schur multiplier of D_8, Q_8, E_1, and E_2 are equal to 2, 1, p^2 and 1, respectively.

Theorem 2.4. [3, Theorem 4.2] Let G be a group and N be a central subgroup of G. Then $N \subseteq Z^*(G)$ if and only if the natural map $H_2(G) \to H_2(G/N)$ is monomorphism.

Beyl *et al.* [3] proved a necessary condition for a group to be capable as follows.

Lemma 2.1. [3, Proposition 1.2] If G is capable and the commutator factor group G/G' of G is of finite exponent, then also $Z(G)$ is bounded and the exponent of $Z(G)$ divides that of G/G'.

The capability of abelian groups has been investigated by various authors. Baer [2] described all capable abelian groups which are direct sums of cyclic groups. The next lemma explains Baer's result for finitely generated abelian groups.

Lemma 2.2. Let G be a finitely generated abelian group written as

$$G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k},$$

such that each integer n_{i+1} is divisible by n_i, where $\mathbb{Z}_0$ denote the infinite cyclic group. Then G is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

It is interesting to know, Hall [4] gave a necessary condition for the capability of finite p-groups of small class. His criterion reduces to what occurs in Baer's result for finitely generated abelian groups, namely that for a capable group in any generating set, there exist two elements of maximal order.

3 Main results

In view of Proposition 2.1, one can obtain some interesting results about nilpotent capable groups. For example, it is easy to conclude that there is no nilpotent capable group of square free order. Moreover, if the order of a nilpotent capable group G is as $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, then α_i should be greater than 1, for each $1 \leq i \leq t$. In this section, the capability of some nilpotent groups of class 2 has been studied. Actually, the explicit structures of nilpotent capable groups in $\mathbf{N}$ are determined. Since for each nilpotent group G of exponent p we have $G' = \Phi(G)$, so this result generalizes the works of Magidin [10, 11] somehow.

Lemma 3.1. Let G be a p-group of nilpotency class 2. If $G' = \Phi(G)$, then there exists a subgroup H of G such that $G = HZ(G)$ and $H' = Z(H)$.

Proof. Since G/G' is an elementary abelian p-group, then the factor group $Z(G)/G'$ has a complement in G/G', say H/G'. Thus $G = HZ(G)$. Now, it is easy to see that $G' = H', G/Z(G) \cong H/Z(H)$ and $Z(H) = Z(G) \cap H$. Therefore, the result holds. $\square$

By using the method applied in the proof of Lemma 3.1, one can easily give another proof for [10, Lemma 4.16].

Let G be a finite p-group such that $\Phi(G) \subseteq Z(G)$. It is easy to see that the derived subgroup of G is an elementary abelian group. Now under some conditions, it can be deduced that the exponent of $Z(G)$ is also p.

Lemma 3.2. If G is a capable p-group such that $\Phi(G) \subseteq G'$, then $Z(G)$ is an elementary abelian p-group.

Proof. Let G be a capable p-group such that $G' = \Phi(G)$. Then G/G' is an elementary abelian p-group, and the result follows by Lemma 2.1. $\square$

Invoking Theorem 2.3 and Lemma 3.2 we have the following corollary.

Corollary 3.1. Let G be a finite capable p-group of nilpotency class 2. If $G' = \Phi(G)$ is a cyclic group, then G is of order p^3 or $Z(G)$ is not cyclic.

The following lemma shorten the proof of the next theorem.

Lemma 3.3. Let G be a finite capable p-group of nilpotency class 2 and $G' = \Phi(G)$ be a cyclic group. Then there exist subgroups H and K of G such that $G = H \times K$, in which H is an extra special p-group of order p^3 and $K \subseteq Z(G)$ is an elementary abelian p-group.

Proof. By the method we used in the proof of Lemma 3.1, there exists a subgroup H of G such that $G = HZ(G)$, $H' = Z(H) = G'$ and also G and H have the same central factor group. Since G' is cyclic and contained in $Z(G)$, then the order of G' is p. Moreover, Theorem 2.2, and the isomorphism $G/Z(G) \cong H/Z(H)$ imply that the order of H is p^3. Now, by assuming K to be the complement of G' in $Z(G)$ the result follows. □

Theorem 3.1. Let G be a finite p-group of nilpotency class 2 and $G' = \Phi(G)$ be a cyclic group. Then G is capable if and only if $G \cong E_1 \times K$ or $G \cong D_8 \times K$, where K is an elementary abelian p-subgroup of $Z(G)$.

Proof. Let G be a capable p-group. By Lemma 3.3, $G \cong H \times K$, where H is an extra special p-group of order p^3 and $K \subseteq Z(G)$ is an elementary abelian p-group. Since E_2 and Q_8 can not be as a direct summand of a capable nilpotent group (see [14, Theorems 4.1 and 5.2]), one can deduce the result. Conversely, we first prove our claim for $p = 2$. Let $G \cong D_8 \times K$, in which K is an elementary abelian 2-group. If K is trivial or $rank(K)$ is greater than 1, then $Z^*(K)$ is trivial by Lemma 2.2. On the other hand the epicenter of D_8 is trivial, so in this case G will be capable. Now, let $K = \mathbb{Z}_2$. Whereas, the epicenter of G is a subgroup of $Z^*(D_8) \times Z^*(\mathbb{Z}_2) = 1 \times \mathbb{Z}_2$, hence G is capable if and only if $Z^*(G) \neq 1 \times \mathbb{Z}_2$. Let, by way of contradiction, the epicenter of G be as $1 \times \mathbb{Z}_2$. Then the factor group $G/Z^*(G)$ is isomorphic to D_8. Using Propositions 2.2 and 2.3, one can easily obtain that the order of $H_2(G)$ is greater than the order of $H_2(G/Z^*(G))$. Therefore the map $H_2(G) \to H_2(G/Z^*(G))$ is not a monomorphism. This contradiction completes the proof in this case. Furthermore, a similar argument as above or [11, Corollary 4.7], shows that $E_1 \times K$ is also capable. Then the result follows. □

Certainly, the exact structure of a capable p-group of order p^n which its derived factor group is elementary abelian of rank p^{n-1}, is given by the above theorem. So, the work of Niroomand and Parvizi [12] is generalized.

We are now ready to prove the main theorem as follows.

Suppose that G belongs to $\mathbf{N}$. Then $G \cong H_1 \times \ldots \times H_t$, in which H_i's are Sylow p_i-subgroups of G. Let H_j be an abelian p_j-subgroup for each $1 \leq j \leq r$, and the nilpotency class of other H_i's be 2. Let $G_l = \mathbb{Z}_{p_l}^{(s_l)}$, where s_l is a natural number grater than 1, and $1 \leq l \leq r$. Also, let $G_m = E_{p_m} \times K_{p_m}$, in which K_{p_m} is an elementary abelian p_m-subgroup of $Z(H_m)$ and E_{p_m} is an extra special p_m-group of order p_m^3 and exponent p_m ($p_m > 2$) for each $r < m \leq t$.

Combining Lemma 2.2 and Theorem 3.1 the following theorem is obtained.

Theorem 3.2. Suppose that G belongs to $\mathbf{N}$. Consider H_i's and G_i's the groups introduced as above.

(i) If $2 \nmid |G|$ or the Sylow 2-subgroup of G is abelian, then G is capable if and only if

$G \cong \prod_{i=1}^{t} G_i$.

(ii) If the Sylow 2-subgroup of G is non-abelian, then G is capable if and only if $G \cong D_8 \times K_2 \times (\prod_{\substack{i=1 \\ i\neq i_0}}^{t} G_i)$, where i_0 is the index of Sylow 2-subgroup of nilpotency class 2 and K_2 is an elementary abelian 2-subgroup of $Z(H_{i_0})$.

Proof. Since $\Phi(G) = G'$ is cyclic and $H_i \trianglelefteq G$, so $\Phi(H_i) = H_i'$ and also H_i' is cyclic for each $1 \leq i \leq t$. Now the result follows by Lemma 2.2 and Theorem 3.1. $\square$

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